On Morse-Smale dynamical systems

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Collaboration

The results were obtained in the collaboration with

Figure: V. Grines

Figure: S. Van Strien

Figure: O. Pochinka

Rough dynamical systems

The concept roughness of a dynamical system was born in Nizhny Novgorod in 1937 (then Gorky). A. Andronov and L. Pontryagin considered a dynamical system

$$\dot{x} = v(x),$$

where v is a C^1-vector field on the plane, $x \in \mathbb{R}^2$ and suggest to call it rough if for any sufficiently small perturbation in the C^1-metric, there exists a homeomorphism close to the identity map which transforms the orbits of the original dynamical system to the orbits of the perturbed system (perturbed system is topologically equivalent to the original one by a conjugating homeomorphism).

Criteria of the roughness

In the paper "A. Andronov and L. Pontryagin. Rough systems. Doklady Akademii Nauk SSSR. 1937. 14 (5): 247–250" for a dynamical system

$$\dot{x} = v(x),$$

where v is a C^1-vector field given on the unit disk and transversal to the boundary, was done a following criteria for its roughness:

- number of the equilibrium points and the periodic orbits is finite and they are hyperbolic;
- there are no saddle connections.

Leontovich-Mayer scheme

The topological classification (division into classes with respect to the topological equivalence) of structurally stable flows (dynamical systems with continuous time) on a bounded part of the plane and on the 2-sphere follows from the results by E. Leontovich-Andronova and A. Mayer. In the papers “E. Leontovich, A. Mayer. On trajectories defining qualitative structure of decomposition of the sphere into trajectories. Dokl. Akad. Nauk SSSR. 1937. 14 (5), 251–257” and “E. Leontovich, A. Mayer. On a scheme defining topological structure of decomposition into trajectories. Dokl. Akad. Nauk SSSR. 1955. 103 (4), 557–560” actually more general class of dynamical systems was considered. The classification was based on the ideas of Poincare-Bendixson to pick a set of specially chosen trajectories so that their relative position (Leontovich-Mayer scheme) fully define the qualitative structure of the decomposition of the phase space of the dynamical system into the trajectories.

Transition to a surface with positive genus

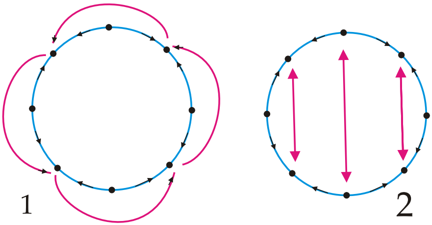
The principal difficulty in generalization of this result in case of arbitrary orientable surfaces of positive genus is the possibility of new types of motion — non-closed recurrent trajectories. The absence of such trajectories for structurally stable flows without singularities on the 2-torus at first was proved by A. Mayer. Actually in the paper “Mayer A.G. Rough transformation of the circle to the circle. Uch. Zap. GGU. 1939. Gorky, Pub. GGU, 12, 215-229.” he introduced the rough notion for cascades (discrete dynamical systems), found the conditions of the roughness for cascades on the circle and also got the topological classification for these cascades.

Rough transformations of circle

Let $R(\mathbb{S}^1)$ be class of rough transformations of the circle which consists of two subclasses $R_+(\mathbb{S}^1)$ and $R_-(\mathbb{S}^1)$ of preserving orientation and reverse orientation diffeomorphisms, accordingly.

1. For each diffeomorphism $\varphi \in R_+(\mathbb{S}^1)$ the non-wandering set $NW(\varphi)$ consists of $2n, n \in \mathbb{N}$ periodic orbits, each of them has period k.

2. For each diffeomorphism $\varphi \in R_-(\mathbb{S}^1)$ the non-wandering set $NW(\varphi)$ consists of $2q, q \in \mathbb{N}$ periodic points, two of them are fixed, others have period 2.

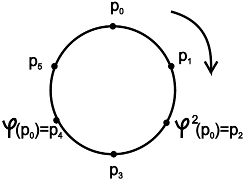


The preserving orientation case

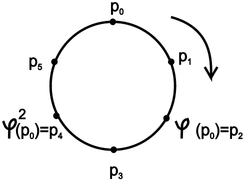
Let $\varphi \in R_+(\mathbb{S}^1)$. Enumerate the periodic points from $NW(\varphi)$: $p_0, p_1, \ldots, p_{2nk-1}, p_{2nk} = p_0$ starting from arbitrary periodic point p_0 clockwise, then $\varphi(p_0) = p_{2nl}$ and (k, l) are coprime.

Two diffeomorphisms $\varphi; \varphi' \in R_+(\mathbb{S}^1)$ with parameters $n, k, l; n', k', l'$ are topologically conjugated if and only if $n = n', k = k'$ and at least one of the following assertions holds:

- $l = l'$,
- $l = k' - l'$.



n=1
k=3
l=2

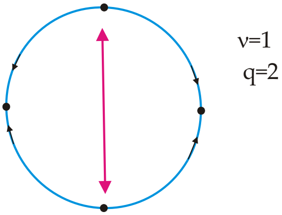


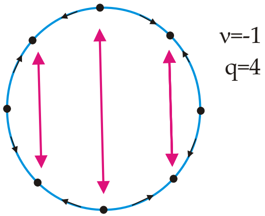
n=1
k=3
l=1

The reversing orientation case

For $\varphi \in R_{-}(\mathbb{S}^1)$ we set $\nu = -1;\ \nu = 0; \nu = +1$ if its fixed point are sources; sink and source; sinks, accordingly. Notice that $\nu = 0$ if q is odd and $\nu = \pm 1$ if q is even.

Two diffeomorphisms $\varphi; \varphi' \in R_{-}(\mathbb{S}^1)$ with parameters $q, \nu;\ q', \nu'$ are topologically conjugated if and only if $q = q'$ and $\nu = \nu'$.





Structural stability

In 1959 M. Peixoto introduced the concept of structural stability of flows to generalize the concept of roughness.

A flow f^t is called structurally stable if, for any sufficiently close flow g^t, there exists a homeomorphism h sending trajectories of the system g^t to trajectories of the system f^t. The original definition of a rough flow involved the additional requirement that the homeomorphism h be C^0-close to the identity map.

Peixoto proved that the concepts of roughness and structural stability for flows on 2-sphere are equivalent. In 1962 Peixoto proved that the conditions 1),2) above plus condition

3) all ω- and α-limit sets are contained in the union of the equilibrium points and the limit cycles

are necessary and sufficient for the structural stability of a flow on arbitrary orientable closed (compact and without boundary) surface and showed that such flows are dense in the space of all C^1-flows.

Morse-Smale systems

An immediate generalization of properties of rough flows on orientable surfaces leads to Morse-Smale systems (continuous and discrete). The non-wandering set of such a system consists of finitely many fixed points and periodic orbits, each of which is hyperbolic and the stable and unstable manifolds W_p^s and W_q^u intersect transversally for any distinct non-wandering points p, q.

Morse-Smale systems are named in 1960 after paper "Morse inequality for Dynamical Systems" Bull. Amer. Math. Soc. 1960, No. 66, 46-49" by S. Smale, where he introduced flows with the above properties (on manifolds of dimension greater than 2) and proved that they satisfy inequalities similar to the Morse inequalities.

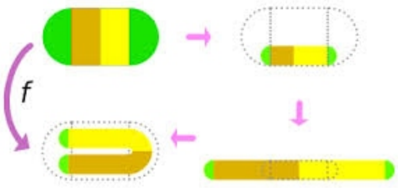
Citation

"We remark that systems satisfying 1)-3) may be very important because of the following possibilities.
(A) It seems at least plausible that system satisfying 1)-3) form an open dense set in the space (with the C^1-topology) of all vector fields on M^n.
(B) It seems likely that conditions 1)-3) are necessary and sufficient for X to be structurally stable in the sense of A. Andronov and L. Pontryagin (1937).
(A) and (B) have been provide for the case M^n is a 2-disk."

S. Smale

Morse-Smale systems do not exhaust the class of all rough systems

Later 1969 S. Smale and J. Palis showed that Morse-Smale systems are structurally stable. However, already in 1961 Smale proved that such systems do not exhaust the class of all rough systems via constructing a structurally stable diffeomorphism on the two-dimensional sphere $\mathbb{S}^2$ with infinitely many periodic points. This diffeomorphism is known now as the Smale's horseshoe.

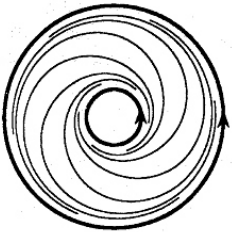


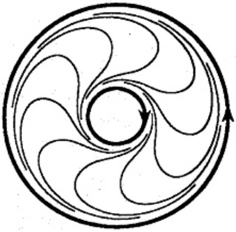
Peixoto's graph

In 1971 M. Peixoto obtained the classification for Morse-Smale flows on arbitrary surfaces. He did this by generalizing the Leontovich-Mayer scheme for such flows to a directed graph whose vertices are in one-to-one correspondence with fixed points and closed trajectories of the flow, and whose edges correspond to the connected components of the invariant manifolds of fixed points and closed trajectories. He proved that the isomorphic class of such directed graph is the complete topological invariant for the class of Morse-Smale systems on surfaces (where the isomorphisms preserve specially chosen subgraphs).

Oshemkov-Sharko approach

A. Oshemkov and V. Sharko in 1998 pointed out a certain inaccuracy concerning the Peixoto invariant due to the fact that an isomorphism of graphs does not distinguish between types of decompositions into trajectories for a domain bounded by two periodic orbits.





Three-colour graph

They therefore suggest to use a three-colour graph, see Figure 4, where vertices correspond to triangular domains and the color of an edge corresponds to passing through a side of triangles of the same color.

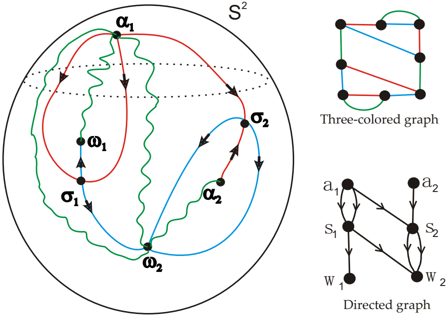


Figure: Directed and three-colour graphs for a Morse-Smale flow on 2-sphere

Gradient-like difeomorphisms

Morse-Smale diffeomorphisms is called gradient-like if it has no heteroclinic points.

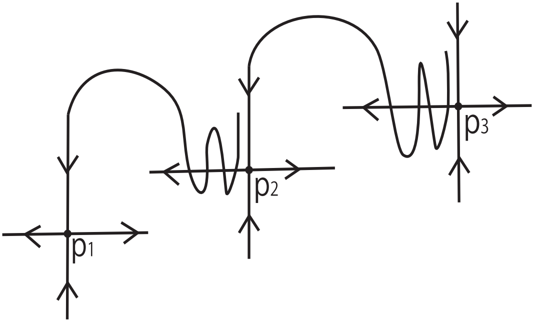


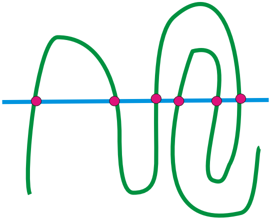
Figure: Heteroclinic points

On topological conjugacy of the gradient-like diffeomorphisms on surfaces

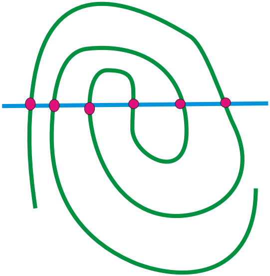
- In 1985 A. Bezdenezhnych, V. Grines showed that for gradient-like diffeomorphism on surfaces a directed graph with an automorphism is again a complete invariant.
- V. Grines, S. Zinina, O. Pochinka in 2014 showed that two gradient-like diffeomorphisms on surface are topologically conjugate if and only if their three-colour graphs equipped by periodic automorphisms are isomorphic and found an efficient algorithm for distinguishing of such graphs.

On topological conjugacy of “beh 1” 2-diffeomorphisms

- In 1993 V. Grines proved that for such diffeomorphisms an invariant similar to Peixoto’s graph carrying an additional information on the heteroclinic intersections (a substitution) is sufficient to describe necessary and sufficient conditions for the topological conjugacy.



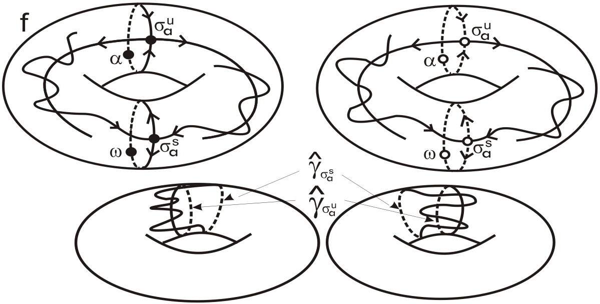
$$\begin{pmatrix} 123456 \\ 123654 \end{pmatrix}$$



$$\begin{pmatrix} 123456 \\ 163452 \end{pmatrix}$$

On topological conjugacy of “beh 1” 2-diffeomorphisms

- R. Langevin in 1993 suggest to consider an orbit space with respect to in a basin of the sink and project to this closed surface the unstable separatries of the saddle points. In 2010 T. Mitryakova and O. Pochinka applied this method to the topological classification of Morse-Smale diffeomorphisms f with $beh(f) \leq 1$ on orientable surfaces and solved the realization problem for such diffeomorphisms.

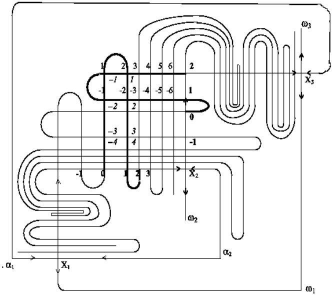


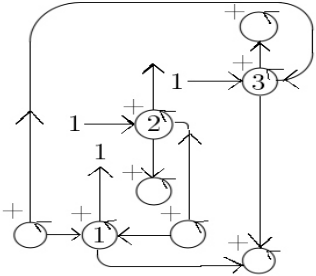
On topological conjugacy of Morse-Smale diffeomorphisms on surfaces

- Ch. Bonatti and R. Langevin in 1998 presented the topological classification of arbitrary structurally stable diffeomorphisms of orientable surfaces using Markov partitions as complete invariant. The main result of that paper consists of a finite combinatorial presentation of the global topological dynamics by the geometrical types of some Markov partitions of the hyperbolic sets and by gluing the domains along their boundary.

On topological conjugacy of Morse-Smale diffeomorphisms on surfaces

- I. Vlasenko in 1998 also presented the topological classification of arbitrary structurally stable diffeomorphisms of orientable surfaces using an equipped oriented graph whose vertices are oriented circles. In this graph to each periodic and heteroclinic point was assigned a vertex and to connecting segments of a separatrices — directed edges.

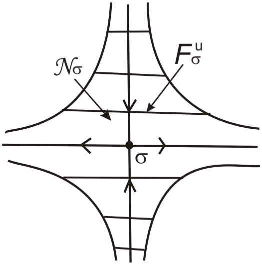




Grines-Pochinka-Van Strien approach (2016)

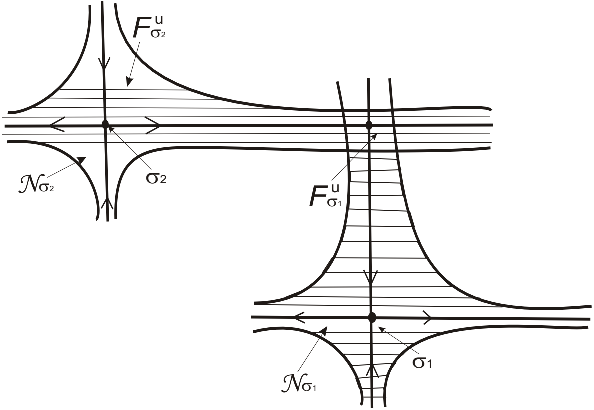
Definition

Let σ be a saddle periodic point for a Morse-Smale diffeomorphism $f : M^2 \to M^2$. A neighborhood N_σ of the point σ with a one-dimensional foliation F^u_σ containing W^u_σ as leaves, is called linearizable if there is a homeomorphism $\mu_\sigma : N_\sigma \to \mathcal{N}$ which conjugates the diffeomorphism $f^{k_\sigma}|_{N_\sigma}$ to the linear diffeomorphism and sends leaves the foliation F^u_σ to the foliation consisting of the horizontal lines.



The compatible system of neighbourhoods

An f-invariant collection N_f of linearizable neighborhoods N_σ of all saddle points $\sigma \in \Sigma$ is called compatible if for if $W^s_{\sigma_1} \cap W^u_{\sigma_2} \neq \emptyset$ for $\sigma_1, \sigma_2 \in \Sigma$ then $(F^u_{\sigma_1,x} \cap N_{\sigma_2}) \subset F^u_{\sigma_2,x}$ for $x \in (N_{\sigma_1} \cap N_{\sigma_2})$.



Heteroclinic rectangle

Definition

A closed 2-disc Π_σ *bounded by segments* $[\sigma, a]^u_\sigma,\ [a, b]^s_{\sigma_1},\ [b, c]^u_{\sigma_2},\ [c, \sigma]^s_\sigma,\ \sigma_1, \sigma_2 \in \Sigma$ *and such that* $int\,\Pi_\sigma \cap \Omega_f = \emptyset$ *is called a heteroclinic rectangle with respect to* σ *if every connected component of the set* $W^s_\Sigma \cap \Pi_\sigma$ *intersects every connected component of the set* $W^u_\Sigma \cap \Pi_\sigma$ *at exactly one point.*

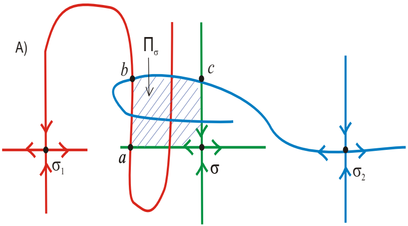


Figure: A) Π_σ is a heteroclinic rectangle. B) Π_σ is not a heteroclinic rectangle

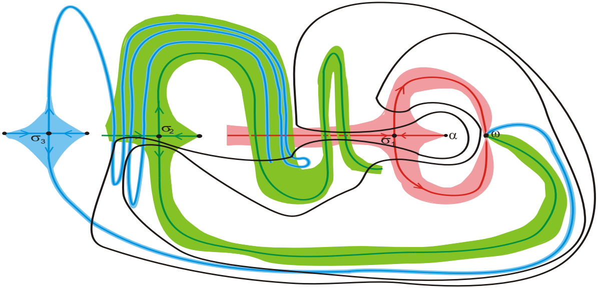
The maximal compatible system of neighbourhoods

Definition

A compatible system of neighbourhoods N_f *is called maximal if every linearizing neighborhood* $N_\sigma \in N_f$ *contains each heteroclinic rectangle* Π_σ*.*

Theorem

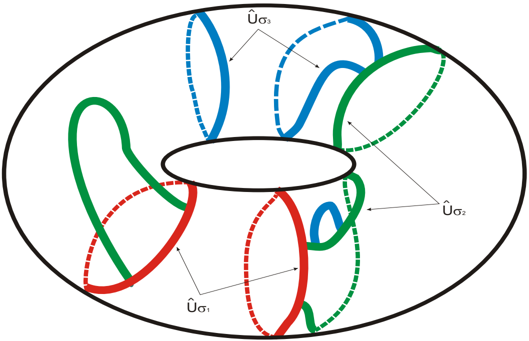
For every diffeomorphism $f \in MS(M^2)$ *there is a maximal compatible system of neighbourhoods.*



The scheme of f

Let Σ_0 be the set of all sinks of f, $V_f = W^s_{\Sigma_0} \setminus \Sigma_0$, $\quad \hat{V}_f = V_f/f$. Denote by $p_f : V_f \to \hat{V}_f$ the natural projection which is a cover in this case. Let η_f be a map composed by induced by p_f homomorphisms from the fundamental groups of connected components of $\hat{V}_f$ to the group $\mathbb{Z}$ and $\hat{U}_f = p_f(N_f)$. A scheme is

$$S_f := (\hat{V}_f, \eta_f, \hat{U}_f).$$



The classification theorem

The schemes S_f and $S_{f'}$ of diffeomorphisms $f, f' \in MS(M^2)$, respectively, are said to be equivalent if there exist a homeomorphism $\hat{\varphi} : \hat{V}_f \to \hat{V}_{f'}$ such that:
1) $\eta_{f'}\hat{\varphi}_* = \eta_f$;
2) $\hat{\varphi}(\hat{U}_f) = \hat{U}_{f'}$ and for every point $\sigma \in \Sigma_i$ there is a point $\sigma' \in \Sigma_i'$ such that $\hat{\varphi}(\hat{L}_\sigma) = \hat{L}_{\sigma'}$ and

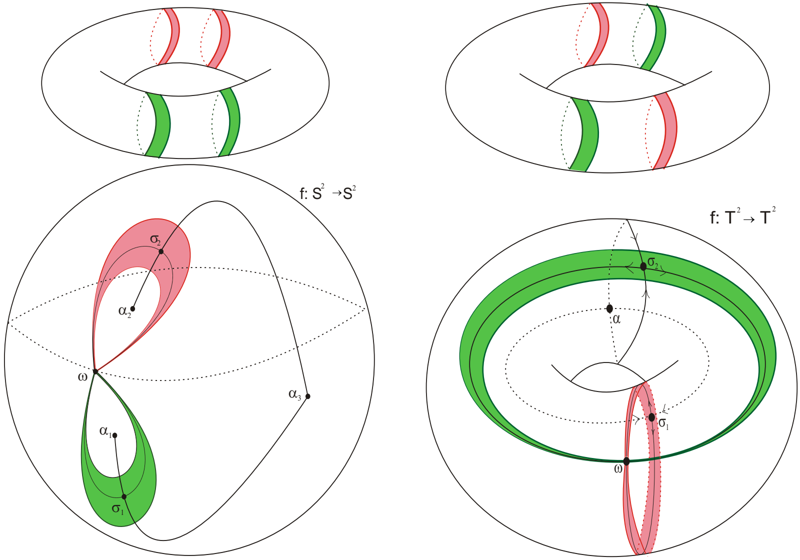
$$q_{\sigma'}^{-1}\hat{\varphi}q_\sigma : S_{\nu_\sigma} \setminus I_\sigma \to S_{\nu_{\sigma'}} \setminus I_{\sigma'}$$

preserves the order of points on the circles.

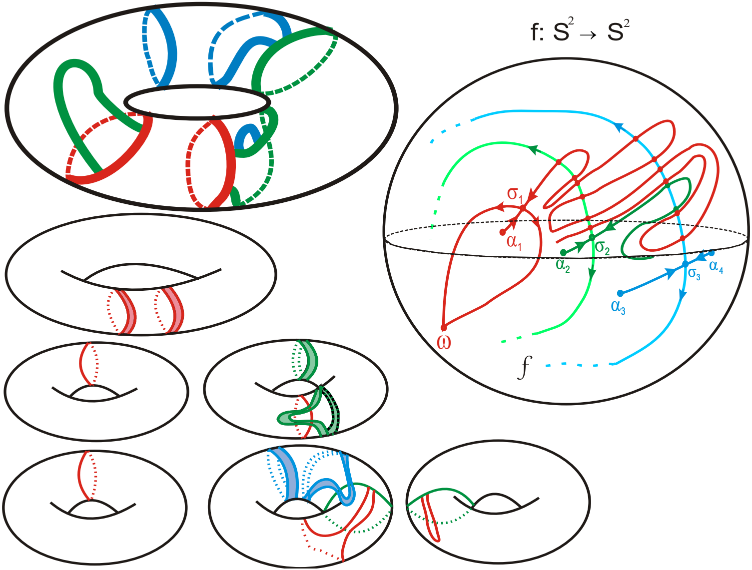
Theorem

Two diffeomorphisms $f, f' \in MS(M^2)$ are topologically conjugate iff their schemes $S_f, S_{f'}$ are equivalent.

The realization

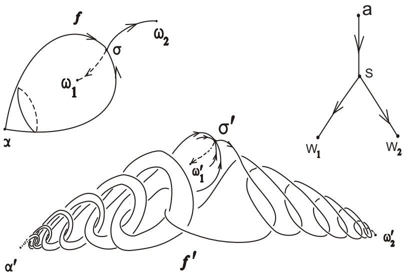


The realization



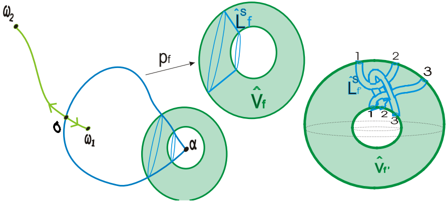
Combinatorial invariants

For wide class of Morse-Smale systems a graph is complete invariant (similar to Leontovich-Andronova and Mayer's scheme or Peixoto's graph for flows). Topological classification of even the simplest examples of Morse-Smale diffeomorphisms on 3-manifolds do not fit into the concept of selecting of the frame of the invariant manifolds of fixed points and periodic orbits. The reason for this surprising effect is the possibility of "wild" behavior of the separatrices of the saddle points. First diffeomorphism with wild separatrices was constructed by D. Pixton in 1977.



Classification of the Pixton's class $\mathcal{P}$

Let $f \in \mathcal{P}$. Set $V_f = W^u_\alpha \setminus \alpha$, $\hat{V}_f = V_f/_f$. Denote $p_f : V_f \to \hat{V}_f$ the natural projection. Then $\hat{V}_f$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, p_f is cover and $\hat{L}^s_f = p_f(W^s_\sigma \setminus \sigma)$ is homeomorphic to $\mathbb{T}^2$.



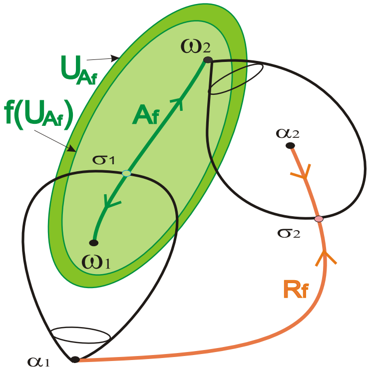
Theorem

(Ch. Bonatti, V. Grines, 2000) Diffeomorphisms $f, f' \in \mathcal{P}$ are topologically conjugated if and only if there is a homeomorphism $\hat{\varphi} : \mathbb{S}^2 \times \mathbb{S}^1 \to \mathbb{S}^2 \times \mathbb{S}^1$ such that $\hat{\varphi}(\hat{L}^s_f) = \hat{L}^s_{f'}$.

Global dynamic of Morse-Smale diffeomorphisms.

Let f be a Morse-Smale diffeomorphism on 3-manifold. Let us denote by $\Omega_q,\ q = 0, 1, 2, 3$ the set of all periodic points with Morse index q.

We set $A_f = \Omega_0 \cup W^u_{\Sigma_1}$, $R_f = \Omega_3 \cup W^s_{\Sigma_2}$. It is possible to prove that A_f (R_f) is a connected set which is an attractor (a repeller) of f.

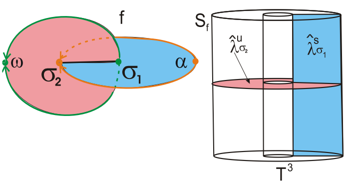


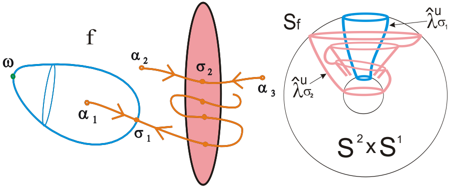
Scheme of Morse-Smale diffeomorphism f

Set $V_f = M^3 \setminus (A_f \cup R_f)$ and $\hat{V}_f = V_f/f$. Denote by $p_f : V_f \to \hat{V}_f$ the natural projection and by $\eta_f : \pi_1(\hat{V}_f) \to \mathbb{Z}$ epimorphism, induced by cover p_f. Set $\hat{L}^s_f = p_f(W^s_{\Omega_1})$ and $\hat{L}^u_f = p_f(W^u_{\Omega_2})$.

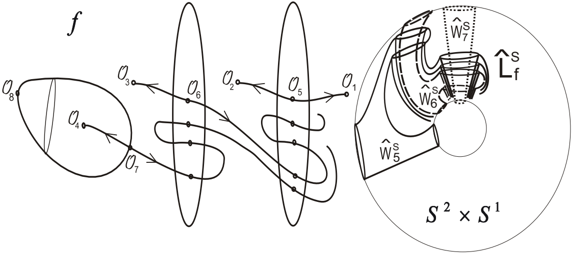
Definition

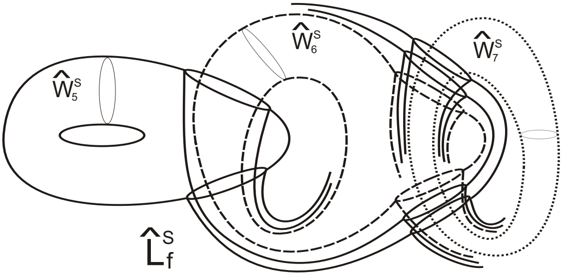
The collection $S_f = (\hat{V}_f, \eta_f, \hat{L}^u_f, \hat{L}^s_f)$ is called scheme of the diffeomorphism f.





Heteroclinic lamination





The results was obtained in collaboration with

Figure: Cr. Bonatti

Figure: F. Laudenbach

Figure: E. Pecou

Figure: V. Grines

Figure: O. Pochinka

Figure: V. Medvedev

The scheme is complete invariant

Definition

The schemes $S_f = (\hat{V}_f, \eta_f, \hat{L}^u_f, \hat{L}^s_f)$ *and* $S_{f'} = (\hat{V}_{f'}, \eta_{f'}, \hat{L}^u_{f'}, \hat{L}^s_{f'})$ *of diffeomorphisms* f, f' *are called equivalent if there is a homeomorphism* $\hat{\varphi} : \hat{V}_f \to \hat{V}_{f'}$ *such that*
1) $\eta_f = \eta_{f'} \hat{\varphi}_*$;
2) $\hat{\varphi}(\hat{L}^u_f) = \hat{L}^u_{f'}$, $\hat{\varphi}(\hat{L}^s_f) = \hat{L}^s_{f'}$.

Theorem

The diffeomorphisms f, f' *are topologically conjugated if and only if their schemes are equivalent.*

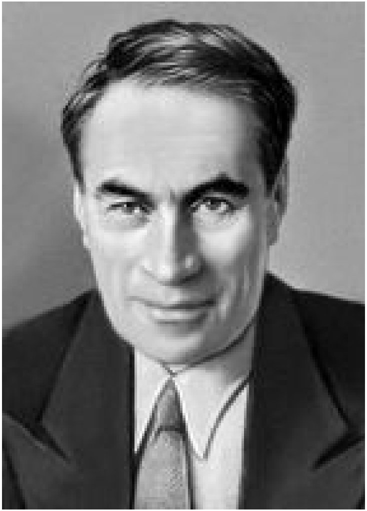


Figure: A. Andronov

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Figure: L. Pontryagin

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An equilibrium point x_0 where $v(x_0) = 0$, is said to be hyperbolic if none of the eigenvalues of the linearization of v at x_0 is purely imaginary. A periodic orbit of a flow is said to be hyperbolic if none of the eigenvalues of the Poincare return map at a point on the orbit has absolute value one.

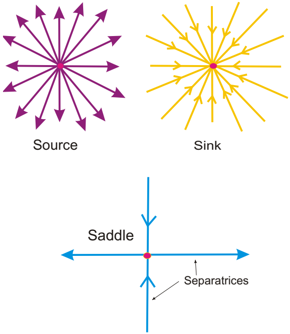

Figure: Hyperbolic equilibrium points

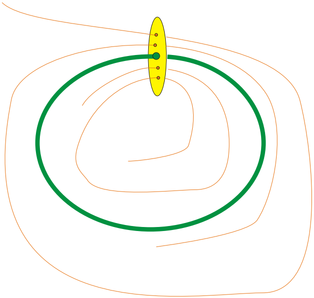

Figure: Hyperbolic periodic orbit

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Saddle connection refers to a situation where an orbit from one saddle point enters the same (homoclinic orbit) or another saddle point (heteroclinic orbit), i.e. the unstable and stable saddle separatrices are connected.

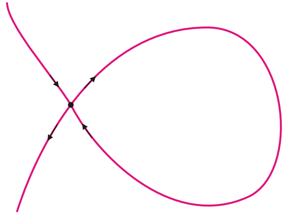

Figure: Homoclinic connection

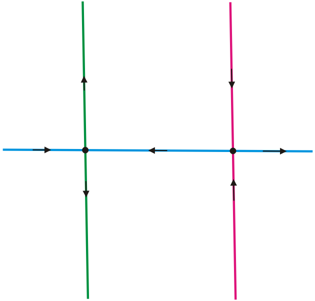

Figure: Heteroclinic connection

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Figure: E. Leontovich-Andronova

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Figure: A. Mayer

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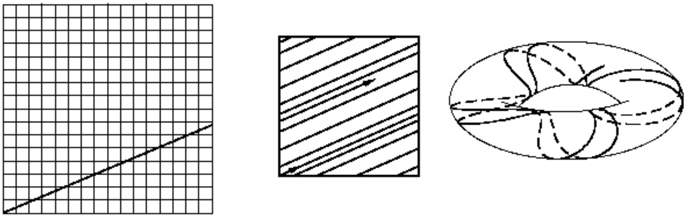

Figure: Irrational winding of the torus

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Figure: M. Peixoto

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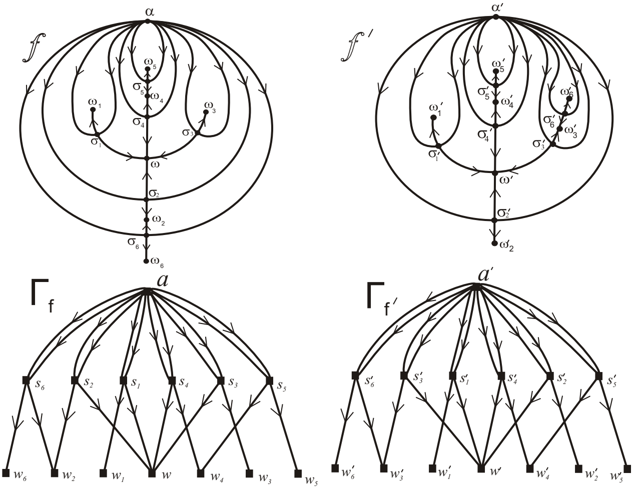

Figure: The diffeomorphisms $f, f' : \mathbb{S}^2 \to \mathbb{S}^2$ have isomorphic graphs but they are not topologically conjugate as their equipped graphs are not isomorphic

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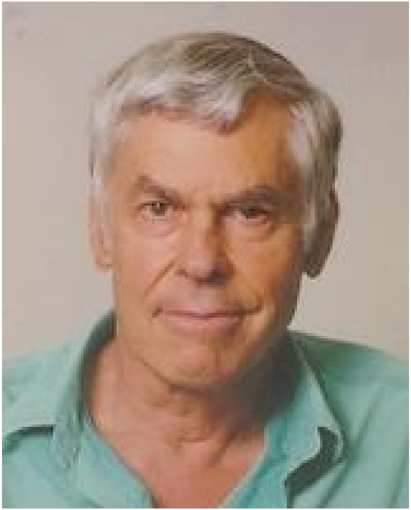

Figure: S. Smale

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Figure: J. Palis

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Figure: A. Oshemkov

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Figure: V. Sharko

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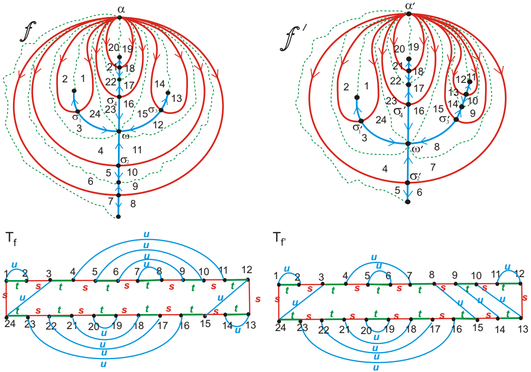

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Figure: A. Bezdenezhnych

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Figure: V. Grines

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Figure: S. Zinina

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Figure: O. Pochinka

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Figure: Ch. Bonatti

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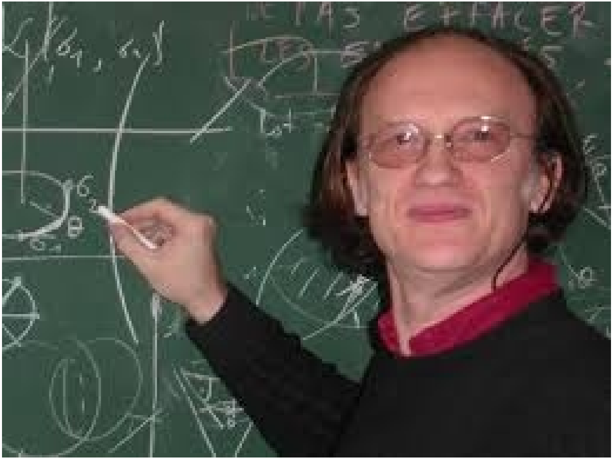

Figure: R. Langevin

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Figure: T. Mitryakova

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Figure: I. Vlasenko

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Figure: D. Pixton

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