Painlevé monodromy varieties: geometry and quantisation

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Based on Chekhov-Mazzocco-R. *arXiv:1511.03851, Int.Math.Res.Notes, 28 Nov. 2106*

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Plan:

- Painlevé equations, Confluence, Isomonodromy and Affine cubics.
- Decorated character varieties.
- Quantisation.

Painlevé equations

The Painlevé equations are non linear second order ODE of the form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = F\left(z, w, \frac{\mathrm{d}w}{\mathrm{d}z}\right), \qquad z \in \mathbb{C},$$

where $F(z, w, y)$ is a rational function of z, w, y and the solutions $w(z; c_1, c_2)$ satisfy

1. Painlevé–Kowalevski property: $w(z; c_1, c_2)$ have no *critical points* that depend on c_1, c_2.
2. Otherwise, they are the only second order ODE (besides the elementary ones: linear, Riccati, elliptic equations,...) such that the movable singularities are poles (i.e., movable essential singularities (branching points) are excluded).
3. For generic c_1, c_2, $w(z; c_1, c_2)$ are new functions, Painlevé Transcendents.

Painlevé property:

- **Example for 1-st ordre ODE**:

$$
\begin{array}{llll}
w' = w & \Longrightarrow & w = e^{z-z_0}, & \checkmark \\
w' = w^2 & \Longrightarrow & w = \dfrac{1}{z_0 - z}, & \checkmark \\
w' = w^3 & \Longrightarrow & w \sim \dfrac{1}{\sqrt{z - z_0}}. & \text{X}
\end{array}
$$

Painlevè list

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 6w^2 + z \qquad\qquad \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 2w^3 + zw + \alpha$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{3w-1}{2w(w-1)}w_z^2 - \frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{\gamma w}{z} + \frac{(w-1)^2}{z^2}\frac{\alpha w^2 + \beta}{w} + \frac{\delta w(w+1)}{w-1}$$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)w_z^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w_z + \\ + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left[\alpha + \beta\frac{z}{w^2} + \gamma\frac{z-1}{(w-1)^2} + \delta\frac{z(z-1)}{(w-z)^2}\right]$$

Painlevé transcendents - paradigmatic integrable systems

- Reductions of soliton equations (KdV, KP, NLS);
- They admit a Hamiltonian formulation;
- They can be expressed as the isomonodromic deformation of some linear differential equation with rational coefficients;
- Recently: P_{II} - has a genuine fully NC analogue (V. Retakh-V.R.)
- More recently: P_{IV} - has a (non genuine) fully NC analogue (M. Cafasso-M. Iglesias)

Confluences of the Painlevé equations-1

Example

Take $w(z)=\epsilon\tilde{w}(\tilde{z})+\frac{1}{\epsilon^5},\quad z=\epsilon^2\tilde{z}-\frac{6}{\epsilon^{10}},\quad \alpha=\frac{4}{\epsilon^{15}}$ then *PII*

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}=2w^3+zw+\alpha$$

becomes

$$\frac{\mathrm{d}^2\tilde{w}}{\mathrm{d}\tilde{z}^2}=6\tilde{w}^2+\tilde{z}+\epsilon^2(2\tilde{w}^3+\tilde{z}\tilde{w}),$$

that for $\epsilon\to 0$ is *PI*.

Confluences of the Painlevé equations-2 (Sakai)

$$
\begin{tikzcd}
 & & P_{III} \arrow[r] \arrow[dr] & P_{III}^{D_7} \arrow[r] \arrow[dr] & P_{III}^{D_8} \\
P_{VI} \arrow[r] & P_V \arrow[ur] \arrow[r] \arrow[dr] & P_V^{deg} \arrow[ur] \arrow[dr] & P_{II}^{JM} \arrow[r] & P_I \\
 & & P_{IV} \arrow[ur] \arrow[r] & P_{II}^{FN} \arrow[ur] &
\end{tikzcd}
$$

All Painlevé equations are **isomonodromic deformation equations** (Jimbo-Miwa1980)

$$\frac{\mathrm{d}B}{\mathrm{d}\lambda} - \frac{\mathrm{d}A}{\mathrm{d}z} = [A, B]$$

$A = A(\lambda; z, w, w_z)$, $B = B(\lambda; z, w, w_z) \in \mathfrak{sl}_2$.
This means that **the monodromy data** of the linear system

$$\frac{\mathrm{d}Y}{\mathrm{d}\lambda} = A(\lambda; z, w, w_z)Y$$

are **locally constant along solutions of the Painlevé equation**.
The monodromy data play the role of initial conditions.
The monodromy data are encoded in an affine cubic surfaces called *monodromy varieties*.

$\mathcal{M}_{0,4,\theta}$

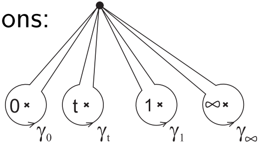
- Painlevé VI corresponds to the rank 2 flat meromorphic connection (= 1-st order fuchsian linear system) on $\mathbb{CP}^1$ with 4 regular singular points $0, t, 1, \infty$.

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} Y = \mathcal{A}(\lambda) Y, \qquad \mathcal{A}(\lambda) = \frac{\mathcal{A}_0}{\lambda} + \frac{\mathcal{A}_t}{\lambda - t} + \frac{\mathcal{A}_1}{\lambda - 1}.$$

$\mathcal{A}_\infty = -\mathcal{A}_0 - \mathcal{A}_1 - \mathcal{A}_\infty$.
Eigenvalues of A_j are $(\theta_j, -\theta_j)$, $j = 0, 1, t, \infty$

- Monodromy matrices $M_0, M_t, M_1, M_\infty \in SL_2(\mathbb{C})$.
- The module space of flat rank 2 connections:

$$\mathcal{M}_{0,4,\theta} = \left\{ M_j \middle| \begin{array}{l} \mathrm{Tr}(M_j) = 2\cos(2\pi\theta_j), \\ M_0 M_t M_1 M_\infty = 1 \end{array} \right\} \Big/ SL_2$$



$\dim \mathcal{M}_{0,4,\theta} = 2$

- As coordinates on $\mathcal{M}_{0,4,\theta}$ one can use σ_i defined by $\mathrm{Tr}(M_j M_k) = 2\cos(2\pi\sigma_i)$, $i, j, k = 0, t, 1$.

Painlevé VI τ function

- Jimbo relation: $G_j = \mathrm{Tr}(M_j) = 2\cos(2\pi\theta_j)$,
  $x_i = \mathrm{Tr}(M_j M_k) = 2\cos(2\pi\sigma_i)$,

$$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + G_1^2 + G_2^2 + G_3^2 + G_\infty^2 + G_1G_2G_3G_\infty =$$
$$= (G_1G_2 + G_3G_\infty)x_1 + (G_3G_2 + G_1G_\infty)x_2 + (G_1G_3 + G_2G_\infty)x_3 + 4$$

- One can use (Bershtein, Lisovyy,...) coordinates σ_1, s_1:

$$s_1^{\pm 1} = \frac{(\omega_2 \pm ix_3 \sin 2\pi\sigma_1) - (\omega_3 \mp ix_2 \sin 2\pi\sigma_1)\, e^{\pm 2\pi i\sigma_1}}{(\cos 2\pi(\theta_t \mp \sigma_1) - G_1)(\cos 2\pi(\theta_1 \mp \sigma_1) - G_\infty)}.$$

  This coordinates are are also closely related to the Nekrasov, Rosly and Shatashvili coordinates α, β.

Riemann Hilbert correspondence $RH : \mathcal{M}_{DR} \to \mathcal{M}_B$

$\mathcal{M}_{DR}$: Logarithmic connection ∇ on a holomorphic rank 2-vector bundle $E \to \Sigma_{0,s}$

In coordinates: $\nabla_{\frac{\mathrm{d}}{\mathrm{d}\lambda}} := \frac{\mathrm{d}}{\mathrm{d}\lambda} - A(\lambda)$

$A(\lambda)$ has s simple poles $\lambda_1, \dots, \lambda_s$. $\mathrm{Res}_{\lambda_j} A(\lambda) \in \mathcal{O}_j \subset \mathfrak{s}l_2(\mathbb{C})$

$\mathcal{M}_B$: $\mathrm{Hom}\left(\pi_1(\Sigma_{0,s}) \to SL_2(\mathbb{C})\right) / _{SL_2(\mathbb{C})}$

Fixed local monodromy.

Theorem (Hitchin '97)

The Riemann Hilbert correspondence $RH : \mathcal{M}_{DR} \to \mathcal{M}_B$ is symplectic. On the l.h.s we have the standard Lie–Poisson structure on $\mathcal{O}_1 \times \cdots \times \mathcal{O}_s \in \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \cdots \times \mathfrak{sl}_2$ obtained by identifying $\mathfrak{sl}_2$ with $\mathfrak{sl}_2^*$, the symplectic leaves are fixed by the conjugacy classes of the residues. On the r.h.s. there is the Atiyah-Bott symplectic structure.

Generalising the Riemann Hilbert correspondence

Question

What happens to $\mathcal{M}$ if we relax the hypotheses on $\mathcal{E}$?

What is the correct notion for $\mathcal{M}$ when higher order poles arise?

- Boalch 2000: 1 irregular and 1 Fuchsian singularity.
- Boalch 2014: Wild character variety.
- Gualtieri-Li-Pym 2015: Stokes groupoid.

Our work: decorated character variety

- Generalised cluster algebra structure.
- Toric Poisson structure
- Explicit quantisation

Painlevé monodromy manifolds Saito and van der Put

$$M_\varphi := \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3]/ < \varphi = 0 >)$$

PVI $$x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 + \omega_1x_1 + \omega_2x_2 + \omega_3x_3 = \omega_4$$

PV $$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1x_1 + \omega_2x_2 + \omega_3x_3 = \omega_4$$

PIV $$x_1x_2x_3 + x_1^2 + \omega_1x_1 + \omega_2x_2 + \omega_2x_3 + 1 = \omega_4$$

PIII $$x_1x_2x_3 + x_1^2 + x_2^2 + \omega_1x_1 + \omega_2x_2 = \omega_1 - 1$$

PII $$x_1x_2x_3 + x_1 + x_2 + x_3 = \omega_4$$

PI $$x_1x_2x_3 + x_1 + x_2 + 1 = 0$$

The (most generic) family of cubics is a variety $M_\varphi = \{(\bar{x},\bar{\omega}) \in \mathbb{C}^3 \times \Omega) : \varphi(\bar{x},\bar{\omega}) = 0\}$ where $\bar{x} = (x_1, x_2, x_3), \quad \bar{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$ and the "$\bar{x}-$forgetful" projection $\pi : M_\varphi \to \Omega : \pi(\bar{x},\bar{\omega}) = \bar{\omega}$. This projection defines a family of affine cubics with generically non–singular fibres $\pi^{-1}(\bar{\omega})$ The cubic surface M_φ has a volume form $\vartheta_{\bar{\omega}}$ given by the Poincaré residue formulae:

$$\vartheta_{\bar{\omega}} = \frac{dx_1 \wedge dx_2}{(\partial\varphi_{\bar{\omega}})/(\partial x_3)} = \frac{dx_2 \wedge dx_3}{(\partial\varphi_{\bar{\omega}})/(\partial x_1)} = \frac{dx_3 \wedge dx_1}{(\partial\varphi_{\bar{\omega}})/(\partial x_2)}. \qquad (1)$$

The volume form is a holomorphic 2-form on the non-singular part of M_φ and it has singularities along the singular locus. This form defines the Poisson brackets on the surface in the usual way as

$$\{x_1, x_2\}_{\bar\omega} = \frac{\partial \varphi_{\bar\omega}}{\partial x_3} \tag{2}$$

The other brackets are defined by circular transposition of x_1, x_2, x_3. For $(i, j, k) = (1, 2, 3)$:

$$\{x_i, x_j\}_{\bar\omega} = \frac{\partial \varphi_{\bar\omega}}{\partial x_k} = x_i x_j + 2\epsilon_i^d x_k + \omega_i^d \tag{3}$$

and the volume form (1) reads as

$$\vartheta_{\bar\omega} = \frac{dx_i \wedge dx_j}{(\partial \varphi_{\bar\omega}/\partial x_k)} = \frac{dx_i \wedge dx_j}{(x_i x_j + 2\epsilon_i^d x_k + \omega_i^d)}. \tag{4}$$

Observe that for any $\varphi \in \mathbb{C}[x_1, x_2, x_3]$ the following formulae define a Poisson bracket on $\mathbb{C}[x_1, x_2, x_3]$:

$$\{x_i, x_{i+1}\} = \frac{\partial \varphi}{\partial x_{i+2}}, \qquad x_{i+3} = x_i, \tag{5}$$

and φ itself is a central element for this bracket, so that the quotient space

$$M_\varphi := \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3]/_{\langle \varphi = 0 \rangle})$$

inherits the Poisson algebra structure $[\mathrm{Nambu} \sim 70]$.
Today we are going to re-parametrize and to quantize it.

PVI Affine Cubic as a toric Poisson

Example

PVI ($\tilde{D}_4$) cubic with only $\omega_4 \neq 0$ admits the log-canonical symplectic structure $\bar{\vartheta} = \frac{du \wedge dv}{uv}$ under isomorphism $\mathbb{C}^* \times \mathbb{C}^*/\imath \to M_\varphi$ by

$$(u, v) \to (x_1 = -(u + \frac{1}{u}), x_2 = -(v + \frac{1}{v}), x_3 = -(uv + \frac{1}{uv}))$$

and $\imath : \mathbb{C}^* \to \mathbb{C}^*$ is the involution $\imath(u) = \frac{1}{u}$, $\imath(v) = \frac{1}{v}$.

Example

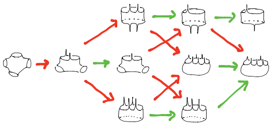
The *PVI* cubic can be "uniformize" by some analogues of theta-functions related to toric mirror data on log-Calabi-Yau surfaces (M. Gross, P. Hacking and S.Keel (see Example 5.12 of "Mirror symmetry for log-Calabi-Yau varieties I, arXiv:1106.4977).

The other Painlevé equations

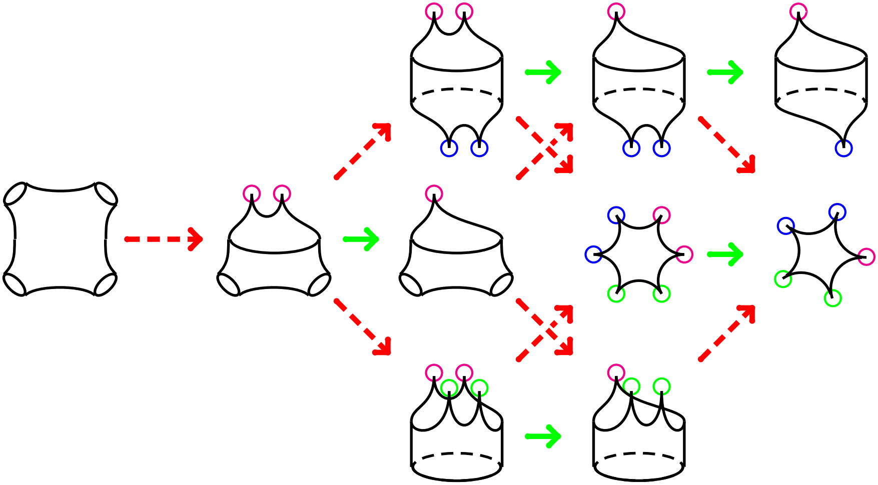
- The PVI monodromy manifold is the $SL_2(\mathbb{C})$–character variety of a four holed Riemann sphere.
- What are the underlying Riemann surfaces for the other Painlevé equations?
- Is there a "toric" (or cluster algebra) structure on it?

Use the confluence scheme of the Painlevé equations.

$$\begin{tikzcd}
 & & P_{III}^{D_6} \arrow[r] \arrow[dr] & P_{III}^{D_7} \arrow[r] \arrow[dr] & P_{III}^{D_8} \\
P_{VI} \arrow[r] & P_V \arrow[ur] \arrow[r] \arrow[dr] & P_V^{deg} \arrow[ur] \arrow[dr] & P_{II}^{JM} \arrow[r] & P_I \\
 & & P_{IV} \arrow[ur] \arrow[r] & P_{II}^{FN} \arrow[ur] &
\end{tikzcd}$$



Geometric surgery and Painlevé confluence



"Confluented "Poisson algebras

$$
\begin{tikzcd}
 & & Poiss_{III}^{D_6} \arrow[dr, hook] & Poiss_{III}^{D_7} \arrow[l, hook'] \arrow[dr, hook] & Poiss_{III}^{D_8} \arrow[l, hook'] \\
Poiss_{VI} \arrow[r, hook] & Poiss_V \arrow[ur, hook] \arrow[dr, hook] & Poiss_V^{deg} \arrow[l, hook'] \arrow[ur, hook] \arrow[dr, hook] & Poiss_{II}^{JM} & Poiss_I \arrow[l, hook'] \\
 & & Poiss_{IV} \arrow[ur, hook] & Poiss_{II}^{FN} \arrow[l, hook'] \arrow[ur, hook] &
\end{tikzcd}
$$

Katz invariant-1

Painlevé monodromy manifolds moduli spaces of flat connections on $\mathbb{P}\setminus S$, where S is a set of isolated singularities:

- $S=\{0,1,t,\infty\}$ for PVI, $S=\{0,1,\infty\}$ for $PV, PIII^{D_6}$, $S=\{0,\infty\}$ for PIV, $S=\{\infty\}$ for PII and PI)
- Points in S are regular (for example in the case of PVI) or irregular.
- Irregular singular points are called “non-ramified” when the connection has non resonant (semi-simple) residue at those points or “ramified” when the residue is resonant (nilpotent).
- These moduli spaces have complex dimension 2 and are classified by the so-called *Katz invariants*.

Katz invariant-2

Consider the flat connection as a local system whose local sections $q(z)$ have the following form:

$$q(z) = a_1(z - z_i)^{-k-1} + \dots a_k(z - z_i)^{-1}, \text{non-ramified case,}$$
$$q(z) = (z - z_i)^{\frac{1}{2}}[a_1(z - z_i)^{-k-\frac{1}{2}} + \dots a_*(z - z_i)^{-1}], \text{ramified case}$$

The number k is called Katz invariant of the singular point z_i.

Poincaré-Katz invariants and Stokes rays

Painlevé eqs	no. of cusps	Katz invariants	no. Stokes rays	pole–orders for φ
PVI	$(0,0,0,0)$	$(0,0,0,0)$	$(0,0,0,0)$	$(2,2,2,2)$
PV	$(0,0,2)$	$(0,0,1)$	$(0,0,2)$	$(2,2,4)$
PV_{deg}	$(0,0,1)$	$(0,0,1/2)$	$(0,0,1)$	$(2,2,3)$
PIV	$(0,4)$	$(0,2)$	$(0,4)$	$(2,6)$
$PIII^{D_6}$	$(-,2,2)$	$(-,1,1)$	$(-,2,2)$	$(-,4,4)$
$PIII^{D_7}$	$(-,1,2)$	$(-,1/2,1)$	$(-,1,2)$	$(-,3,4)$
$PIII^{D_8}$	$(-,1,1)$	$(-,1/2,1/2)$	$(-,1,1)$	$(-,3,3)$
PII^{FN}	$(0,3)$	$(0,3/2)$	$(0,3)$	$(2,5)$
PII^{MJ}	6	3	6	8
PI	5	$5/2$	5	7

Table: For each Painlevé isomonodromic problem, this table displays the number of cusps on each hole for the corresponding Riemann surface, the Katz invariants associated to the corresponding flat connection, the number of Stokes rays in the linear system defined by the flat connection and the number of poles of the quadratic differential φ defined by the linear system.

Diagonalizable (non-ramified) case

Notation: the fundamental matrix at an irregular singular point λ_k has the form

$$Y_k = G_k(\lambda)\lambda^{\Lambda_k}\begin{pmatrix} e^{Q_k(\lambda)} & 0 \\ 0 & e^{-Q_k(\lambda)} \end{pmatrix}$$

	$A(\lambda)$	Casimirs	extended exponents	$dim(\mathcal{C})$
PV	$\frac{A_0}{\lambda}+\frac{A_1}{\lambda-1}+A_\infty$	eigen(A_0),eigen(A_1),Λ_∞	$Q_\infty=\frac{t}{2}\lambda$	7
PIV	$\frac{A_0}{\lambda}+A_1+A_\infty\lambda$	eigen(A_0),Λ_∞	$Q_\infty=\lambda^2+\frac{t}{2}\lambda$	8
$PIII^{D_6}$	$\frac{A_0}{\lambda^2}+\frac{A_1}{\lambda}+A_\infty$	Λ_0, Λ_∞	$Q_\infty=\frac{t}{2}\lambda, Q_0=\frac{t}{2}\frac{1}{\lambda}$	8
PII^{MJ}	$A_0+A_1\lambda+A_\infty\lambda^2$	Λ_∞	$Q_\infty=\lambda^3+\frac{t}{2}\lambda$	9

Table: Here $Q_k(\lambda)$ is polynomial in $(\lambda-\lambda_k)$ of order $n-1$ with n being the order of λ_k and Λ_k is the formal monodromy (diagonal). Expand $A(\lambda)$ near λ_k to calculate $Q_k(\lambda)$ and Λ_k, then diagonalize it using the gauge transformation $G_k(\lambda)$..

Non-diagonalizable (ramified) case

The same notations.

	$A(\lambda)$	Casimirs	extended exponents	$dim(\mathcal{C})$
PV_{deg}	$\frac{A_0}{\lambda}+\frac{A_1}{\lambda-1}+A_\infty$	eigen(A_0),eigen(A_1),Λ_∞	$Q_\infty=\frac{t}{2}\sqrt{\lambda}$	5
$PIII^{D_7}$	$\frac{A_0}{\lambda^2}+\frac{A_1}{\lambda}+A_\infty$	Λ_0,Λ_∞	$Q_0=\frac{1}{\sqrt{\lambda}}$, $Q_\infty=\frac{t}{2}\lambda$	6
$PIII^{D_8}$	$\frac{A_0}{\lambda^2}+\frac{A_1}{\lambda}+A_\infty$	$\Lambda_0,\ \Lambda_\infty$	$Q_\infty=\sqrt{\lambda}$, $Q_0=\frac{1}{\sqrt{\lambda}}$	4
PII^{FN}	$\frac{A_0}{\lambda}+A_1+A_\infty\lambda$	eigen(A_0), Λ_∞	$Q_\infty=\lambda^{3/2}+\frac{t}{2}\sqrt{\lambda}$	6
PI	$A_0+A_1\lambda+A_\infty\lambda^2$	Λ_∞	$Q_\infty=\lambda^{5/2}+\frac{t}{2}\sqrt{\lambda}$	7

Table: Here
$\dim PV_{deg}=\dim PV-2,\quad \dim PIII^{D_7}=\dim PIII-2,\quad \dim PIII^{D_8}=\dim PIII^{D_7}-2,\quad \dim PII^{FN}=\dim PIV-2,\quad \dim PI=\dim PII^{JM}-2.$

Geometric confluence scheme of the Painlevé equations: application. (P. Gavrylenko, O. Lisovyy, arXiv:1608.00958)

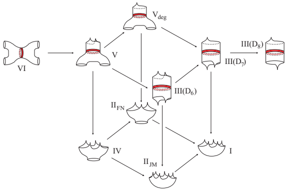


Figure 3: CMR confluence diagram for Painlevé equations.

Figure 4: Some solvable RHPs in rank $N = 2$: Gauss hypergeometric (3 regular punctures), Whittaker (1 regular + 1 of Poincaré rank 1) and Bessel (1 regular + 1 of rank $\frac{1}{2}$).

Basic ideas:history

- The character variety of a Riemann sphere with 4 holes $\mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus \{0, t, 1, \infty\}); \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$ is the monodromy cubic of the Painlevé VI (Goldman-Toledo).
- The "wild" character varieties, fissions and meromorphic connections (Ph. Boalch 2011-2015).
- Quasi-Poisson structures (A. Alexeev, Y. Kosmann-Schwarzbach, E.Meinrenken, M. Van den Bergh 1999 -2005).
- Moduli spaces for quilted surfaces. (D. Li-Bland, P. Severa 2013).
- Stokes groupoides (M. Gualtieri, S. Li, B. Pym 2014).

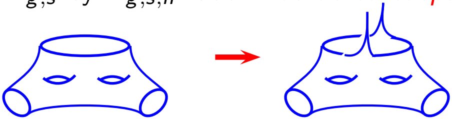
Basic ideas: Painlevé context

- Poisson structures and Isomonodromic deformations on the sphere and torus (Korotkin-Samtleben, Chekhov-Mazzocco 1995, 2010).
- Wild character variety for *PV* (J.P. Ramis, E. Paul. 2015).
- The confluent Painlevé monodromy manifolds are "decorated character varieties" (Chekhov-Mazzocco -R.2015).
- The real slice of the $\mathrm{SL}_2(\mathbb{C})$ character variety is the Teichmüller space.
- The shear coordinates on the Teichmüller space can be complexified $\Rightarrow$ coordinate description for the character variety.
- To visualize the confluence and the "decoration" we shall introduce two moves correspond to certain asymptotics in the (complexified) shear coordinates.

Decorated character variety

Mimic the regular case: $\mathrm{Hom}\left(\pi_1(\Sigma_{g,s}) \to SL_2(\mathbb{C})\right)/_{SL_2(\mathbb{C})}$

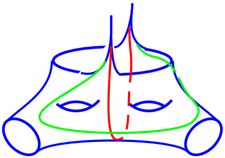
- Replace $\Sigma_{g,s}$ by $\Sigma_{g,s,n}$: add n *bordered cusps.*



Quadratic differential $\mathrm{d}\varphi := \det A(z) dz^{\otimes 2}$ with s poles of order $n_1, \dots, n_s$, $n = \sum_{k=1}^{s}(n_k - 2)$ [Gaiotto, Moore and Neitzke '13]

- Replace π_1 by *the fundamental groupoid of arcs:*

$$\mathfrak{U} := \{\gamma_{ij} : [0,1] \to \Sigma_{g,s,n} | \, \gamma_{ij}(0) = m_i, \, \gamma_{ij}(1) = m_j\}/_{\text{homotopy}}.$$



Decorated character variety

Definition

Decorated character variety (L.Chekhov, M. Mazzocco and V.R.- 2015):

$$\mathrm{Hom}\left(\mathfrak{U}, SL_2(\mathbb{C})\right) / _{\prod_{j=1}^{n} U_j}$$

Lemma

The decorated character variety is an affine variety of $\dim = 6g - 6 + 3s + 2n$.

New character (a new U_i-invariant function on this variety):

$$\begin{aligned} \mathrm{tr}_K : \quad & SL_2(\mathbb{C}) \to \mathbb{C} \\ & M \mapsto \mathrm{Tr}(MK), \qquad \text{where} \quad K = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

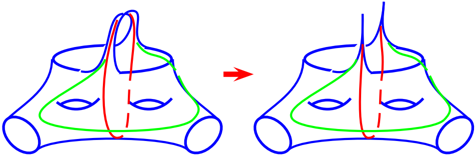
$$\mathrm{tr}_K(M_{ij}) = \mathrm{tr}_K(U_j M_{ij} U_j).$$

tr is defined on the isotopy groups isomorphic to π_1:

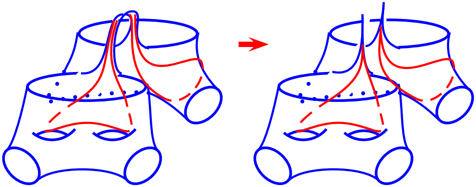
$$\Pi_j := \{\gamma_{jj} | \gamma_{jj} : [0,1] \to \tilde{\Sigma}_{g,s,n},\ \gamma_{jj}(0) = \gamma_{jj}(1) = m_j, \}/\{\mathrm{homotopy}\}.$$

Chewing-gum moves

- **Hooking holes:**



- **Pinching two sides of the same hole:**



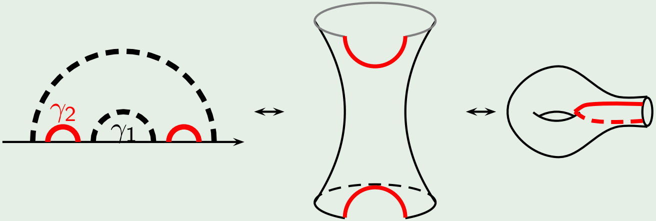
Bordered cusps á la Fomin-Shapiro-Thurston.

Poincaré uniformsation

$$\Sigma = \mathbb{H}/\Delta,$$

where Δ is a *Fuchsian group*, i.e. a discrete sub-group of $\mathbb{P}SL_2(\mathbb{R})$.

Examples



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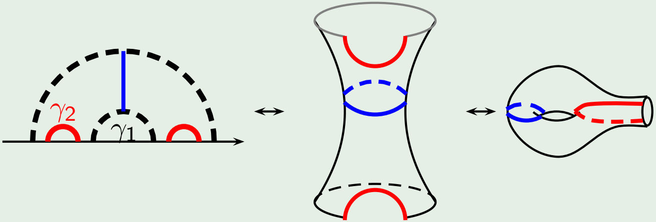


Theorem

Elements in $\pi_1(\Sigma_{g,s})$ are in 1-1 correspondence with conjugacy classes of closed geodesics.

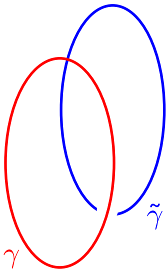
Coordinates: geodesic lengths

Theorem

The geodesic length functions ($G_\gamma := \mathrm{Tr}\gamma = 2\cosh(\mathrm{l}_\gamma)$) form an algebra with multiplication:

$$G_\gamma G_{\tilde\gamma} = G_{\gamma\tilde\gamma} + G_{\gamma\tilde\gamma^{-1}}.$$

$$(\mathrm{Tr}(AB) + \mathrm{Tr}(AB^{-1}) = \mathrm{Tr}(A)\mathrm{Tr}(B) \quad \forall A, B \in \mathrm{SL}_2(\mathbb{C}))$$



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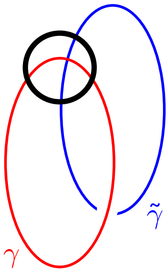

$=$

Coordinates: geodesic lengths

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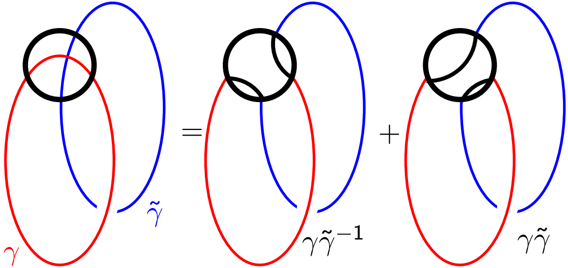
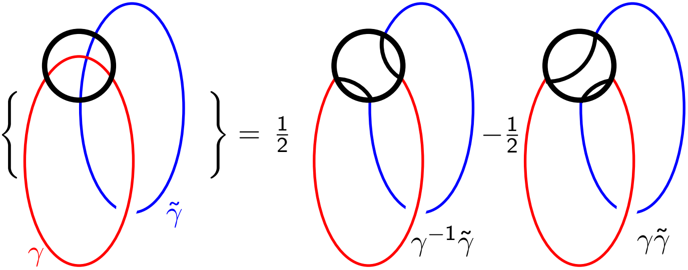
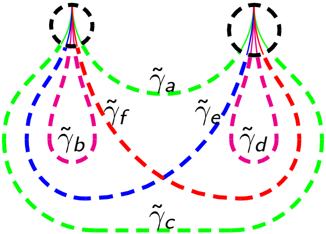


Poisson structure

$$\{G_\gamma, G_{\tilde\gamma}\} = \frac{1}{2} G_{\gamma\tilde\gamma} - \frac{1}{2} G_{\gamma\tilde\gamma^{-1}}.$$



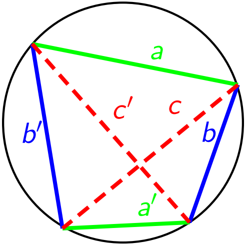
Ptolemy Relation-1



$$G_{\tilde{\gamma}_e} G_{\tilde{\gamma}_f} = G_{\tilde{\gamma}_a} G_{\tilde{\gamma}_c} + G_{\tilde{\gamma}_b} G_{\tilde{\gamma}_d}$$

Ptolemy Relation-2

$$aa' + bb' = cc'$$



Cluster algebra

- We call a set of n numbers $(x_1, \dots, x_n)$ a cluster of rank n.
- A seed consists of a cluster and an exchange matrix B, i.e. a skew–symmetrisable matrix with integer entries.
- A mutation is a transformation $\mu_i : (x_1, x_2, \dots, x_n) \to (x_1', x_2', \dots, x_n')$, $\mu_i : B \to B'$ where

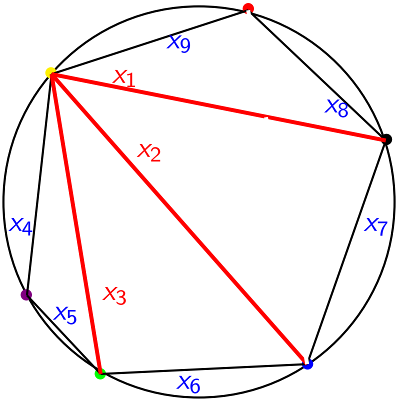
$$x_i x_i' = \prod_{j:b_{ij}>0} x_j^{b_{ij}} + \prod_{j:b_{ij}<0} x_j^{-b_{ij}}, \quad x_j' = x_j \,\forall j \neq i.$$

Definition

A cluster algebra of rank n is a set of all seeds $(x_1, \dots, x_n, B)$ related to one another by sequences of mutations $\mu_1, \dots, \mu_k$. The cluster variables $x_1, \dots, x_k$ are called exchangeable, while $x_{k+1}, \dots, x_n$ are called frozen. [Fomin-Zelevnsky 2002].

Example

Cluster algebra of rank 9 with 3 exchangeable variables x_1, x_2, x_3 and 6 frozen ones $x_4, \ldots, x_9$.



Outline

Are all cluster algebras of geometric origin?

- Introduce bordered cusps
- Geodesics length functions on a Riemann surface with bordered cusps form a cluster algebra.

All Berenstein-Zelevinsky cluster algebras are geometric

Cusped laminations

Definition

A cusped lamination is a lamination made of arcs that can only meet at the cusps.

Definition

A cusped lamination is complete if all closed geodesic functions and all λ–lengths of arcs in the Riemann surface are Laurent polynomials of the λ–lengths of the arcs in the cusped lamination.

Theorem

Given a Riemann surface $\Sigma_{g,s,n}$ of any genus g, any number of holes s and at least one cusp $n \geq 1$, there always exists a complete cusped lamination of $6g - 6 + 3s + 2n$ arcs [Chekhov-M.Mazzocco; ArXiv:1509.07044].

Poisson brackets after chewing-gum

Theorem

The Poisson algebra of the λ-lengths of a complete cusped lamination is toric. [ArXiv:1509.07044].

$$\{\lambda_{s_i,t_j},\lambda_{p_r,q_l}\} = \lambda_{s_i,t_j}\lambda_{p_r,q_l}\mathcal{I}_{s_i,t_j,p_r,q_l}$$

$$\mathcal{I}_{s_i,t_j,p_r,q_l} = \frac{\mathrm{sign}_{i-r}\delta_{s,p} + \mathrm{sign}_{j-r}\delta_{t,p} + \mathrm{sign}_{i-l}\delta_{s,q} + \mathrm{sign}_{j-l}\delta_{t,q}}{4}$$

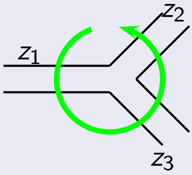
- In our case the Poisson structure on the decorated character variety is an example of a "Trace-Poisson" quadratic structure of Massuyeau G., Turaev and Avan J., Ragoucy E., V.R. defined on the representation space $\mathrm{Hom}\,(\mathfrak{U}, SL_2(\mathbb{C}))$ - work in progress.

Poisson bracket in shear coordinates

Non-cusped case:

Theorem

Chekhov-Fock '99



The Poisson bracket:

$$\{z_i, z_{i+1}\} = 1, \qquad z_{i+3} = z_i$$

is the Goldman bracket.

Cusped case:

Theorem

The same Poisson bracket gives rise to the asymptotic limit of the Goldman bracket. [Chekhov-M.Mazzocco'15]

Poisson bracket for the lamination

Lemma

All λ-lengths in a complete cusped lamination are monomials in $e^{+Z_\alpha/2}$ and $e^{+\pi_i/2}$, where $\{Z_\alpha, \pi_i\}$ are the extended shear coordinates of the fat graph dual to this lamination.

Theorem

The Poisson bracket for a complete cusped lamination has the following form:

$$\{\lambda_{s_i,t_j}, \lambda_{p_r,q_l}\} = \lambda_{s_i,t_j}\lambda_{p_r,q_l}\mathcal{I}_{s_i,t_j,p_r,q_l}$$

$$\mathcal{I}_{s_i,t_j,p_r,q_l} = \frac{\mathrm{sign}_{i-r}\delta_{s,p} + \mathrm{sign}_{j-r}\delta_{t,p} + \mathrm{sign}_{i-l}\delta_{s,q} + \mathrm{sign}_{j-l}\delta_{t,q}}{4}$$

Complexification

- To each arc we associate a matrix $M_{ij} \in SL_2(\mathbb{R})$.
- The λ–lengths in the cusped lamination are monomials in the exponentiated shear coordinates.
- By complexifying $z_0, \dots, z_{6g-6+3s+2n}$ we associate a matrix $M_{ij} \in SL_2(\mathbb{C})$.
- The characters are the complexified λ-lengths.
- We postulate:

$$\{z_i, z_j\} = \{\overline{z}_i, \overline{z}_j\} := \{z_i, z_j\}_{\mathbb{R}}, \qquad \{z_i, \overline{z}_j\} \equiv 0$$

Theorem

The $SL_2(\mathbb{C})$ decorated character variety is a Poisson algebra on an affine space of dimension $6g - 6 + 3s + 2n$.

Extended Riemann-Hilbert correspondence: $\mathcal{M}_{DR}^{irr} \to \mathcal{M}_{B}^{irr}$

$\mathcal{M}_{DR}^{irr}$: Irregular connections ∇ on a holomorphic rank 2-vector bundle $E \to \Sigma_{g,s}$

$\mathcal{M}_{B}^{irr} := \mathrm{Hom}\left(\pi_{\mathfrak{a}}(\Sigma_{g,s,n}), SL_2(\mathbb{C})\right) / _{\prod_{j=1}^{n} U_j}$

Conjecture

The Riemann Hilbert correspondence $RH : \mathcal{M}_{DR}^{irr} \to \mathcal{M}_{B}^{irr}$ is a Poisson map.

PV – example-1

The character variety of PV is 7-dimensional (rather than 2-dimensional like in PVI case). The fat-graph admits a complete cusped lamination as displayed in the figure below. A full set of coordinates on the character variety is given by the five elements in the lamination and the two parameters G_1 and G_2 which determine the perimeter of the two non-cusped holes.

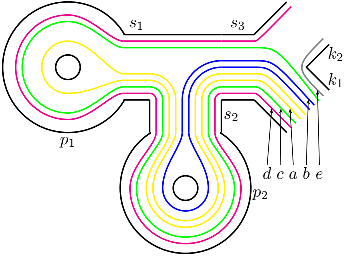


FIGURE 6. The system of arcs for PV.

$PV-$ example-2

They satisfy the following Poisson relations:

$$\{a,b\}=ab,\quad \{a,c\}=0,\quad \{a,d\}=-\frac{1}{2}ad,\quad \{a,e\}=\frac{1}{2}ae,\tag{6}$$

$$\{b,c\}=0,\quad \{b,d\}=-\frac{1}{2}bd,\quad \{b,e\}=\frac{1}{2}be,\tag{7}$$

$$\{c,d\}=-\frac{1}{2}cd,\quad \{c,e\}=\frac{1}{2}ce,\quad \{d,e\}=0,\tag{8}$$

so that the element $G_3G_\infty=de$ is a Casimir.

$PV-$ example-3

The symplectic leaves are determined by the values of the three Casimirs G_1, G_2 and $G_3 G_\infty$. On each symplectic leaf, the PV monodromy manifold is the subspace defined by those functions of a, b, c (and of the Casimir values $G_1,\ G_2,\ G_3 G_\infty$) which commute with $G_3 = e$. We determine the exponentiated shear coordinates in terms of a, b, c, d, e and then deduce he expressions of x_1, x_2, x_3 in terms of the lamination. We obtain the following expressions:

$$x_1 = -e\frac{a}{c} - d\frac{b}{c}, \qquad x_2 = -e\frac{b}{c} - G_1 d\frac{b}{a} - d\frac{b^2}{ac} - d\frac{c}{a}, \tag{9}$$

$$x_3 = -G_2\frac{c}{b} - G_1\frac{c}{a} - \frac{b}{a} - \frac{c^2}{ab} - \frac{a}{b}, \tag{10}$$

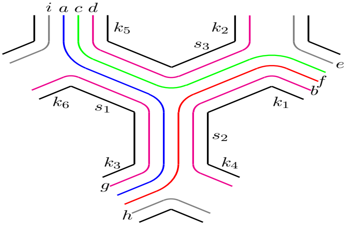
which automatically satisfy $PV-$ monodromy variety cubic equation.

$PV-$ example-4

Due to the Poisson relations (6) the functions that commute with e are exactly the functions of $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$. Such functions may depend on the Casimir values G_1, G_2 and $G_3 G_\infty$ and e itself, so that $d = G_\infty$ becomes a parameter now. It is easy to prove that x_1, x_2, x_3 are algebraically independent functions of $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ so that x_1, x_2, x_3 form a basis in the space of functions which commute with e. The "reduced" $2D$ decorated character variety is the affine cubic family:

$$\pi : \mathrm{Spec}(\mathbb{C}[G_1, G_2, G_3, G_3^{-1}, x_1, x_2, x_3]/x_1x_2x_3 + x_1^2 + x_2^2 -$$
$$-(G_1 + G_2 G_3)x_1 - (G_2 + G_1 G_3)x_2 - G_3 x_3 + 1 + G_3^2 + G_1 G_2 G_3) -$$
$$\mapsto \mathrm{Spec}(\mathbb{C}[G_1, G_2, G_3, G_3^{-1}]). \tag{11}$$

The decorated character variety associated with PII^{JM} has 6 cusps on the boundary is 9-dimensional. The fat-graph admits a complete cusped lamination as displayed in the figure below.



Quantisation-1

Oblomkov: the quantisation of the D_4 affine cubic surface M_φ coincides with spherical subalgebra of the generalised rank 1 double affine Hecke algebra H (or Cherednick algebra of type $C_1 C_1^\nu$):

Theorem

Denote by X_1, X_2, X_3 the quantum Hermitian operators corresponding to x_1, x_2, x_3 as above. The quantum commutation relations are:

$$q^{-\frac{1}{2}} X_i X_{i+1} - q^{\frac{1}{2}} X_{i+1} X_i = \left(\frac{1}{q} - q\right) \epsilon_k^{(d)} X_k - (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) \omega_k^{(d)} \quad (12)$$

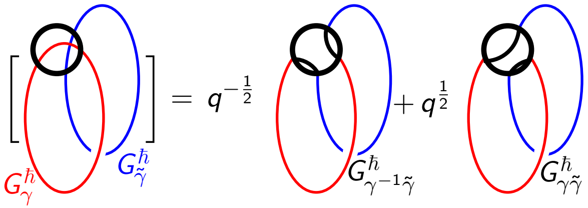
$\epsilon_i^{(d)}$ and $\omega_i^{(d)}$ are the same as in the classical case. The quantum operators satisfy the following quantum cubic relations:

$$q^{\frac{1}{2}} X_3 X_1 X_2 - q\epsilon_3^{(d)} X_3^2 - q^{-1}\epsilon_1^{(d)} X_1^2 - q\epsilon_2^{(d)} X_2^2 +$$

$$q^{\frac{1}{2}} \epsilon_3^{(d)} \omega_3 X_3 + q^{-\frac{1}{2}} \omega_1^{(d)} X_1 + q^{\frac{1}{2}} \omega_2^{(d)} X_2 - \omega_4^{(d)} = 0.$$

Quantisation-2

For standard geodesic lengths $G_\gamma \to G^\hbar_\gamma$ [Chekhov-Fock '99]:



$$[G^\hbar_\gamma, G^\hbar_{\tilde\gamma}] = q^{-\frac{1}{2}}\, G^\hbar_{\gamma^{-1}\tilde\gamma} + q^{\frac{1}{2}}\, G^\hbar_{\gamma\tilde\gamma}$$

For arcs $g_{s_i,t_j} \to g^\hbar_{s_i,t_j}$:

$$q^{\mathcal{I}_{s_i,t_j,p_r,q_l}} g^\hbar_{s_i,t_j} g^\hbar_{p_r,q_l} = g^\hbar_{p_r,q_l} g^\hbar_{s_i,t_j} q^{\mathcal{I}_{p_r,q_l,s_i,t_j}}$$

This identifies the geometric basis of the quantum cluster algebras introduced by Berenstein - Zelevinsky**.**

Quantization-2

To produce the quantum Painlevé cubics, we introduce the Hermitian operators S_1, S_2, S_3 subject to the commutation inherited from the Poisson bracket of $\tilde{s}_i$:

$$[S_i, S_{i+1}] = i\pi\hbar\{\tilde{s}_i, \tilde{s}_{i+1}\} = i\pi\hbar, \quad i = 1, 2, 3,\ i + 3 \equiv i.$$

Observe that thanks to this fact, the commutators $[S_i, S_j]$ are always numbers and therefore we have

$$\exp(aS_i)\exp(bS_j) = \exp\left(aS_i + bS_i + \frac{ab}{2}[S_i, S_j]\right),$$

for any two constants a, b. Therefore we have the Weyl ordering:

$$e^{S_1+S_2} = q^{\frac{1}{2}} e^{S_1} e^{S_2} = q^{-\frac{1}{2}} e^{S_2} e^{S_1}, \quad q \equiv e^{-i\pi\hbar}.$$

Quantization-2

Theorem

(L. Chekhov-M. Mazzocco-V.R)

Denote by X_1, X_2, X_3 the quantum Hermitian operators corresponding to x_1, x_2, x_3 as above. The quantum commutation relations are:

$$q^{-\frac{1}{2}} X_i X_{i+1} - q^{\frac{1}{2}} X_{i+1} X_i = \left(\frac{1}{q} - q\right) \epsilon_k^{(d)} X_k - (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) \omega_k^{(d)} \quad (13)$$

where $\epsilon_i^{(d)}$ and $\omega_i^{(d)}$ are the same as in the classical case. The quantum operators satisfy the following quantum cubic relations:

$$q^{\frac{1}{2}} X_3 X_1 X_2 - q \epsilon_3^{(d)} X_3^2 - q^{-1} \epsilon_1^{(d)} X_1^2 - q \epsilon_2^{(d)} X_2^2 +$$

$$q^{\frac{1}{2}} \epsilon_3^{(d)} \omega_3 X_3 + q^{-\frac{1}{2}} \omega_1^{(d)} X_1 + q^{\frac{1}{2}} \omega_2^{(d)} X_2 - \omega_4^{(d)} = 0.$$

Quantization-2

The Hermitian operators X_1, X_2, X_3 corresponding to x_1, x_2, x_3 are introduced as follows: consider the classical expressions for x_1, x_2, x_3 in terms of s_1, s_2, s_3 and p_1, p_2, p_3. Write each product of exponential terms as the exponential of the sum of the exponents and replace those exponents by their quantum version. For example (the case $\tilde{D}_5$): the classical x_1 is

$$x_1 = -e^{s_2+s_3} - e^{-(\tilde{s}_2+\tilde{s}_3)} - G_2 e^{\tilde{s}_3} - G_3 e^{-\tilde{s}_2},$$

and its quantum version is defined as

$$X_1 = -e^{S_2} - (e^{p_2/2} + e^{-p_2/2})e^{S_3} - e^{S_3-S_2} - e^{S_3+S_2} =$$

$$-e^{S_2} - (e^{p_2/2} + e^{-p_2/2})e^{S_3} - q^{-1/2}e^{-S_2}e^{S_3} - q^{1/2}e^{S_2}e^{S_3}.$$

Quantization-2

- Our theorem and close results of Marta Mazzocco show that we can interpret the Cherednik algebra and their close "relatives" as a quantisation of the (group algebra of the) monodromy group of the Painlevé equations. Here $q := e^{-i\pi\hbar}$ and $q^n \neq 1$.
- The Askey-Wilson $AW(3)$ (or Zhedanov algebra) can be obtained from (13) for a special constant choice after a proper "rescaling".

"Physical Motivations"

- Standard Model$SU(3) \times SU(2) \times U(1)$ Gauge Theory
- SUSY desired phenomena are inherited from String Theory
- Superstring Theory: $\mathbb{R}^{1,9} 10D = 1 + 3 + 6$ Dirichlet $p-$ branes: $p+1-$subvarieties in $\mathbb{R}^{1,9}$ on which open strings can end;
- $D-$brane world: live on $D3-$brane $\perp$ $6D-$affine variety $\mathcal{M}$. $1+3D-$world-volume with SUSY YM and product gauge group.

$D-$brane algebras and superpotentials. Basic principles:

- One can associate an algebra to the category of $D-$branes at a singular point p. In every known example, the collection of possible $D-$branes at p can be described as a collection of QFT with the same Lagrangian for each of the theories.
- More precisely, one does specify the "matter representation" (as a collection of adjoint and bifundamental fields for the gauge groups G_i) and one specifies a superpotential $W-$ the trace of a polynomial in the matter fields.
- To such data one can assign a quiver whose vertices label the groups G_i and whose directed edges specify the bifundamental and adjoint fields in the matter representation.

Quiver Theory

- Action

$$\int d^4x[\int d^4\theta\Psi_i^\dagger e^V\Psi_i+(\frac{1}{4g^2}\int d^2\theta\mathrm{Tr}\mathcal{W}_\alpha\mathcal{W}^\alpha+\int \mathrm{d}^2\theta\mathrm{W}(\bar{\psi})+\mathrm{h.c.})]$$

W = superpotential;

$V(\varphi_i;\bar{\varphi}_i)=\sum_i \mid \frac{\partial W}{\partial\varphi_i}\mid^2+\frac{g^2}{4}(\sum_i q_i\mid\varphi_i\mid^2)^2$

- Encode in a Quiver:

k nodes $\iff \mathcal{V}^{n_1},\dots\mathcal{V}^{n_k} \iff \prod_{j=1}^k U(n_j)$ gauge group;

Each arrow $i\to j \iff$ bifundamental fields X_{ij} of $U(n_i)\times U(n_j)$;

Each loop $i\to i \iff$ adjoint fields φ_i of $U(n_i)$;

Superpotential $W \iff$ linear combination of cycles: $\sum_i c_i$ gauge invariant operators;

Relations $\iff$ jacobian of $W(\varphi_i,X_{ij})$.

Vacuum:$\rightsquigarrow V(\varphi_i;\bar{\varphi}_i)=0\Rightarrow\frac{\partial W}{\partial\varphi_i}=0;\ \sum_i q_i\mid\varphi_i\mid^2=0.$

Superpotential algebra

- From the quiver, we directly get the path algebra, which is the algebra of all paths on the quiver (i.e., all ordered monomials in matter fields).
- A universal feature of this family of theories is the relations in the path algebra determined by what are called "F−term constraints" in physics: $\frac{\partial W}{\partial \varphi_i} = 0$
- These are the algebra relations dictated by $\frac{\partial W}{\partial X_j}$. So, given a field theory description of the family of D-branes in the form above, the D-brane algebra is

$$A = \text{path algebra of quiver}/(\frac{\partial W}{\partial X_j}).$$

- This is called a superpotential algebra, which is a Calabi - Yau algebra.

Elementary example

- First example, we consider the case in which P is a smooth point. In physics language, the conformal fields theory is the $N = 4$ SUSY Yang-Mills theory, written in $N = 1$ language. The $N = 4$ gauge multiplet decomposes as an $N = 1$ gauge multiplet plus three complex scalar fields X, Y, Z each transforming in the adjoint representation of the group.
- The superpotential is

$$W = \mathrm{tr}(X(YZ - ZY)).$$

- The $F-$ term constraint in this case tells us

$$YZ = YZ, \quad XZ = ZX \quad \text{and} \quad XY = YX.$$

- Thus, we find

$$\mathcal{A} = \mathbb{C}[X, Y, Z],$$

the (commutative) polynomial algebra in three variables.

Example 2. Sklyanin algebra-1

- The most famous example of 3-Calabi-Yau algebra is the following graded associative algebra associated with an elliptic curve $\mathcal{E}$ (possibly degenerated).
- This algebra denotes by $Q_3(\mathcal{E}, a, b, c)$ where $(a, b, c) \in \mathbb{C}^3$ such that $Q_3(\mathcal{E}, a, b, c) = \mathbb{C} < X, Y, Z > /J_W$ with

$$J_W =< aYZ+bZY+cX^2, aZX+bXZ+cY^2, aXY+bYX+cZ^2 >$$

- The ideal J_W can be written as a non-commutative jacobian ideal $J_W =< \partial_X, \partial_Y, \partial_Z >\in \mathbb{C} < X, Y, Z >$ for superpotential

$$W = aXYZ + bYXZ + c(X^3 + Y^3 + Z^3)$$

Example 2. Sklyanin algebra-2

- Here we consider W as a cyclic word of three variables X, Y, Z, i.e. like an element of the quotient $A_\natural := \mathbb{C} < X, Y, Z > / [\mathbb{C} < X, Y, Z >, \mathbb{C} < X, Y, Z >]$ with
- cyclic derivatives $\partial_X, \quad \partial_Y, \quad \partial_Z$ where

$$\partial_j : A_\natural \to \mathbb{C} < X, Y, Z >, j = X, Y, Z$$

defines for any cyclic word $\varphi \in A_\natural$ by

$$\partial_j \varphi := \sum_{k | i_k = j} X_{i_k+1} X_{i_k+2} \ldots X_{i_N} \ldots X_{i_1} X_{i_2} .. X_{i_k-1} \in \mathbb{C} < X, Y, Z >$$

Example 2. Sklyanin algebra-3

Etingof-Ginzburg:

- One can identify the Sklyanin algebra $Q_3(\mathcal{E}, 1, -q, \frac{c}{3})$ with the flat deformation of the Poisson algebra $(\mathbb{C}[x,y,z], \{-,-\}_\varphi)$ as above with $\varphi = \frac{1}{3}(x^3 + y^3 + z^3) + \tau xyz$ and $W = XYZ - qYXZ + \frac{c}{3}(X^3 + Y^3 + Z^3)$.
- The coordinate ring $B_\varphi = \mathbb{C}[x,y,z]/\varphi\mathbb{C}[x,y,z]$ of the affine surface $\varphi = 0$ inherits a Poisson algebra structure.
- There is a degree 3 central element $\Phi \in Z(Q_3(\mathcal{E}, 1, -q, \frac{c}{3}))$ and the quotient of the Sklyanin 3-Calabi-Yau algebra by two-sided ideal $< \Phi >$ is a flat deformation of the Poisson algebra B_φ.

Superpotentials of marginal and relevant deformations-1

- There is a "physical interpretation" of the Sklyanin superpotential (Berenstein-Leigh) as a marginal deformation of the superpotential from the Example 1:

$$W + W_{marg} =$$

$$= g\,\mathrm{tr}(X[Y,Z]) + \mathrm{tr}(aXYZ + bYXZ + \frac{c}{3}(X^3 + Y^3 + Z^3)) \in A_\natural.$$

- The structure of the vacua of D-brane gauge theories relates to the Non-Commutative Geometry also via another superpotentials (relevant deformations) having the form

$$W_{rel} = \mathrm{tr}(\frac{m_1}{2}X^2 + \frac{m_2}{2}(Y^2 + Z^2) + e_1X + e_2Y + e_3Z)$$

Superpotentials of marginal and relevant deformations-2

- The "vacua" of the theory with $W_{tot} = W + W_{marg} + W_{tel}$ superpotential corresponds to solutions of

$$\partial_i W_{tot} = 0, i = X, Y, Z.$$

- The defining equations (for $a = 1, b = -q$):

$$\left\{ \begin{array}{rl} X_1 X_2 - q X_2 X_1 = & -c X_3^2 - m_2 X_3 - e_3 \\ X_2 X_3 - q X_3 X_2 = & -c X_1^2 - m_1 X_1 - e_1 \\ X_3 X_1 - q X_1 X_3 = & -c X_2^2 - m_2 X_2 - e_2 \end{array} \right. \quad (14)$$

This relations contain our (13) (again, after a special constant choice and a "rescaling").

Etingof-Ginzburg ideology-1:

- Let $M = \mathbb{C}^3$ considering as the simplest Calabi-Yau manifold and $\varphi \in \mathcal{A} = \mathbb{C}[x_1, x_2, x_3]$ defines the Poisson bracket of jacobian type as above.
- $M_\varphi : \varphi(x_1, x_2, x_3) = 0$ is an affine surface in M and the coordinate ring $\mathcal{B}_\varphi := \mathbb{C}[M_\varphi] = \mathcal{A}/(\varphi)$ is a commutative Poisson algebra with the structure induced by φ
- Let $\varphi^{\tau,\nu} = \tau x_1 x_2 x_3 + \frac{\nu}{3}(x_1^3 + x_2^3 + x_3^3) + P(x_1) + Q(x_2) + R(x_3) = 0$ be the family of affine surfaces containing the E_6 del Pezzo. Here $\mathrm{degP}, \mathrm{degQ}$ and $\mathrm{degQ} < 3$.

Etingof-Ginzburg ideology-2:

- Let $A = \mathbb{C} < X_1, X_2, X_3 >$ and $A_\natural$ be defined as above and $\Phi^{q,\nu}_{P,Q,R} = X_1X_2X_3 - qX_2X_1X_3 + \nu(X_1^3 + X_2^3 + X_3^3) + P(X_1) + Q(X_2) + R(X_3) \in A_\natural$
- $\mathfrak{U}(\Phi^{q,\nu}_{P,Q,R})$ is a filtered algebra defined by three inhomogeneous "jacobian" relations:

$$X_iX_j - qX_jX_i = \nu X_k^2 + \frac{\mathrm{d}P(Q,R)}{\mathrm{d}X_k}, (i,j,k) = (1,2,3) \quad (15)$$

- The superpotential $\Phi^{q,\nu}_{P,Q,R} = \Phi^{q,\nu} + \Phi_{P,Q,R}$ where $\Phi^{q,\nu} = X_1X_2X_3 - qX_2X_1X_3 + \nu(X_1^3 + X_2^3 + X_3^3) \in A_\natural^{(3)}$ and $\Phi_{P,Q,R} \in A_\natural^{(\leq 2)}$ is a 3-CY-superpotential (for generic parameters)

Etingof-Ginzburg ideology-3:

$$
\begin{tikzcd}
\mathcal{A}_\varphi \arrow[r, rightsquigarrow, "\text{fl. def.}"] \arrow[d] & \mathfrak{U}(\Phi^{q,\nu}_{P,Q,R}) \arrow[d] \\
\mathcal{B}_\varphi \arrow[r, rightsquigarrow] & B(\Phi^{q,\nu}_{P,Q,R}, \Psi) = \mathfrak{U}(\Phi^{q,\nu}_{P,Q,R})/(\Psi).
\end{tikzcd}
$$

In our case $\Phi^{q,0}_{P,Q,R} := X_1X_2X_3 - qX_2X_1X_3$

$$
\Psi^{q,\epsilon,\omega} = X_1X_2X_3 - q^2X_2X_1X_3 + \epsilon_1^{(d)}\frac{q-1}{\sqrt{q}}X_1^2 + \epsilon_2^{(d)}q^{3/2}(q-1)X_2^2 + \epsilon_3^{(d)}\frac{q^3-1}{\sqrt{q}}X_3^2 - -\omega_1^{(d)}(q-1)X_1 - \omega_2^{(d)}q(q-1)X_2 - \omega_3^{(d)}(q^2-1)X_3 \tag{16}
$$

Conclusion

- New notion of decorated character variety
- In the case of the Painlevé differential equations, each decorated character variety is a Poisson manifold of dimension $3s + 2n - 6$, where s is the number of holes and $n \geq 1$ is the number of cusps.
- In each case the decorated character variety admits a special Poisson submanifold defined by the set of functions which Poisson commute with the frozen cluster variables.
- This submanifold is defined as a cubic surface
- By quantisation: two different approach: spherical subalgebras of DAHA or quantum cluster algebras of geometric type.

Many thanks for your attention!!!