Bihamiltonian Integrable Hierarchies And Their Tau Structures

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Bihamiltonian Cohomologies and Integrable Hierarchies II: the Tau Structures, eprint arXiv:1701.03222.

Outline of the talk

1. Introduction
2. Flat exact bihamiltonian structures of hydrodynmic type
3. The principal hierarchy and its tau structure
4. Tau-symmetric integrable Hamiltonian deformations of the principal hierarchy
5. Tau-symmetric bihamiltonian deformations of the principal hierarchy

1. Introduction

We consider a class of bihamiltonian integrable hierarchies which generalize the Korteweg–de Vires hierarchy

$$u_{t_0} = u_x,$$
$$u_{t_1} = u u_x + \frac{\epsilon^2}{12} u_{xxx}, \dots$$

with the bihamiltonian structure

$$\{u(x), u(y)\}_1 = \delta'(x-y),$$
$$\{u(x), u(y)\}_2 = u(x)\delta'(x-y) + \frac{1}{2}\, u_x \delta(x-y) + \frac{\epsilon^2}{8}\delta'''(x-y).$$

First Hamiltinian structure: Gardner 1971, Faddeev & Zakharov 1971
Bihamiltonian structure: Magri 1978

We emphasize on properties of the so called tau structures of these integrable hierarchies. For the KdV hierarchy, the tau structure is obtained by an appropriate choice of the densities h_p of the Hamiltonians

$$H_p = \int h_p(u, u_x, \dots)dx, \quad p \geq -1$$

such that the KdV hierarchy can be represented as

$$\frac{\partial u}{\partial t_p} = \{u(x), H_p\}_1 = (p + \frac{1}{2})^{-1}\{u(x), H_{p-1}\}_2, \quad p \geq 0,$$

and the following tau-symmtry condition is satisfied:

$$\frac{\partial h_{p-1}}{\partial t_q} = \frac{\partial h_{q-1}}{\partial t_p}, \quad p,\ q \geq 0.$$

The first few densities of Hamiltonians

$$
\begin{aligned}
h_{-1} &= u, \\
h_0 &= \frac{u^2}{2} + \epsilon^2 \frac{u''}{12}, \\
h_1 &= \frac{u^3}{6} + \frac{\epsilon^2}{24}(u'^2 + 2u\,u'') + \epsilon^4 \frac{u^{IV}}{240}, \\
h_2 &= \frac{u^4}{24} + \frac{\epsilon^2}{24}(u\,u'^2 + u^2 u'') \\
&\qquad + \frac{\epsilon^4}{480}(3u''^2 + 4u' u''' + 2u\,u^{IV}) + \frac{\epsilon^6}{6720} u^{VI}.
\end{aligned}
$$

The tau-symmetry condition leads to the existence of a set of differential polynomials

$$\Omega_{p,q}(u, u_x, \dots) = \Omega_{q,p}(u, u_x, \dots), \quad p, q \geq 0$$

such that

$$\frac{\partial h_{p-1}}{\partial t_q} = \partial_x \Omega_{p,q}, \quad h_p = \Omega_{0,p+1}.$$

This in turn implies the existence, for any solution $u = u(t)$ of the KdV hierarchy, of a tau function $\tau(t_0, t_1, \dots)$ such that

$$\epsilon^2 \frac{\partial^2 \log \tau(t)}{\partial t_p \partial t_q} = \Omega_{p,q}(u, u_x, \dots)|_{u=u(t)}.$$

In particular, we have

$$u = \epsilon^2 \partial_x^2 \log \tau.$$

Some aspects of tau functions

1. R. Hirota: Bilinear equations for soliton equations, 1970's.
2. M. Sato, M. Jimbo, T. Miwa: The notion of tau functions, 1980's
   Solution of KP equation and Grassmannians
   Monodromy preserving deformation equations of linear ODEs
   Hamiltonian systems
3. Witten, Kontsevich: Partition function of 2d gravity, 1990's.
   Generating function of intersection numbers on the
   Degline–Mumford modulis space of stable algebraic curves
4. Dubrovin, Z.: Tau structures and Frobenius manifolds, 2000's
5. This talk: tau structures for semisimple flat exact BHS

Main questions:

1. To describe the class of bihamiltonian structures which possess tau structures.
2. The ambiguity in the choice of tau structures.

Bihamiltonian structure which does not possess a tau structure

$$\{u(x), u(y)\}_1 = \delta'(x-y) - \frac{\epsilon^2}{8}\delta'''(x-y),$$
$$\{u(x), u(y)\}_2 = u(x)\delta'(x-y) + \frac{1}{2}\, u_x\delta(x-y)$$

for the Camassa-Holm equation

$$u_t = mm_x - \frac{\epsilon^2}{12}m_x m_{xx} - \frac{\epsilon^2}{24}mm_{xxx}$$
$$u = m - \frac{\epsilon^2}{8}m_{xx}$$

Different tau functions for the KdV hierarchy

For the KdV hierarchy, apart from the above defined tau function τ, another well-known tau function $\tilde{\tau}$ is defined by the relation

$$v = \epsilon \partial_x (\log \tau - \log \tilde{\tau}) = \epsilon \frac{\partial}{\partial x} \left(\log \frac{\tau}{\tilde{\tau}} \right),$$

where v is related to the original unknown function u of the KdV hierarchy by the Miura transformation

$$u = v^2 + \epsilon v_x.$$

Note that one need to use the above representation of v in terms of two tau functions to get Hirota's the bilinear equations for the mKdV hierarchy

$$\partial_{t_0} v = v_x, \quad \partial_{t_1} = -\frac{1}{2} v^2 v_x + \frac{\epsilon^2}{12} v_{xxx}, \dots.$$

We will start from the study of a class of so called flat exact bihamiltonian structures of hydrodynamic type, show the existence of an integrable hierarchy of systems of hydrodynamic type, and the tau structures. Then we study the deformations of these bihamiltonian structures which possess tau structures, and the classification of tau structures.

2. Flat exact bihamiltonian structures of hydrodynmic type

Bihamiltonian structures of hydrodynamic type

Let (P_1, P_2) be a bihamiltonian structure of hydrodynamic type on the loop space $\mathcal{L}(M^n)$. Then in the local coordinates $v^1, \dots, v^n$ the compatible Hamiltonian operators have the expressions

$$P_a^{\alpha\beta} = g_a^{\alpha\beta}(v)\partial_x + \Gamma_{a,\gamma}^{\alpha\beta}(v)v_x^\gamma, \quad a = 1, 2.$$

Here $(g_a^{\alpha\beta})$ are symmetric and nondegenerate, and $(g_a^{\alpha\beta})^{-1}$ are flat metrics on M, $\Gamma_{a,\gamma}^{\alpha\beta}$ are the contravariant components of the Levi-Civita connections of these metrics respectively.

Semisimple bihamiltonian structures of hydrodynamic type

The bihamiltonian structure (P_1, P_2) of hydrodynamic type is called semisimple if the roots $u^1(v), \dots, u^n(v)$ of the characteristic equation

$$\det(g_2^{\alpha\beta}(v) - u\, g_1^{\alpha\beta}(v)) = 0$$

are pairwise distinct for generic point of v and are non-constant. They can be used as local coordinates, called canonical coordinates of the semisimple bihamiltonian structure.

In the canonical coordinates the two flat metrics are diagonal

$$g_1^{ij} = f^i(u)\delta_{ij}, \quad g_2^{ij} = u^i f^i(u)\delta_{ij}.$$

Exact bihamiltonian structures

The Bihamiltonian structure (P_1, P_2) is called exact if there exist a vector field Z on the loop space such that

$$[Z, P_1] = 0, \quad [Z, P_2] = P_1.$$

Let (P_1, P_2) be semisimple, then modulo a bihamiltonian vector field Z must take the form

$$Z = \sum_{i=1}^{n} \frac{\partial}{\partial u^i}.$$

(Falqui, Lorenzoni, 2012)

We call Z the unity vector field of the exact bihamiltonian structure.

Flat exact bihamiltonian structures of hydrodynamic type

The exact bihamiltonian structure $(P_1, P_2; Z)$ is called flat exact if the unity vector field Z is flat with respect to the first flat metric g_1, i.e.

$$\nabla Z = 0.$$

Lemma

Z is flat if and only if the first diagonal metric $ds_1^2 = \sum_{i=1}^n f_i(u)(du^i)^2$ with $f_i := (f^i)^{-1}$ $(i = 1, \ldots, n)$ satisfy the following Egoroff conditions:

$$\frac{\partial f_i}{\partial u^j} = \frac{\partial f_j}{\partial u^i}, \quad \forall\, 1 \leq i, j \leq n.$$

i.e. the flat diagonal metric g_1 is a Egoroff metric.

Relation to equations of isomonodromy deformations

Let γ_{ij} be the rotation coefficients of the first flat metric g_1:

$$\gamma_{ij}(u) = \frac{1}{2\sqrt{f_i f_j}} \frac{\partial f_i}{\partial u^j}, \quad i \neq j, \quad \gamma_{ii} = 0.$$

Denote $\Gamma = (\gamma_{ij})$, and define

$$U = \mathrm{diag}(u^1, \dots, u^n), \quad V = [\Gamma, U].$$

Then V satisfies equations of isomonodromy deformations

$$\frac{\partial V}{\partial u^i} = [V_i, V], \quad V_i = \mathrm{ad}_{E_i} \mathrm{ad}_U^{-1}(V), \quad (E_i)_{jk} = \delta_{ij}\delta_{ik}.$$

of the linear system of differential equations with rational coefficients

$$\frac{dY}{dz} = \left(U + \frac{V}{z} \right) Y.$$

Lemma

Let the anti-symmetric matrix V be a solution to the above isomonodromy deformation equations defined on a neighborhhod of a point $u_0 = \left(u_0^1, \dots, u_0^n\right)$ such that $u_0^i \neq u_0^j$, $\forall\ i \neq j$, and let $H_1(u)$, ..., $H_n(u)$ be a solution to the system

$$\frac{\partial H_i}{\partial u^j} = \gamma_{ij} H_j, \quad i \neq j, \quad \frac{\partial H_i}{\partial u^i} = -\sum_j \gamma_{ij} H_j,$$

for $i, j = 1, \dots, n$ such that $H_1(u_0) \cdot \ldots \cdot H_n(u_0) \neq 0$. Then the pair of metrics

$$ds_1^2 = \sum_{i=1}^n H_i^2(u) \left(du^i\right)^2, \quad ds_2^2 = \sum_{i=1}^n \frac{H_i^2(u)}{u^i} \left(du^i\right)^2$$

along with the vector field $Z = \sum_{i=1}^n \frac{\partial}{\partial u^i}$ defines a flat exact semisimple bihamiltonian structure of hydrodynamic type on a neighborhood of u_0. Moreover, any flat exact semisimple bihamiltonian structure of hydrodynamic type can be locally obtained by this construction.

Irreducibility

Definition

The semisimple bihamiltonian structure (P_1, P_2) is called reducible at $u \in M$ if there exists a partition of the set $\{1, 2, \ldots, n\}$ into the union of two nonempty nonintersecting subsets I and J such that

$$\gamma_{ij}(u) = 0, \quad \forall i \in I, \ \forall j \in J.$$

(P_1, P_2) is called irreducible on a certain domain $D \subset M$, if it is not reducible at any point $u \in D$.

We will impose this irreducibility condition on the class of semisimple flat exact bihamiltonian structures.

Relation to Frobenius manifold structure

When one chooses the solution $(H_1, \dots, H_n)$ (Lamé coefficients) of the equations of isomonodromy equations to be an eigenvector of V, the resulting bihamiltonian structure corresponds to the flat metric and intersection form of a Frobenius manifold structure (Boris Dubrovin, 1994).

Let $(M, \cdot, \eta, e, E)$ be a Frobenius manifold. Then the pair of metrics

$$g_1^{\alpha\beta}(v) = \langle dv^\alpha, dv^\beta \rangle = \eta^{\alpha\beta},$$
$$g_2^{\alpha\beta}(v) = (dv^\alpha, dv^\beta) = i_E \left(dv^\alpha \cdot dv^\beta \right) =: g^{\alpha\beta}(v)$$

on T^*M defines a flat exact bihamiltonian structure with $Z = e$. For a semisimple Frobenius manifold the canonical coordinates of the Frobenius manifold coincide with that of the bihamiltonian structure.

Relation to Frobenius manifold structure

More bihamiltonian structures can be obtained from those of the above example by a Legendre type transformation

$$\hat{v}_\alpha = b^\gamma \frac{\partial^2 F(v)}{\partial v^\gamma \partial v^\alpha}, \quad \hat{v}^\alpha = \eta^{\alpha\beta} \hat{v}_\beta.$$

Here $F(v)$ is the potential of the Frobenius manifold and $b = b^\gamma \frac{\partial}{\partial v^\gamma}$ is a flat invertible vector field on it. The new metrics on T^*M by definition have the *same* Gram matrices in the new coordinates

$$\langle d\hat{v}^\alpha, d\hat{v}^\beta \rangle = \eta^{\alpha\beta}, \quad \left(d\hat{v}^\alpha, d\hat{v}^\beta \right) = g^{\alpha\beta}(v).$$

Relation to Frobenius manifold structure

Applying the Legendre transformation to the potential $F(v)$ of the Frobenius manifold one obtains a new solution $\hat{F}(\hat{v})$ to the WDVV associativity equations defined from

$$\frac{\partial^2 \hat{F}(\hat{v})}{\partial \hat{v}^\alpha \partial \hat{v}^\beta} = \frac{\partial^2 F(v)}{\partial v^\alpha \partial v^\beta}.$$

The new unit vector field is given by

$$\hat{e} = b^\gamma \frac{\partial}{\partial \hat{v}^\gamma}.$$

The new solution to the WDVV associativity equations defines on M another Frobenius manifold structure if the vector $b = b^\gamma \frac{\partial}{\partial v^\gamma}$ satisfies

$$[b, E] = \lambda \cdot b$$

for some $\lambda \in \mathbb{C}$. Otherwise the quasihomogeneity axiom does not hold true.

Relation to Frobenius manifold structure

Theorem

For an arbitrary Frobenius manifold M the pair of flat metrics obtained above by a Legendre transformation defines on M a flat exact bihamiltonian structure of hydrodynamic type. Conversely, any irreducible flat exact semisimple bihamiltonian structure of hydrodynamic type can be obtained in this way.

3. The principal hierarchy and its tau structure

Let us given a flat exact semisimple bihamiltonian structure of hydrodynamic type $(P_1, P_2; Z)$, and let $v^1, \dots, v^n$ be a system of flat coordinates of the first metric g_1 such that

$$Z = \frac{\partial}{\partial v^1}.$$

In the flat coordinates, the first Hamiltonian structure P_1 is given by the Hamiltonian operator

$$\eta^{\alpha\beta} \frac{\partial}{\partial x}.$$

Calibration of a bihamiltonian structure of hydrodynamic type

Definition

A collection of smooth functions

$$\{h_{\alpha,p} \mid \alpha = 1, \dots, n;\ p = -1, 0, 1, \dots\}$$

is called a calibration of $(P_1, P_2; Z)$ *if*

i) $H_{\alpha,p} = \int(h_{\alpha,p})$ *are bihamiltonian conserved quantities, i.e.*

$$[P_2, [P_1, H_{\alpha,p}]] = 0.$$

ii) *The recursion:* $Z(h_{\alpha,p}) = h_{\alpha,p-1}$ $(p \geq 0)$.

iii) *Normalization:* $h_{\alpha,-1} = v_\alpha = \eta_{\alpha\gamma} v^\gamma$, *and* $\frac{\partial}{\partial t^{1,0}} = \frac{\partial}{\partial x}$.

Definition

The hierarchy of quasi-linear PDEs

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}} = \eta^{\alpha\gamma}\frac{\partial}{\partial x}\left(\frac{\partial h_{\beta,q}(v)}{\partial v^\gamma}\right), \quad 1 \le \alpha, \beta \le n,\ q \ge 0$$

is called the principal hierarchy of the flat exact bihamiltonian structure $(P_1, P_2; Z)$ *with the calibration* $\{h_{\alpha,p}\}$.

Theorem

Let $\{h_{\alpha,p}\}$ be a calibration of $(P_1, P_2; Z)$, then the associated principal hierarchy is an integrable bihamiltonian hierarchy of hydrodynamic type with

$$\frac{\partial v^\alpha}{\partial t^{1,0}} = v^\alpha_x.$$

Moreover, for any (α, p), (β, q), we have

$$\frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} = \frac{\partial h_{\beta,q-1}}{\partial t^{\alpha,p}},$$

and there exists differential polynomials $\Omega_{\alpha,p;\beta,q}$ such that

$$\frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} = \frac{\partial h_{\beta,q-1}}{\partial t^{\alpha,p}} = \partial\Omega_{\alpha,p;\beta,q}.$$

Definition

A collection of smooth functions

$$\{\Omega_{\alpha,p;\beta,q} \mid \alpha, \beta = 1, \ldots, n;\ \ p, q = 0, 1, 2, \ldots\}$$

is called a tau structure of the flat exact bihamiltonian structure $(P_1, P_2; Z)$ *and the principal hierarchy associated to a fixed calibration* $\{h_{\alpha,p}\}$ *if the following conditions are satisfied:*

i) $\partial_x \Omega_{\alpha,p;\beta,q} = \frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} = \frac{\partial h_{\beta,q-1}}{\partial t^{\alpha,p}}$.

ii) $\Omega_{\alpha,p;\beta,q} = \Omega_{\beta,q;\alpha,p}$.

iii) $\Omega_{\alpha,p+1;1,0} = h_{\alpha,p}$.

Construction of a calibration and principal hierarchy

For the given flat exact semisimple bihamiltonian structure of hydrodynamic type $(P_1, P_2; Z)$, let us fix a system of linearly independent solutions $\psi_{i\alpha,0}(u)$, $\alpha = 1, \dots, n$ to the linear differential equations

$$\frac{\partial \psi_{i\alpha,0}}{\partial u^j} = \gamma_{ij}\, \psi_{j\alpha,0}, \quad i \neq j,$$
$$\frac{\partial \psi_{i\alpha,0}}{\partial u^i} = -\sum_j \gamma_{ij}\, \psi_{j\alpha,0},$$

which satisfy the conditions

$$\psi_{i1,0} = H_i = f_i^{1/2}, \quad i = 1, \dots, n.$$

A Frobenius manifold structure

- The symmetric non-degenerate constant matrix

$$\eta_{\alpha\beta} = \sum_{i=1}^{n} \psi_{i\alpha,0}(u)\psi_{i\beta,0}(u), \quad (\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1}.$$

- The local coordinates $v^1, \dots, v^n$ defined by

$$dv^\alpha = \sum_{i=1}^{n} \eta^{\alpha\gamma} H_i \psi_{i\gamma,0} du^i, \quad i = 1, \dots, n.$$

  are flat coordinates of the first metric, $ds_1^2 = \eta_{\alpha\beta} dv^\alpha dv^\beta$.
- The structure constants of the Frobenius algebra are given by

$$c_{\alpha\beta\gamma} = \sum_{i=1}^{n} \frac{\psi_{i\alpha,0}\psi_{i\beta,0}\psi_{i\gamma,0}}{\psi_{i1,0}}, \quad c^\gamma_{\alpha\beta} = \eta^{\gamma\xi} c_{\alpha\beta\gamma}.$$

- The unity vector field $Z = \frac{\partial}{\partial v^1}$.

Construction of a calibration and principal hierarchy

Let

$$\psi_{i\alpha}(u;z)=\sum_{p=0}^{\infty}\psi_{i\alpha,p}(u)z^p,\quad \alpha=1,\dots,n$$

be a system of solutions to the differential equations

$$\begin{aligned}\frac{\partial\psi_{i\alpha}}{\partial u^j}&=\gamma_{ij}\psi_{j\alpha},\quad i\neq j,\\ \frac{\partial\psi_{i\alpha}}{\partial u^i}&=z\,\psi_{i\alpha}-\sum_j\gamma_{ij}\psi_{j\alpha}\end{aligned}$$

with the leading term $\psi_{i\alpha,0}(u)$ and normalization

$$\sum_{i=1}^{n}\psi_{i\alpha}(u;z)\psi_{i\beta}(u;-z)=\eta_{\alpha\beta}.$$

Construction of a calibration and principal hierarchy

Introduce a family of functions

$$\Omega_{\alpha,p;\beta,q}(u) = \Omega_{\beta,q;\alpha,p}(v)$$

by

$$\frac{\sum_{i=1}^{n} \psi_{i\alpha}(u;z)\psi_{i\beta}(u;w) - \eta_{\alpha\beta}}{z+w} = \sum_{p,\,q=0}^{\infty} \Omega_{\alpha,p;\beta,q}(u) z^p w^q.$$

and define the functions

$$h_{\alpha,p} = \Omega_{1,0;\alpha,p+1}.$$

Theorem

1. *The set of functions* $\{h_{\alpha,p}\}$ *and* $\{\Omega_{\alpha,p;\beta,q}\}$ *give a calibration and a tau structure of the flat exact semisimple bihamiltonian structure of hydrodynamic type* (P_1, P_2, Z).

2. *The principal hierarchy also possesses the Galilean symmetry*

$$\frac{\partial v^\alpha}{\partial s} = \delta^\alpha_1 + \sum_{\beta,q} t^{\beta,q+1}\frac{\partial v^\alpha}{\partial t^{\beta,q}}.$$

4. Tau-symmetric integrable Hamiltonian deformations of the principal hierarchy

Given a flat exact semisimple bihamiltonian structure of hydrodynamic type $(P_1, P_2; Z)$, with a fixed

- Calibration $\{h_{\alpha,p}\}$,
- Tau structure $\{\Omega_{\alpha,p;\beta,q}\}$.

The principal hierarchy can be represented, in the flat coordinates $v^1, \dots, v^n$ of the first flat metric g_1, by

$$\begin{aligned}\frac{\partial v^\alpha}{\partial t^{\beta,q}} &= \{v^\alpha(x), H_{\alpha,p}\}_1 \\ &= \eta^{\alpha\gamma}\frac{\partial}{\partial x}\left(\frac{\partial h_{\beta,q}}{\partial v^\gamma}\right), \quad \alpha,\beta = 1,\dots,n,\ q \ge 0.\end{aligned}$$

Definition

The pair $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$ is called a tau-symmetric integrable deformation of $(P_1, \{h_{\alpha,p}\})$ if it satisfies the following conditions:

i) *$\tilde{P}_1$ and $\tilde{h}_{\alpha,p}$ have the form*

$$\tilde{P}_1 = P_1 + \epsilon^2 P_1^{[2]} + \epsilon^3 P_1^{[3]} + \dots,$$
$$\tilde{h}_{\alpha,p} = h_{\alpha,p} + \epsilon^2 h_{\alpha,p}^{[2]} + \epsilon^3 h_{\alpha,p}^{[3]} + \cdots,$$

where $\tilde{P}_1$ is a Hamiltonian structure.

ii) *Define $\tilde{H}_{\alpha,p} = \int(h_{\alpha,p})$, then*

$$\{\tilde{H}_{\alpha,p}, \tilde{H}_{\beta,q}\}_{\tilde{P}_1} = 0.$$

iii) *The deformed hierarchy $\tilde{\partial}_{\beta,q}$ satisfy the tau-symmetry condition*

$$\tilde{\partial}_{\alpha,p}\left(\tilde{h}_{\beta,q-1}\right) = \tilde{\partial}_{\beta,q}\left(\tilde{h}_{\alpha,p-1}\right).$$

The deformed hierarchy has the properties

i) $\tilde{\partial}_{1,0} v^\alpha = v_x^\alpha$.

ii) $\tilde{H}_{\alpha,0}$ are Casimirs of $\tilde{P}_1$.

Lemma

For any deformation $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$ *of* $(P_1, h_{\alpha,p}\})$*, there exists a unique collection of differential polynomials* $\{\tilde{\Omega}_{\alpha,p;\beta,q}\}$ *satisfying the following conditions:*

i) $\tilde{\Omega}_{\alpha,p;\beta,q} = \Omega_{\alpha,p;\beta,q} + \epsilon^2\Omega^{[2]}_{\alpha,p;\beta,q} + \epsilon^3\Omega^{[3]}_{\alpha,p;\beta,q} + \cdots.$

ii) $\partial_x\tilde{\Omega}_{\alpha,p;\beta,q} = \tilde{\partial}_{\alpha,p}\left(\tilde{h}_{\beta,q-1}\right).$

iii) $\tilde{\Omega}_{\alpha,p;\beta,q} = \tilde{\Omega}_{\beta,q;\alpha,p}$, *and* $\tilde{\Omega}_{\alpha,p+1;1,0} = \tilde{h}_{\alpha,p}$.

iv) $\tilde{\partial}_{\gamma,r}\tilde{\Omega}_{\alpha,p;\beta,q} = \tilde{\partial}_{\beta,q}\tilde{\Omega}_{\alpha,p;\gamma,r}$.

Here $\alpha,\beta,\gamma = 1,\ldots,n$ *and* $p,q,r \geq 0$*. This collection of differential polynomials* $\{\tilde{\Omega}_{\alpha,p;\beta,q}\}$ *is called a tau structure of* $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$*.*

Equivalent deformations

Definition

Suppose $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$ *and* $(\hat{P}_1, \{\hat{h}_{\alpha,p}\})$ *are two deformations of* $(P_1, \{h_{\alpha,p}\})$. *Define* $\tilde{H}_{\alpha,p} = \int \left(\tilde{h}_{\alpha,p}\right)$ *and* $\hat{H}_{\alpha,p} = \int \left(\hat{h}_{\alpha,p}\right)$. *If there exists a Miura transformation* e^{ad_Y} *such that*

$$\hat{P}_1 = e^{\mathrm{ad}_Y}\left(\tilde{P}_1\right), \quad \hat{H}_{\alpha,p} = e^{\mathrm{ad}_Y}\left(\tilde{H}_{\alpha,p}\right),$$

then we say that $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$ *and* $(\hat{P}_1, \{\hat{h}_{\alpha,p}\})$ *are equivalent.*

Theorem

Suppose $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$ *and* $(\hat{P}_1, \{\hat{h}_{\alpha,p}\})$ *are two equivalent deformations related by a Miura transformation* e^{ad_Y}, *and they have tau structures* $\{\tilde{\Omega}_{\alpha,p;\beta,q}\}$ *and* $\{\hat{\Omega}_{\alpha,p;\beta,q}\}$ *respectively. Then there exists a differential polynomial* G *such that*

$$\hat{h}_{\alpha,p} = e^Y\left(\tilde{h}_{\alpha,p}\right) + \partial\hat{\partial}_{\alpha,p}G,$$

$$\hat{\Omega}_{\alpha,p;\beta,q} = e^Y\left(\tilde{\Omega}_{\alpha,p;\beta,q}\right) + \hat{\partial}_{\alpha,p}\hat{\partial}_{\beta,q}G.$$

Let $\tilde{\tau}$ be the tau function of the deformed principal hierarchy corresponding to $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$, then a tau function $\hat{\tau}$ of the deformed principal hierarchy corresponding to $(\hat{P}_1, \{\hat{h}_{\alpha,p}\})$ can be chosen to satisfy the relation

$$\log\hat{\tau} = \log\tilde{\tau}(t) + G(v)|_{v=v(t)}.$$

Deformation of the unity vector field

Definition

The triple $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\}, \tilde{Z})$ *is a deformation of* $(P_1, \{h_{\alpha,p}\}, Z)$ *if*

i) *The pair* $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\})$ *is a deformation of* $(P_1, \{h_{\alpha,p}\})$*.*

ii) *The vector field* $\tilde{Z}$ *has the form*

$$\tilde{Z} = Z + Z^{[2]} + Z^{[3]} + \dots,$$

and satisfies conditions $[\tilde{Z}, \tilde{P}_1] = 0$ *and*

$$Z(\tilde{h}_{\alpha,-1}) = \eta_{\alpha,1},\ Z(\tilde{h}_{\alpha,p+1}) = \tilde{h}_{\alpha,p},\ \alpha = 1, \dots, n,\ p \geq 0.$$

The Galilean symmetry

Theorem

Let $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\}, \tilde{Z})$ be a deformation of $(P_1, \{h_{\alpha,p}\}, Z)$, then the associated deformed principal hierarchy possesses the Galilean symmetry

$$\frac{\partial \log \tilde{\tau}}{\partial s} = \frac{1}{2}\eta_{\alpha\beta} t^{\alpha,0} t^{\beta,0} + \sum_{\alpha,p} t^{\alpha,p+1} \frac{\partial \log \tilde{\tau}}{\partial t^{\alpha,p}}.$$

5. Tau-symmetric bihamiltonian deformations of the principal hierarchy

Deformations of semisimple bihamiltonian structure of hydrodynamic type

i) Uniquely parameterized by central invariants $c_1(u^1), \ldots, c_n(u^n)$. (Dubrovin, Liu, Z. 2006)

ii) Existence of deformation with a given set of central invariants.

(Liu, Z. 2012 for KdV case, Carlet, Posthuma, Shadrin 2015 for general case)

iii) The space of bihamiltonian conserved quantities and bihamiltonian vector fields of the deformed bihamiltonian structure are isomorphic to those of the original bihamiltinian structure.

(Dubrovin, Liu, Z. 2017)

A theorem of Falqui & Lorenzoni on exactness

Theorem (Falqui, Lorenzoni 2012)

A deformation $(\tilde{P}_1, \tilde{P}_2)$ of the exact semisimple bihamiltonian structure of hydrodynamic type $(P_1, P_2; Z)$ is exact if and only if its central invariants $c_1, \dots, c_n$ are constant functions. Moreover, there exists a Miura type transformation g such that

$$g(\tilde{P}_1) = P_1, \quad g(\tilde{P}_2) = P_2 + \epsilon^2 Q_1 + \epsilon^4 Q_2 + \dots,$$

and the unity vector field $\tilde{Z}$ is transformed to the undeformed one

$$g(\tilde{Z}) = Z = \sum_{i=1}^{n} \frac{\partial}{\partial u^i}.$$

Let $(P_1, P_2; Z)$ be a flat exact semisimple bihamiltonian structure of hydrodynamic type, with a fixed calibration and a tau structure

$$\{h_{\alpha,p}\}, \quad \{\Omega_{\alpha,p;\beta,q}\}.$$

Given a set of constants $c_1, \dots, c_n$ we have a unique, up to Miura type transformations, deformation $(\tilde{P}_1, \tilde{P}_2; \tilde{Z})$ with

$$\tilde{P}_1 = P_1, \quad \tilde{P}_2 = P_2 + \epsilon^2 Q_1 + \epsilon^4 Q_2 + \dots,$$

$$\tilde{Z} = \sum_{i=1}^{n} \frac{\partial}{\partial u^i}.$$

Then we have a unique deformation

$$\tilde{H}_{\alpha,p} = H_{\alpha,p} + \epsilon^2 H^{[1]}_{\alpha,p} + \epsilon^4 H^{[2]}_{\alpha,p} + \dots$$

of the bihamiltonian conserved quantities $H_{\alpha,p} = \int(h_{\alpha,p})$.

Theorem

The deformation $(\tilde{P}_1, \tilde{P}_2; \tilde{Z})$ and the associated deformed principal hierarchy possess a tau structure.

Construction of the tau structure

The densities of the Hamiltonians are given by

$$\tilde{h}_{\alpha,p} = \delta_Z \tilde{H}_{\alpha,p+1} = \sum_{i=1}^{n} \frac{\delta \tilde{H}_{\alpha,p+1}}{\delta u^i}, \quad \alpha = 1, \dots, n, \ p = -1, 0, 1, 2, \dots.$$

The differential polynomials $\tilde{\Omega}_{\alpha,p;\beta,q}$ are given by

$$\frac{\partial \tilde{h}_{\alpha,p-1}}{\partial t^{\beta,q}} = \partial_x \tilde{\Omega}_{\alpha,p;\beta,q}.$$

Ambiguity of the choice of tau structure

The deformation $(\tilde{P}_1, \{\tilde{h}_{\alpha,p}\}, \tilde{Z})$ constructed in the above theorem depends on the choice of $\tilde{P}_2$.
If we start from another deformation $(\hat{P}_1, \hat{P}_2; \hat{Z})$ which has the same central invariants as $(\tilde{P}_1, \tilde{P}_2; \tilde{Z})$, then

$$\hat{P}_a = e^{\mathrm{ad}_Y}\left(\tilde{P}_a\right), \quad a = 1, 2.$$

Since $\hat{P}_1 = P_1$, we have $[P_1, Y] = 0$, so there exists a local functional K such that $Y = [P_1, K]$.

Lemma

There exist a differential polynomial g such that

$$\delta_Z K = \partial_x g.$$

Theorem

The tau structures are related by the formulae

$$\hat{h}_{\alpha,p} = e^{Y}\left(\tilde{h}_{\alpha,p}\right) + \partial\hat{\partial}_{\alpha,p}G,$$
$$\hat{\Omega}_{\alpha,p;\beta,q} = e^{Y}\left(\tilde{\Omega}_{\alpha,p;\beta,q}\right) + \hat{\partial}_{\alpha,p}\hat{\partial}_{\beta,q}G.$$

where

$$G = \sum_{i=1}^{\infty}\frac{1}{i!}Y^{i-1}\left(g\right).$$

Works to be done

To provide a constructive approach to fix a representative of the deformations of a flat exact semisimple bihamiltonian structure of hydrodynamic type with constant central invariants, and to fix the function G (thus the tau structure), so that the deformation of the principal hierarchy can be used as candidate of integrable hierarchies that control 2D TFT.

Thanks!