On a complexification of the moduli space of Bohr - Sommerfeld lagrangian cycles

Nikolai Tyurin*
BLTPh JINR (Dubna) and NRU HSE (Moscow)

Let (M, ω) – be a compact simply connected symplectic manifold of real dimension $2n$ with integer symplectic form such that $[\omega] \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R})$. Consider prequantization data (L, a), where $L \to M$ – is a complex line bundle with a fixed hermitian structure h and $a \in \mathcal{A}_h(L)$ – a hermitian connection whose curvature form satsifies $F_a = 2\pi i \omega$. Choose and fix an appropriate topological type top$S$ of smooth oriented n - dimensional manifold and a homology class $[S] \in H_n(M, \mathbb{Z})$, then it leads to the derivation of the corresponding moduli space $\mathcal{B}_S$ of Bohr - Sommerfeld lagrangian cycles of fixed topological type, which has been constructed in [1]. This moduli space is an infinite dimensional Frechet smooth real manifold, locally isomorphic to space $C^\infty(S, \mathbb{R})$ modulo constants. The points of $\mathcal{B}_S$ can be presented by lagrangian submanifolds of fixed topological type satisying the Bohr - Sommerfeld condition: the restriction of the prequantization data $(L, a)|_S$ admits covariantly constant sections. The details of the construction can be found in [1].

The moduli space $\mathcal{B}_S$ can be exploited in the lagrangian approach to Geometric Quantization: the corresponding "quantum" mechanics can be directly described while the measurment process, which any quantum mechanical system must admit, can be defined only after an appropriate "complexification" of the moduli space $\mathcal{B}_S$ is given. This problem is in the focus of our studies.

Consider the space $\Gamma(M, L)$ of all smooth section of the prequantization bundle and its projectivization $\mathbb{P}\Gamma(M, L)$. One can attach to every element of the last space certain complex 1 -form, defined on the open complement $M \backslash D_\alpha$ where D_α is the zeroset of the corresponding section. Namely for every $[\alpha] \in \mathbb{P}\Gamma(M, L)$ take a smooth section α, defined up to scaling, and consider the following expression

$$\rho(\alpha) = \frac{\nabla_a \alpha}{\alpha} = \frac{< \nabla_a \alpha, \alpha >_h}{< \alpha, \alpha >_h} \in \Omega^1_{M \backslash D_\alpha} \otimes \mathbb{C},$$

where D_α corresponds to α. Evidently this 1 -form $\rho(\alpha)$ does not depend on the scaling $c\alpha$, therefore this form corresponds not to a section but to the class $[\alpha] \in \mathbb{P}\Gamma(M, L)$.

The main properties of this 1 - form $\rho(\alpha)$ are the following: its real part $\text{Re}\rho(\alpha)$ is exact, and the differential of the imaginary part satisfies $\text{dIm}\rho(\alpha) = 2\pi\omega$ (see [2]).

Then we say that

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Definition 1. *Lagrangian submanifold $S \subset M$ is special Bohr - Sommerfeld with respect to α (or α - SBS for short) iff the restriction* $\mathrm{Im}\rho(\alpha)|_S$ *identically vanishes.*

In particular it implies that S does not intersect zeroset D_α.

Remark. In paper [2] we originally used another defitnion of SBS and then deduced the vanishing condition for the imaginary part as a proposition. However in the present situation we skip the story and use the vanishing condition as the definition.

SBS - condition, introduced above, leads to the following

Definition 2. *Subset $\mathcal{U}_{SBS} \subset \mathcal{B}_S \times \mathbb{P}\Gamma(M, L)$ is formed by the pairs $(S, [\alpha])$, such that S satisfies α - SBS condition.*

By the very definition the subset $\mathcal{U}_{SBS}$ admits two canonical projections $p : \mathcal{U}_{SBS} \to \mathbb{P}\Gamma(M, L)$ and $q : \mathcal{U}_{SBS} \to \mathcal{B}_S$. The first projection has been studied in [2], there one establishes that the fibers of the projections are discrete; the image of the projection is an open subset in the projective space; the differential dp has trivial kernel at generic point (Theorem 1, [2]). It follows that $\mathcal{U}_{SBS}$ is weakly Kahler: the standard Kahler form from the projective space $\mathbb{P}\Gamma(M, L)$ can be lifted using the differential dp.

In preprint [3] we use the correspondence $[\alpha] \longleftrightarrow \rho(\alpha)$ to prove that the differential dp is an isomorphism: any tangent to $\mathbb{P}\Gamma(M, L)$ vector can be canonically lifted to a tangent one to the moduli space $\mathcal{U}_{SBS}$; in particular it implies the existence of the complex structure on $\mathcal{U}_{SBS}$, lifted by the differential dp.

However the space $\mathcal{U}_{SBS}$ is not a complexification of the moduli space $\mathcal{B}_S$, since it is too big. Indeed, the "dimension" of the moduli space $\mathcal{B}_S$ equals to the dimension of $C^\infty(S, \mathbb{R})$ modulo constants while the "dimension" of $\mathcal{U}_{SBS}$ is the same as for $\mathbb{P}\Gamma(M, L)$, so it is of order $C^\infty(M, \mathbb{C})$ modulo constants. On the other hand it exists a nantural map $\tau : \mathcal{U}_{SBS} \to T\mathcal{B}_S$, which splits the second projection q, namely one has the representation $q = \tau \circ \pi$, where $\pi : T\mathcal{B}_S \to \mathcal{B}_S$ is the canonical projection to the base. The map τ is given by a simple and exact expression: since Definition 1 implies that for any point $(S, [\alpha]) \in \mathcal{U}_{SBS}$ the restriction $\rho(\alpha)|_S$ is an exact real 1 - form, but from [1] we know that any exact real 1 - form is a tangent vector to $\mathcal{B}_S$ at point S, consequently we get a simple formula

$$\tau(S, [\alpha]) = (\rho(\alpha))|_S \in T_S\mathcal{B}_S,$$

and it is easy to see that applying π we get the image $q(S, [\alpha]) = S \in \mathcal{B}_S$.

In [3] we show that a generic fiber $\tau^{-1}(S, \mathrm{d}f) \subset \mathcal{U}_{SBS}$ is a complex subset in $\mathcal{U}_{SBS}$ with respect to the complex structure, given by the previous construction. It should lead to a construction of derived complex structures on the tangent bundle $T\mathcal{B}_S$ itself: suppose that one finds an appropriate section of the fibered map $\tau : \mathcal{U}_{SBS} \to T\mathcal{B}_S$, which is Kahler orthogonal to the fibers or at least which is a complex subset in $\mathcal{U}_{SBS}$. Right now we can not present a good candidate, hoping that it will be found in the nearest future.

References:

[1] A. Gorodentsev, A. Tyurin, *"Abelian lagrangian algebraic geometry"*, Izvestiya: mat. 65:3 (2001), pp 15 -50;

[2] N. Tyurin, *"Special Bohr - Sommerfeld lagrangian submanifolds"*, Izvestiya: mat. 80:6 (2016), pp 274–293;

[3] N. Tyurin, *"On a complexification of the moduli space of Bohr - Sommerfeld lagrangian cycles"*, arXiv:1807.11351.