Separatrix map for slow-fast Hamiltonian systems

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Dynamics in Siberia 2021

Slow-fast Hamiltonian systems

- Consider a Hamiltonian system on $M \times N$ with a symplectic structure

$$\omega_\varepsilon = \omega + \varepsilon^{-1}\Omega, \qquad \varepsilon \ll 1.$$

  ω and Ω – symplectic structures on M and N.
  Hamilton's equations:

$$\dot{z} = J\partial_z H(z,w), \quad \dot{w} = \varepsilon J\partial_w H(z,w)$$

  $z \in M$ – fast variables, $w \in N$ – slow variables.

- A slowly time dependent system

$$\dot{z} = J\partial_z H(z,\tau), \qquad \dot{\tau} = \varepsilon \ll 1$$

  can be represented as a slow-fast system

$$\hat{H}(z,\tau,h) = H(z,\tau) + h, \quad \hat{\omega}_\varepsilon = \omega + \varepsilon^{-1} dh \wedge d\tau,$$

  on the energy level $\{\hat{H} = 0\}$.

Adiabatic invariant

- For $\varepsilon = 0$ we obtain a frozen system

$$\dot{z} = J\nabla H_w(z), \quad H_w(z) = H(z, w)$$

  depending on a parameter w.
- Suppose trajectories γ of the frozen system on energy levels $\{H_w = E\}$ are periodic. $I(w, E) = A(\gamma)$ – the Maupertuis action of γ. Period $\tau(w, E) = \partial_E I(w, I)$.
- For small ε the action is an adiabatic invariant: on $\Sigma_E = \{H = E\}$ it changes with average rate $O(\varepsilon^2)$.
- Averaged system:

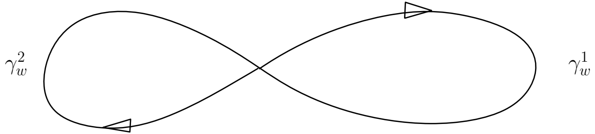
$$\dot{w} = -\varepsilon \frac{J\partial_w I(w, E)}{\tau(w, E)}$$

- Period map $S : w \to w - \varepsilon J\nabla I(w, E) + o(\varepsilon)$.

Adiabatic invariant near a separatrix

- Averaging doesn't work for trajectories passing near equilibria of the frozen system but there is an analog of the monodromy map – separatrix map.
- Suppose the frozen system has 1 dof and $z_0(w)$ is a hyperbolic equilibrium with a figure 8 separatrix $\Gamma_w = \gamma_w^1 \cup \gamma_w^2$ – union of 2 homoclinics.
- In the complement of Γ_w there is an adiabatic invariant.
- For $E \to h(w) = H(z_0(w), w)$ inside γ_w^k it behaves as

$$I_k(w, E) = A(\gamma_w^k) + \frac{(h(w) - E) \ln |h(w) - E|}{\lambda(w)} + \cdots$$



Jumps of the adiabatic invariant

- While a trajectory stays away from the separatrix, the slow variable shadows a level curve of an adiabatic invariant.
- For 1 dof A. Neishtadt (1986) showed that when the slow variable crosses the curve $Z_E = \{h = E\}$ (then the trajectory of the fast system shadows the separatrix), the adiabatic invariant has quasi-random jumps of order ε. Then the slow variable shadows a level curve of another adiabatic invariant till it crosses Z_E again and so on.
- We obtain a partial multidimensional analog of Neishtadt's result for trajectories shadowing a chain of homoclinics.
- The behavior of shadowing trajectories is described by an analog of the separatrix map of B.Chirikov, A.Neishtadt, D.Treschev.

Remarks

- The problem is related to Arnold's diffusion problem near a multiple resonance.
- V.Gelfreich and D.Turaev (2008) showed that if the frozen system has compact uniformly hyperbolic chaotic invariant sets $\Lambda_{w,E} \subset \{H_w = E\}$, there exist trajectories with quasirandom change of the slow variable with average rate of order ε.
- w shadows a trajectory of a composition of maps $w \to J\nabla I_k(w, E)$, $I_k(w, E) = A(\gamma^k_{w,E})$, where $\gamma^k_{w,E} \subset \Lambda_{w,E}$ – hyperbolic periodic orbits. Tool – Shilnikov's separatrix map.
- This result does not work on critical frozen energy levels.

Setup

- Suppose the frozen system has a hyperbolic equilibrium $z_0(w)$ such that

$$\nabla h(w) \neq 0, \qquad h(w) = H(z_0(w), w).$$

- Then energy levels

$$\Sigma_E = \{(z, w) : H(z, w) = E\}$$

  are smooth at $(z_0(w), w)$.
- If $(z_0(w), w) \in \Sigma_E$, then w belongs to the uncertainty manifold $Z_E = \{h = E\}$.
- Goal: study trajectories on $(z(t), w(t)) \in \Sigma_E$ with $w(t)$ near Z_E.

Poincaré function

- Let ϕ_w^t be the flow of the frozen system and

$$W^{\pm}(w) = \{x : \phi_w^t(x) \to z_0(w) \text{ as } t \to \pm\infty\}$$

the stable and unstable manifolds.

- Let $\gamma_w^k : \mathbb{R} \to W^-(w) \cap W^+(w)$ be transverse homoclinics. Poincaré's functions $P_k(w) = A(\gamma_w^k)$.
- Melnikov's function:

$$\nabla P_k(w) = -\int_{-\infty}^{+\infty} \partial_w \hat{H}(\gamma_w^k(t), w)\, dt, \quad \hat{H}(z, w) = H - h(w).$$

- For a trajectory $(z(t), w(t)) \in \Sigma_E$ shadowing a homoclinic γ_w^k for $-T \le t \le T$,

$$\Delta w = \int_{-T}^{T} \dot w \, dt = \varepsilon \int_{-T}^{T} J\partial_w \hat H \, dt + \varepsilon \int_{-T}^{T} J\nabla h \, dt.$$

- For large T the first integral converges to $-J\nabla P_k(w)$, and the second grows as $2TJ\nabla h$.
- If the trajectory comes $O(\sqrt{\varepsilon})$-close to $z_0(w)$ at $t = T$, then $h(w) - E \sim \varepsilon$ and

$$2T \sim \frac{\ln |h(w) - E|}{\alpha(w)}, \quad \alpha(w) = \min |\mathrm{Re}\,(\mathrm{eigenvalues})|$$

- Hence $\Delta w \approx -\varepsilon J\nabla I_k(w, E)$, where

$$I_k(w, E) = P_k(w) + \frac{(h(w) - E) \ln |h(w) - E|}{\alpha(w)}$$

– analog of the adiabatic invariant for 1 dof fast system.

- The separatrix map corresponding to the homoclinic γ_w^k is

$$S_\varepsilon^k(w) = w - \varepsilon J \nabla I_k(w, E) + o(\varepsilon).$$

- S_ε^k is an analog of the separatrix map introduced by B.Chirikov and D.Treschev for nearly integrable time periodic systems with 1 dof, and by A.Neishtadt for slowly time dependent systems with 1 dof.
- For 1 dof fast system, the separatrix map is the Poincaré map for an appropriate choice of the section. Not so for higher dimension.
- Compositions of the separatrix maps are not always defined.

Relation to the scattering map

- For small ε the system has a normally hyperbolic invariant manifold $N_\varepsilon = \{(z_\varepsilon(w), w) : w \in N\}$.
- The Hamiltonian flow ϕ_ε^t on N_ε:

$$\dot{w} = \varepsilon J \nabla h(w) + o(\varepsilon).$$

- The scattering map $F_\varepsilon^k : N_\varepsilon \to N_\varepsilon$ corresponding to the homoclinic γ_w^k is given by $F_\varepsilon^k(w) = w_+$ if $W^-(z_\varepsilon(w), w) \cap W^+(z_\varepsilon(w_+), w_+)$ contains a curve close to γ_w^k. Then

$$F_\varepsilon^k(w) = w - \varepsilon J \nabla P_k(w) + o(\varepsilon).$$

- The separatrix map S_ε^k is a superposition of the flow ϕ_ε^t on N_ε and the scattering maps F_ε^k.

Leading homoclinics

$\alpha(w) = \min\{|\mathrm{Re}\,\lambda| : \lambda$ eigenvalue of $z_0(w)\}$.
Generically there are 2 cases:

- $\pm\alpha(w)$ are real simple eigenvalues.
- $\pm\alpha(w) \pm i\beta(w)$ are complex simple eigenvalues.

$W^{\pm}_{\mathrm{strong}}(w)$ – strong stable and unstable manifolds of $z_0(w)$ corresponding to strong eigenvalues with $|\mathrm{Re}\,\lambda| > \alpha(w)$.
We call a homoclinic orbit γ_w of the frozen system leading if $\gamma_w(\mathbb{R}) \not\subset W^{+}_{\mathrm{strong}}(w) \cup W^{-}_{\mathrm{strong}}(w)$.
Generic homoclinics are leading and transverse.

Real case

For a leading homoclinic there exist

$$v_\pm(\gamma_w) = \lim_{t\to\pm\infty} e^{-\alpha(w)|t|}\dot\gamma_w(t) \neq 0.$$

We call γ_w positive (negative) if

$$\omega(v_+(\gamma_w), v_-(\gamma_w)) > 0 \qquad (< 0).$$

Example: natural systems

$$H_w(q,p) = \frac{1}{2}\|p\|^2 + V_w(q)$$

- If q_0 is a point of nondegenerate maximum of V_w, then $z_0 = (q_0, 0)$ is a hyperbolic equilibrium with real eigenvalues.
- A homoclinic $\gamma(t) = (q(t), p(t))$ is positive (negative) if

$$\lim_{t\to+\infty} \frac{\dot q(t)}{\|\dot q(t)\|} = \mp \lim_{t\to-\infty} \frac{\dot q(t)}{\|\dot q(t)\|}.$$

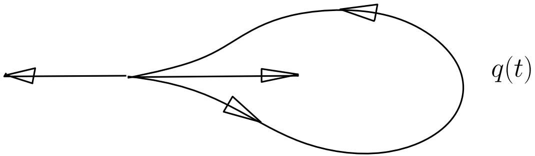


Figure: Positive homoclinic in the configuration space

Shadowing a positive homoclinic

$\Omega_+(\varepsilon) = \{a \le h - E \le b\varepsilon\}$. Let $\varepsilon > 0$ be small enough.

Theorem

For any $w_0 \in \Omega_+$ there exist a sequence $(t_i)_{i=1}^n$ and a trajectory $(z(t), w(t)) \in \Sigma_E$ such that:

- *$w(0) = w_0$ and $z(t)$ shadows the homoclinic chain $(\gamma_{w_i}^k)_{i=0}^n$, $w_i = w(t_i)$.*
- *$d(z(t), z_0(w(t)))$ has a local minimum $\sim \sqrt{\varepsilon}$ at $t = t_i$.*
- *The sequence w_i shadows a trajectory of the separatrix map:*

$$\Delta w_i = w_{i+1} - w_i = -\varepsilon J \nabla I_k(w_i, E) + o(\varepsilon)$$

- $\Delta t_i = t_{i+1} - t_i = \frac{|\ln \varepsilon|}{\alpha(w_i)} + O(1)$.

n is determined by the condition $w_i \in \Omega_+$ for $i = 1, \dots, n$.

Complex case

In the complex case we don't need the homoclinic to be positive.

Theorem

For any sequence $m_i \in \mathbb{N}$ and $w_0 \in \Omega_+$ there exist a sequence $(t_i)_{i=1}^n$ and a trajectory $(z(t), w(t)) \in \Sigma_E$ such that:

- $w(0) = w_0$ *and* $z(t)$ *shadows the homoclinic chain* $(\gamma_{w_i})_{i=0}^n$.
- $d(z(t), z_0(w(t)))$ *has a local minimum* $\sim \sqrt{\varepsilon}$ *at* $t = t_i$.
- *The sequence* w_i *shadows a trajectory of the separatrix map:* $\Delta w_i = w_{i+1} - w_i = -\varepsilon J \nabla I_k(w_i, E) + o(\varepsilon)$,
- $\Delta t_i = t_{i+1} - t_i = \frac{|\ln \varepsilon|}{\alpha(w_i)} + \frac{2\pi m_i}{\beta(w_i)} + O(1)$.

The index m_i shows the number of revolutions near $z_0(w_i)$ between shadowing of the homoclinic.

Shadowing a homoclinic chain

Suppose the leading eigenvalues are real. We call a code (k_i) positive if $s_{k_i,k_{i+1}} = \omega(v_+(\gamma_w^{k_i}), v_-(\gamma_w^{k_{i+1}})) > 0$ for all i.

Theorem

For any positive code $(k_i)_{i=1}^n$ and any $w_0 \in \Omega_+$ there exist a sequence $(t_i)_{i=1}^n$ and a trajectory $(z(t), w(t)) \in \Sigma_E$ such that $w(0) = w_0$ and:

- *$d(z(t), z_0(w(t)))$ has a local minimum $\sim \sqrt{\varepsilon}$ at $t = t_i$.*
- *The trajectory shadows the homoclinic chain $(\gamma_{w_i}^{k_i})$ and*

$$\Delta t_i = t_{i+1} - t_i = \frac{|\ln \varepsilon|}{\alpha(w_i)} + O(1).$$

$$\Delta w_i = w(t_{i+1}) - w(t_i) = -\varepsilon J \nabla I_{k_i}(w_i, E) + o(\varepsilon),$$

The orbit w_i moves along a trajectory with Hamiltonian I_{k_1}, then along a trajectory with Hamiltonian I_{k_2}, and so on. Similarly for the complex case.

Remarks

- The time interval $0 \le t \le T \sim n|\ln \varepsilon|$ is relatively short: for longer time the trajectory will exit the domain

$$\Omega_+(\varepsilon) = \{a\varepsilon \le h - E \le b\varepsilon\}.$$

  One can get similar results in a larger domain $a\varepsilon \le |h - E| \le \delta \ll 1$, but this is not written yet.
- If there are homoclinics $\gamma_w^{1,2}$ such that the Poisson brackets $\{h, P_1\} > 0$ and $\{h, P_2\} < 0$, there exist shadowing trajectories with $w(t) \in \Omega_+$ for long time.

Trajectories with "random" behavior

In the real case let $\gamma_w^{1,2}$ be positive homoclinics such that $s_{1,2} > 0$, $s_{2,1} > 0$. Suppose also $\{h, P_1\} > 0$ and $\{h, P_2\} < 0$.

Corollary

Let $u : [0, T] \to \mathbb{R}_+$ be a continuous function. For small $\varepsilon > 0$ there exist a code $(k_i)_{i=1}^n$, a sequence $(t_i)_{i=0}^n$ and a trajectory $(z(t), w(t)) \in \Sigma_E$, $0 \le \tau \le T$, such that

- *$z(t)$ shadows the homoclinic chain $(\gamma_{w_i}^{k_i})_{i=1}^N$ with $\Delta w_i = -\varepsilon J \nabla I_{k_i}(w_i, E)$.*
- $|h(w_i) - \varepsilon u(t_i)| = o(\varepsilon)$.

The time interval T is independent of ε. In the complex case no positivity assumption is needed.

Some references

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