Density of thin film billiard reflection pseudogroup in Hamiltonian symplectomorphism pseudogroup

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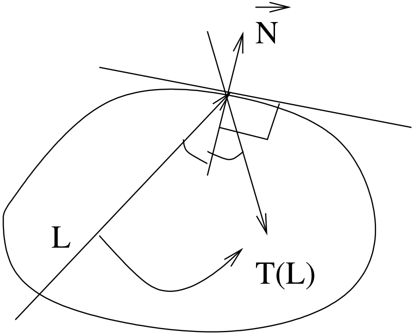
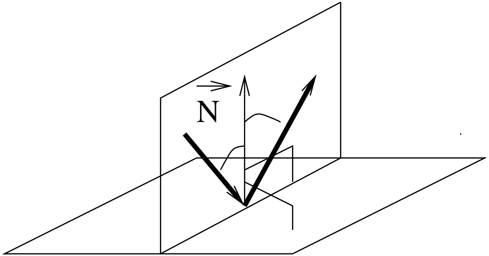
Action of billiard reflection on the space of oriented lines

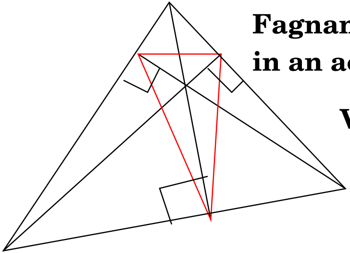




Problem 1 Does every triangular billiard have at least one periodic orbit?

Introduction to billiards. Some famous open problems

Problem 1 Does every triangular billiard have at least one periodic orbit?



Fagnano 3–periodic orbit in an acute–angled triangular billiard

Vertices = bases of heights

Richard Schwarz: partial progress for obtuse-angled triangular billiards based on numerical experiments.

Problem 2 (Ivrii's Conjecture). In every billiard with smooth boundary the set of **periodic oriented lines** has **zero Lebesgue measure**?

Partial results: 3-periodic case: **Rychlik, Stojanov, Vorobets (any dim)...**
4-periodic in **dim** $= 2$: **A.G. and Yuri Kudryashov.**

Symplectic properties of billiards

M – 2n-dimensional manifold, ω - 2-form on M.
The form ω is **symplectic,** if ω is **non-degenerate,** and $d\omega = 0$.

Standard symplectic form on $\mathbb{R}^{2n}_{q_1,\dots,q_n,p_1,\dots,p_n}$: $\omega = \sum_{j=1}^{n} dp_j \wedge dq_j$.

Example. For every manifold N the cotangent bundle T^*N is symplectic.

Namely, let $\pi : T^*N \to N$ projection, $x \in N$, $\beta \in T^*_x N$: lin. form on $T_x N$.
The canonical 1-form α on T^*N, the **Liouville form,** is defined as follows.

$$\text{For every } \ \zeta \in T_{(x,\beta)}(T^*N) \ \text{ set } \quad \alpha(\zeta) := \beta(\pi_* \zeta).$$

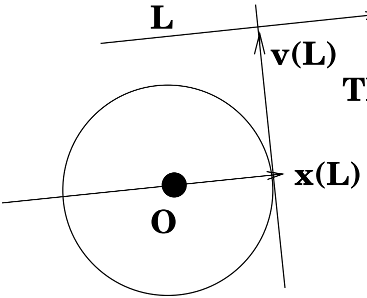
$\omega := d\alpha$ is the **standard symplectic form** on T^*N.

Let N be a **Riemannian manifold.** Then $T^*N \simeq TN$.
The **Riemannian Liouville form** α and **symplectic form** ω on TN:

$$x \in N, \ w \in T_x N, \ \zeta \in T_{(x,w)} TN; \quad \alpha(\zeta) := < w, \pi_* \zeta >, \quad \omega := d\alpha.$$

Symplectic properties of billiards

Example. The **space** E_n **of oriented lines** in $\mathbb{R}^n$ is canonically diffeomorphic to the **tangent bundle** TS^{n-1} to the unit sphere; $\dim E_n = 2(n-1)$.
The pullback of the standard symplectic form on TS^{n-1} is called the **standard symplectic form** on the space of oriented lines E_n.



The isomorphism E_n --> TS^{n-1}

L --> (x,v)

Equivalent definition: Melrose symplectic reduction.
Let ω - standard symplectic form on $T\mathbb{R}^n$. Consider the hypersurface

$$S := T_1\mathbb{R}^n = \{(x,v) \in T\mathbb{R}^n \mid ||v|| = 1\} \subset (T\mathbb{R}^n, \omega)$$

$$\mathrm{Ker}(\omega|_S) - \text{ line field on } S.$$

$$\Sigma \subset S - \text{ a cross-section } \simeq \text{ open subset } U \subset E_n.$$

$$\omega|_\Sigma = \text{ a symplectic structure on } \Sigma \simeq U.$$

The symplectic form ω_U is **independent** on the choice of Σ for given U.

Theorem (Melrose). Billiard reflection from every hypersurface in $\mathbb{R}^n$ acting on the space E_n is a **symplectomorphism:** preserves the form ω.

Melrose made this construction for every **Riemannian manifold**:
the canonical symplectic form on the local space of geodesics.
It is invariant under billiard reflection from any hypersurface.

Symplectic properties of billiards

Reflection from any hypersurface (mirror) acting on the space E_n of oriented lines is a symplectomorphism.

Problem (A.Yu.Plakhov, D.V.Treschev, S.L.Tabachnikov). Which symplectomorphisms can be realized as compositions of mirror reflections?

Very few symplectomorphisms can be realized:
- a **symplectomorphism** between domains in $\mathbb{R}^{2(n-1)}$ is locally defined by a **generating function of** $2(n-1)$ **variables;**
- a reflecting **hypersurface** is a graph of **function of** $n-1$ **variables.**

Consider two **parallel beams of lines** in $\mathbb{R}^3$ identified with two domains $U, V \subset \mathbb{R}^2$. They are **Lagrangian surfaces** in E_3.

Theorem (A.Yu.Plakhov, D.V.Treschev, S.L.Tabachnikov, 2016). Every orientation-preserving diffeomorphism $U \to V$ is a **composition of 6 reflections.**

Hamiltonian symplectomorphisms

A symplectomorphism F of a symplectic manifold (M, ω) is **Hamiltonian,** if there exists a function $H : M \times [0, 1] \to \mathbb{R}$ such that F is the time 1 flow map of corresponding non-autonomous Hamiltonian differential equation:

$$\dot{x} = \nabla_\omega H(x, t), \ \nabla_\omega := \text{ the skew gradient of } H \text{ in } x \in M.$$

A diffeomorphism of a manifold with boundary has **compact support,** if it equals identity outside a compact subset and near the boundary.

Each symplectomorphism of a **symplectic** topological **disk** is Hamiltonian.

The group of **symplectomorphisms** of **top. disk** with compact support is **path-connected** by smooth paths and **contractible.**

Theorem (M.Gromov). The same holds for symplectomorphisms of $\mathbb{R}^4$.

In **higher dimensions** it is **not known,** whether a similar statement is true.

Let $V \subset (M, \omega)$ be a domain. A symplectomorphism $F : V \to F(V) \subset M$ is called M-**Hamiltonian,** if there exists a smooth family of symplectomorphisms $F_t : V \to V_t = F(V_t) \subset M$, $t \in [0,1]$, $F_0 = Id$, $F_1 = F$, such that for every $t \in [0,1]$
the derivative $\frac{dF}{dt}$ is a **Hamiltonian** vector field on V_t.

Example. Let M be a symplectic topological **cylinder,**
$V \Subset M$ be a **subcylinder:** a deformation retract of M.
Then **not every** symplectomorphism $F : V \to F(V) \Subset M$ isotopic to identity is M-Hamiltonian.

Necessary condition for F to be M-Hamiltonian: case of cylinders.
Consider a boundary component $L \subset \partial V$. Then
the **signed area** of the domain bounded by L and $F(L)$ should be **zero.**

Main results: global case.

Let $\alpha > 0$, $k \in \mathbb{N}$.

Two hypersurfaces in $\mathbb{R}^{n+1}$ are (α, k)-close, if they admit parametrizations by the same source n-manifold that are α-close in C^k-norm.

Let $\gamma \subset \mathbb{R}^{n+1}$ be a strictly convex C^∞-smooth closed hypersurface,

$$\Pi := \{\text{the oriented lines intersecting } \gamma \text{ transversally}\} \subset E_{n+1}.$$

$\Pi :=$ **phase cylinder** of billiard inside the convex domain bounded by γ.

Theorem (A.G.) For $\alpha > 0$ arbitrarily small, $k \in \mathbb{N}$ arbitrarily large and arbitrary domain $V \subset \Pi$ each Π-**Hamiltonian** symplectomorphism $F : V \to F(V) \subset \Pi$ is a C^∞-**limit** of compositions of **reflections** (and their **inverses**) from closed C^∞-smooth hypersurfaces (α, k)-close to γ.

In other words: the C^∞-**closure** of **pseudogroup** generated by **reflections** from hypersurfaces (α, k)-close to γ contains the **pseudogroup** of Π-**Hamiltonian symplectomorphisms** between domains in Π.

Thin film billiards (after Ron Perline)

Perline, R. *Geometry of Thin Films.* J. Nonlin. Sci. 29 (2019), 621–642.

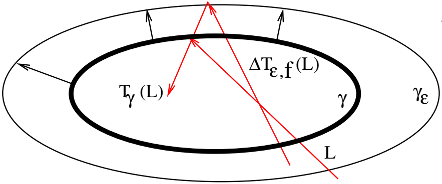
Let $\gamma \subset \mathbb{R}^{n+1}$ be a C^∞-smooth closed strictly convex hypersurface,

Let $\Pi :=$ {oriented lines intersecting γ transversally} $\subset E_{n+1}$.
$\vec{N} = \vec{N}(x) :=$ the **exterior** unit **normal** vector field on γ. Let $f \in C^\infty(\gamma)$,

$$\gamma_\varepsilon = \gamma_{\varepsilon,f} := \{x + \varepsilon f(x)\vec{N}(x) \mid x \in \gamma\},$$

$T_\gamma, T_{\gamma_\varepsilon} : E_{n+1} \to E_{n+1}$ the reflections from γ and γ_ε acting on lines,

$$\Delta T_{\varepsilon,f} := T_{\gamma_\varepsilon}^{-1} \circ T_\gamma, \quad \Delta T_{0,f} = Id.$$



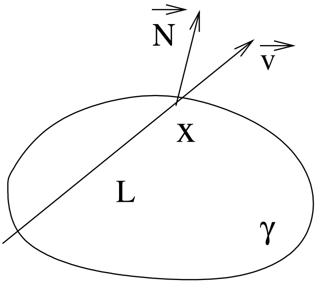
$$V_f := \frac{d}{d\varepsilon}(\Delta T_{\varepsilon,f})|_{\varepsilon=0}$$

is a **symplectic vector field on** Π

The **field** V_f is **Hamiltonian.**

$V_f := \frac{d}{d\varepsilon}(\Delta T_{\varepsilon,f})|_{\varepsilon=0}$ is a **Hamiltonian** **vector field** on Π.

$L \in \Pi$ a line.



L <––> (x,v), ||v||=1

Theorem (Ron Perline).

The **Hamiltonian function** of the vector field V_f is

$H_f(x,v) := -2\langle v, \vec{N}\rangle f(x)$

Goal. Find the **Lie algebra** generated by fields V_f, $f \in C^\infty(\gamma)$.

$<=>$ Find the **Lie algebra** generated by **functions** $H_f(x,v)$ under **Poisson bracket:** $\{F, G\} := \omega(\nabla_\omega F, \nabla_\omega G)$,

where $\nabla_\omega H$ is the Hamiltonian vector field for the function H.

Main result: density of Lie algebra generated by fields V_f

$$\{F, G\} := \omega(\nabla_\omega F, \nabla_\omega G), \qquad \nabla_\omega\{F, G\} = [\nabla_\omega f, \nabla_\omega g].$$

Hamiltonian function for V_f: $H_f(L) = H_f(x, v) = -2 < v, \vec{N}(x) > f(x)$.

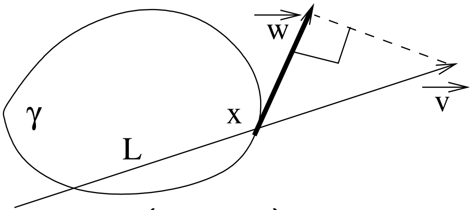
Main Theorem (A.G.) Consider the **Lie algebra** $\mathfrak{H} \subset C^\infty(\Pi)$ generated by all the functions $H_f(x, v)$ under the Poisson bracket $\{,\}$, $f \in C^\infty(\gamma)$.

The algebra $\mathfrak{H}$ is C^∞-**dense** in the algebra of all the C^∞-functions on Π.

Main Theorem $=>$ for every $\alpha > 0$, $k \in \mathbb{N}$ the **closure of pseudogroup** gener. by **reflections** from hypersurf. (α, k)-close to γ **contains** the **pseudogroup of Π-Hamiltonian symplectomorphisms** between domains in Π.

We **find the algebra $\mathfrak{H}$ explicitly** and prove its density.

Unit ball bundle model: $\Pi \simeq T_{<1}\gamma = \{(x, w) \in T\gamma \mid ||w|| < 1\}$.



Theorem (Melrose). This is a **symplectomorphism** $\Pi \to T_{<1}\gamma$ with respect to the standard symplectic forms on Π and on $T\gamma$.

The Hamiltonian function $H_f(L)$ in the model $\Pi \simeq T_{<1}\gamma$:

$$H_f(x, w) = -2 < v, \vec{N}(x) > f(x) = -2\sqrt{1 - ||w||^2} f(x).$$

Theorem. Let γ be an **arbitrary Riemannian manifold.** Then the **Lie algebra** $\mathfrak{H}$ generated by the functions $H_f(x, w)$ with respect to $\{,\}$ is C^∞**-dense** in the space of C^∞-smooth functions on $T_{<1}\gamma$.

Lie algebra $\mathfrak{H}$ generated by functions $H_f(x,w) = -2\sqrt{1-||w||^2}f(x)$

$$T_{<1}\gamma \simeq T\gamma : \quad (x,w) \in T_{<1}\gamma, \quad w \mapsto y := \frac{w}{\sqrt{1-||w||^2}} \in T_x\gamma.$$

This is **not** a symplectomorphism.

We describe $\mathfrak{H}$ in coordinates (x,y), in which $H_f = -2\frac{f(x)}{\sqrt{1+||y||^2}}$.

$S^k(T^*\gamma) := \{$functions on $T\gamma$ whose **restrictions** to the **fibers** $T_x\gamma$ are degree k **homogeneous polynomials** in $y\}$; $S^0(T^*\gamma) = C^\infty(\gamma)$.

$$\Lambda_k := \frac{1}{\sqrt{1+||y||^2}} S^k(T^*\gamma) \subset C^\infty(T\gamma).$$

We deal with functions in Λ_k as functions on $T_{<1}\gamma \simeq T\gamma$.

The **vector space** Λ_0 is spanned by functions H_f.

Theorem. Let γ be a Riemannian manifold not diffeomorphic to a circle. Then $\mathfrak{H} = \oplus_{k=0}^{\infty}\Lambda_k$.

Special case: $\gamma = S^1$, s – natural parameter, $|S^1| = 2\pi$;
$T_{<1}S^1 = S^1_s \times (-1,1)_w$, $\omega = dw \wedge ds$, $y = \frac{w}{\sqrt{1-|w|^2}}$.

$\Lambda_k =$ **vector space** of functions $H_{k,f}(s,y) := \dfrac{y^k}{\sqrt{1+y^2}} f(s)$, $f \in C^\infty(S^1)$.

$\mathfrak{H} =$ the **Lie algebra** in $C^\infty(T_{<1}S^1) \simeq C^\infty(S^1_s \times \mathbb{R}_y)$ generated by Λ_0.

$$\Lambda_{k,0} := \{H_{k,f} \in \Lambda_k \mid \int_0^{2\pi} f(s)ds = 0\}.$$

$$\mathfrak{G}_{glob,0} := \Lambda_1 \oplus (\oplus_{k \in 2\mathbb{Z}_{\geq 0}} \Lambda_k) \oplus (\oplus_{k \in 2\mathbb{Z}_{\geq 1}+1} \Lambda_{k,0}).$$

$\text{Odd}(y) :=$ the space of odd polynomials in y.

Theorem. One has $\mathfrak{H} = \mathfrak{G}_{glob,0} \oplus \Psi$ (the vector space sum) for the vector subspace $\Psi \subset \frac{1}{\sqrt{1+y^2}} \text{Odd}(y)$ defined below. Namely,
for every $P(y) \in \text{Odd}(y)$ set $\widetilde{P}(x) := x^{-\frac{1}{2}} P(x^{\frac{1}{2}})$. One has

$$\Psi := \left\{ \frac{P(y)}{\sqrt{1+y^2}} \mid P(y) \in \text{Odd}(y),\ P'(0) = 0,\ \widetilde{P}'(-1) = 0 \right\}.$$

Density of the Lie algebra thus obtained

Main result

$=>$ In all the cases, except for $\gamma = S^1$ one has $\mathfrak{H} = \oplus_{k\geq 0}\Lambda_k$.

$=>$ And in the case, when $\gamma = S^1$ this is **almost true:** description of $\mathfrak{H}$.

Proposition. The Lie algebra $\oplus_{k\geq 0}\Lambda_k$ is **dense** in $C^\infty(T_{<1}\gamma)$.
The same holds for the algebra $\mathfrak{H}$ in the case, when $\gamma = S^1$.

Proof. Deduced from **Weierstrass** Theorem on C^∞-density of polynomials.

Sketch of proof of main results: description of Lie algebra $\mathfrak{H}$.

The functions H_f form the vector space Λ_0.
Goal: show that **the Lie algebra** $\mathfrak{H}$ generated by Λ_0 is $\oplus_{k=0}^{+\infty}\Lambda_k$.

$$\{\Lambda_0, \Lambda_0\} = \Lambda_1, \quad \{\Lambda_1, \Lambda_1\} = \Lambda_1 \quad => \quad \Lambda_1 \text{ is a Lie algebra.}$$

$$\{\Lambda_d, \Lambda_k\} \subset \Lambda_{d+k-1} \oplus \Lambda_{d+k+1}.$$

$$\pi_d : \oplus_k \Lambda_k \to \Lambda_d$$

$$\pi_{k+1}(\{\Lambda_0, \Lambda_k\}) = \Lambda_{k+1} \text{ whenever } k \neq 2. \qquad (1)$$

If (1) were true for all k, then the proof would have been done.
But in general, (1) does not hold for $k = 2$:
- for $\gamma = S^1$;
- in higher dimensions.

For $\gamma = S^1 = \mathbb{R}_s/2\pi\mathbb{Z}$, metric ds^2, one has

$\Lambda_k =$ **vector space** of functions $H_{k,f}(s,y) := \frac{y^k}{\sqrt{1+y^2}} f(s),\ f \in C^\infty(S^1)$.

$$\{H_{d,f}, H_{k,g}\} = H_{d+k-1,h_{d+k-1}} + H_{d+k+1,h_{d+k+1}} \tag{2}$$

$$h_{d+k-1} = (d+k)fg' - k(fg)',\ h_{d+k+1} = (d+k-2)fg' - (k-1)(fg)'. \tag{3}$$

Claim 1. $\pi_3(\{\Lambda_0, \Lambda_2\}) = \Lambda_{3,0} = \{\frac{h(x)y^3}{\sqrt{1+y^2}} \mid \int_0^{2\pi} h(x)dx = 0\}$.

Proof. (2), (3) => $\{H_{0,f}, H_{2,g}\} = H_{1,2f'g} + H_{3,-(fg)'}$, and $\int_0^{2\pi}(fg)'ds = 0$.

$f \in C^\infty(S^1) \mapsto$ **average:** $\widehat{f} := \frac{1}{2\pi}\int_0^{2\pi} f(x)dx \in \mathbb{R}$.

Claim 2. $(d+k-2)\widehat{h}_{d+k-1} = (d+k)\widehat{h}_{d+k+1}$. **Proof.** See (3).

$$\{H_{d,f}, H_{k,g}\} = H_{d+k-1,h_{d+k-1}} + H_{d+k+1,h_{d+k+1}}$$

Claim 2: $(d+k-2)\widehat{h}_{d+k-1} = (d+k)\widehat{h}_{d+k+1}.$

$$\Lambda_{k,0} := \{H_{k,f} \in \Lambda_k \mid \widehat{f} = 0\}.$$

Claim 3: $\pi_{k+1}\{\Lambda_0, \Lambda_k\} = \pi_{k+1}\{\Lambda_0, \Lambda_{k,0}\} = \Lambda_{k+1}$ for $k \neq 2$.

Claims 2, 3 $\Rightarrow \mathfrak{H} = \Lambda_1 \oplus (\oplus_{k\in 2\mathbb{Z}_{\geq 0}}\Lambda_k) \oplus (\oplus_{k\in 2\mathbb{Z}_{\geq 1}+1}\Lambda_{k,0}) \oplus \Psi,$

$$\Psi = \text{Span}\left\{ \frac{R_j(y)}{\sqrt{1+y^2}} \mid R_j(y) := jy^{2j-1} + (j-1)y^{2j+1}, \ \ j \geq 2 \right\}.$$

For $P(y) = \sum_{i=1}^{m} a_i y^{2i+1}$, $\widetilde{P}(x) := x^{-\frac{1}{2}} P(x^{\frac{1}{2}})$; $\widetilde{R}_j(x) = jx^{j-1} + (j-1)x^j$.

Miracle 1. $\widetilde{R}_j'(-1) = 0.$ One has $\frac{P(y)}{\sqrt{1+y^2}} \in \Psi \iff \widetilde{P}'(-1) = 0.$

Implies the main result for $\gamma = S^1$.

Case of higher dimensions

$$\{\Lambda_d, \Lambda_k\} \subset \Lambda_{d+k-1} \oplus \Lambda_{d+k+1},$$

$$\pi_{k+1}\{\Lambda_0, \Lambda_k\} = \Lambda_{k+1} \quad \text{for } k \neq 2. \tag{4}$$

This is **not true** for $k = 2$: $\pi_3\{\Lambda_0, \Lambda_2\} \subset \Lambda_3, \neq \Lambda_3$.

$$G^{\pm} : \Lambda_0 \otimes_{\mathbb{R}} \Lambda_4 \to \Lambda_{4\pm1} : \ G^{+}(F\otimes H) := \pi_5\{F, H\}, \ G^{-}(F\otimes H) := \pi_3\{F, H\}.$$

Miracle 2. If $dim\gamma \geq 2 => G^{-}(\ker G^{+}) = \Lambda_3 => \Lambda_3 \subset \{\Lambda_0, \Lambda_4\}$.

(4) + **Miracle 2** $=> \mathfrak{H} = \oplus_{k\geq0}\Lambda_k$. **Proves main result.**

We have proved that:

each Π-Hamiltonian symplectomorphism is a C^∞-limit of **compositions of reflections and their inverses.**

Reflections are done from hypersurfaces close to γ.

Open question. What about the case of **pseudo-semigroup:** compositions of **just reflections, without inverses?**

Related to **Ivrii's Conjecture.**

Thank you for your attention!