Dynamics of a Circular Cylinder and Two Point Vortices in a Perfect Fluid

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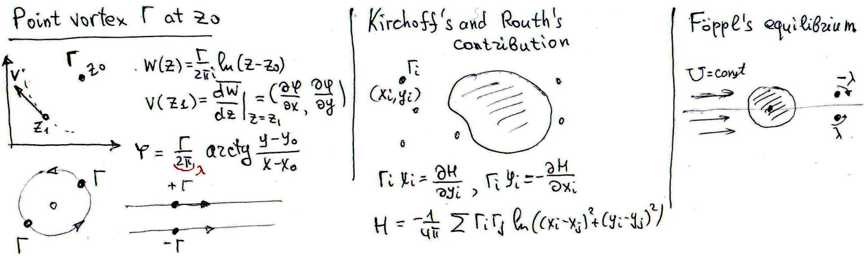
"DYNAMICS IN SIBERIA" 2021

3, March, 2021, Sobolev Institute of Mathematics,
4 Acad. Koptyug Ave, Novosibirsk

1. Point vortices interacting with rigid bodies (historical highlights).

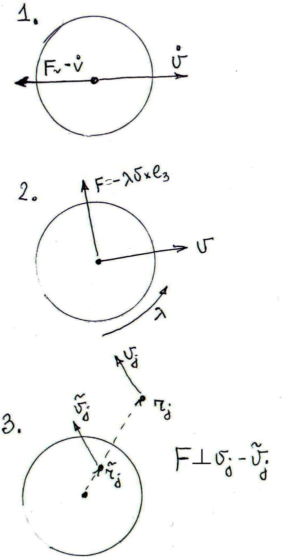
- General equations and conservation laws (Helmholtz, Kelvin)
- These general equations are Hamiltonian (Kirchhoff) and even in the presence of interfaces (Lin)
- Vortices on surfaces (Gromeka, Zermelo, Bogomolov, Boatto, Dritschel, Koiller)
- The motion and stability of point vortices exterior to and within a circular domain (Greenhill, Havelock, von Karman, Kurakin).
- Vortices and bodies *interact dynamically* (Koiller, Ramodanov, Shashikanth, Borisov, Mamaev, Sokolov)



2. Forces on a cylinder in a perfect fluid.

1. Force due to the added-mass effect
   $\boldsymbol{F} = \pi R^2 \varrho \dot{\boldsymbol{v}}$ or $\boldsymbol{F} \sim \dot{\boldsymbol{v}}$.
2. Joukowski's lifting force
   $\boldsymbol{F} = -\varrho\Gamma \boldsymbol{v} \times \boldsymbol{e}_3 = -\lambda \boldsymbol{v} \times \boldsymbol{e}_3$ (the vector $\boldsymbol{e}_3$ is orthogonal to the flow)
3. Force exerted by vortex of intensity λ_j at position $\boldsymbol{r}_j$. This force is proportional to the difference of the two velocities: of the vortex itself and of its inverse image

$$\boldsymbol{F} = i\lambda_j \left(\dot{\boldsymbol{r}}_j - \dot{\tilde{\boldsymbol{r}}}_j \right)$$



Integrable Case of the Rigid Body and Single Vortex. Governing Equations

$$\dot{\boldsymbol{r}}_1 = -\boldsymbol{v} + \operatorname{grad} \widetilde{\varphi}(\boldsymbol{r})\Big|_{\boldsymbol{r}=\boldsymbol{r}_1}, \quad \dot{\boldsymbol{r}}_c = \boldsymbol{v},$$

$$a\dot{v}_1 = \lambda v_2 - \lambda_1(\dot{\widetilde{y}}_1 - \dot{y}_1), \quad a\dot{v}_2 = -\lambda v_1 + \lambda_1(\dot{\widetilde{x}}_1 - \dot{x}_1),$$

here $\boldsymbol{r}_c = (x_c, y_c)$ — coordinates of the center, $\boldsymbol{v} = (v_1, v_2)$ — velicity, $\boldsymbol{r}_1 = (x_1, y_1)$ — vortex coordinates, $\widetilde{\boldsymbol{r}}_1 = (\widetilde{x}_1, \widetilde{y}_1) = \frac{R^2 \boldsymbol{r}_1}{r_1^2}$ — inverse image of the vortex , R — cylinder radius, a — mass of the cylinder, $\lambda = \frac{\Gamma}{2\pi}$, $\lambda_1 = \frac{\Gamma_1}{2\pi}$ — circulation and vortex intensity, $\widetilde{\varphi}(\boldsymbol{r})$ — perfect liquid potential $\varphi(\boldsymbol{r})$ regularized in $\boldsymbol{r} = \boldsymbol{r}_1$

$$\varphi(\boldsymbol{r}) = -\frac{R^2}{r^2}(\boldsymbol{r}, \boldsymbol{v}) - \lambda \operatorname{arctg} \frac{y}{x} + \lambda_1 \left(\operatorname{arctg}\left(\frac{y - \widetilde{y}_1}{x - \widetilde{x}_1}\right) - \operatorname{arctg}\left(\frac{y - y_1}{x - x_1}\right)\right).$$

Integrable Case Rigid Body and Single Vortex. Governing Equations

$$\dot{\zeta}_i = \{\zeta_i, H\} = \sum_k \{\zeta_i, \zeta_k\} \frac{\partial H}{\partial \zeta_k}, \tag{1}$$

here ζ_i — phase vector $\zeta = \{x_1, y_1, v_1, v_2, x_c, y_c\}$, Hamiltonian

$$H = \frac{1}{2} a v^2 + \frac{1}{2} \lambda_1^2 \ln(r_1^2 - R^2) - \frac{1}{2} \lambda_1 \lambda \ln r_1^2,$$

Phase space of the system

$$\mathcal{M} = \mathbb{R}^6 \setminus \mathcal{B}, \qquad \mathcal{B} = \{(x_1, y_1) : x_1^2 + y_1^2 \leq R^2\}.$$

Poisson structure

$$\{v_1, x_1\} = \frac{1}{a}\frac{r_1^4 - R^2(x_1^2 - y_1^2)}{r_1^4}, \ \{v_1, y_1\} = \{v_2, x_1\} = -\frac{1}{a}\frac{2R^2 x_1 y_1}{r_1^4},$$
$$\{v_2, y_1\} = \frac{1}{a}\frac{r_1^4 + R^2(x_1^2 - y_1^2)}{r_1^4}, \ \{v_1, v_2\} = \frac{\lambda}{a^2} - \frac{\lambda_1}{a^2}\frac{r_1^4 - R^4}{r_1^4},$$
$$\{x_1, y_1\} = -\frac{1}{\lambda_1}, \ \{x_c, v_1\} = \{y_c, v_2\} = \frac{1}{a}.$$

Poisson bracket is nondegenerate

First Integrals

$$Q = av_2 + \lambda x_c - \lambda_1 x_1 \left(\frac{R^2}{\boldsymbol{r}_1^2} - 1 \right), \quad P = av_1 - \lambda y_c + \lambda_1 y_1 \left(\frac{R^2}{\boldsymbol{r}_1^2} - 1 \right),$$

$$K = a(v_1 y_c - v_2 x_c) - \frac{1}{2}\lambda \boldsymbol{r}_c^2 - \frac{1}{2}\lambda_1 \boldsymbol{r}_1^2 + \frac{1}{2}\left(\frac{R^2}{\boldsymbol{r}_1^2} - 1 \right)(\boldsymbol{r}_1, \boldsymbol{r}_c)$$

$$\{Q, P\} = \lambda, \quad \{K, Q\} = P, \quad \{K, P\} = -Q.$$

$$F = a\boldsymbol{v}^2 + \lambda_1 \left[2a \left(1 - \frac{R^2}{\boldsymbol{r}_1^2} \right)(x_1 v_2 - y_1 v_1) + (\lambda_1 - \lambda)\boldsymbol{r}_1^2 + \lambda_1 \frac{R^4}{\boldsymbol{r}_1^2} \right],$$

$$F = 2\lambda K + P^2 + Q^2 + 2R^2\lambda_1^2,$$

Reduced system

$$
\begin{aligned}
\dot{\boldsymbol{r}}_1 &= -\boldsymbol{v} + \operatorname{grad} \widetilde{\varphi}(\boldsymbol{r})\Big|_{\boldsymbol{r}=\boldsymbol{r}_1}, \\
a\dot{v}_1 &= \lambda v_2 - \lambda_1(\dot{\widetilde{y}}_1 - \dot{y}_1), \\
a\dot{v}_2 &= -\lambda v_1 + \lambda_1(\dot{\widetilde{x}}_1 - \dot{x}_1),
\end{aligned}
\tag{2}
$$

Reduced system

$$\dot{\zeta}_i = \{\zeta_i,\, H\} = \sum_k \{\zeta_i,\, \zeta_k\} \frac{\partial H}{\partial \zeta_k}, \tag{3}$$

where $\zeta = \{x_1,\, y_1,\, v_1,\, v_2\}$,

$$H = \frac{1}{2} a v^2 + \frac{1}{2} \lambda_1^2 \ln(r_1^2 - R^2) - \frac{1}{2} \lambda_1 \lambda \ln r_1^2, \tag{4}$$

Phase space

$$\mathcal{M}_1 = \mathbb{R}^4 \setminus \mathcal{B}, \qquad \mathcal{B} = \{(x_1, y_1) : x_1^2 + y_1^2 \leq R^2\}.$$

Integrability

System (3) has additional first integral

$$F = a\boldsymbol{v}^2 + \lambda_1 \left[2a \left(1 - \frac{R^2}{\boldsymbol{r}_1^2} \right) (x_1 v_2 - y_1 v_1) + (\lambda_1 - \lambda)\boldsymbol{r}_1^2 + \lambda_1 \frac{R^4}{\boldsymbol{r}_1^2} \right],$$

which is commute with hamiltonian (4) with respect to (4). Using Arnold–Liouville's theorem, one can assert that the compact connected component of the integral manifold $\mathcal{P} = \{H = h,\ F = f\}$ is diffeomorphic to a two-dimensional torus.

Invariant Manifold

Invariant relations

$$F_1 = 0, \quad F_2 = 0, \tag{5}$$

$$F_1 = x_1 v_1 + y_1 v_2,$$

$$F_2 = a\left[(x_1^2 + y_1^2)^2 - R^4\right](x_1 v_2 - y_1 v_1)^2 + \{(\lambda_1 - \lambda)\times$$

$$\times\left[(x_1^2 + y_1^2)^3 - (a + R^2)(x_1^2 + y_1^2)^2\right] - R^2(a\lambda + \lambda_1 R^2)(x_1^2 + y_1^2) + \lambda_1 R^6\}\times$$

$$\times(x_1 v_2 - y_1 v_1) - \lambda_1(x_1^2 + y_1^2)(x_1^2 + y_1^2 - R^2)\left[\lambda R^2 + (\lambda_1 - \lambda)(x_1^2 + y_1^2)\right].$$

Theorem

The critical set $\mathcal{C}$ of the momentum map $\mathcal{F}$ exhausted by solutions of (5)*.*
The set $\mathcal{N}$ is 2D invariant submanifold of initial hamiltonian system.

Common Level of the First Integrals

We introduce the following variables, which are integrals of the symmetry field generated by the integral K

$$p_1 = a(x_1 v_1 + y_1 v_2), \quad p_2 = a(x_1 v_2 - y_1 v_1), \quad r = x_1^2 + y_1^2.$$

In the new variables, the integrals H and F take, respectively, the form

$$\begin{aligned} H &= \frac{p_1^2 + p_2^2}{2ar} + \frac{1}{2}\lambda_1[\lambda_1 \ln(r - R^2) - \lambda \ln r], \\ F &= \frac{p_1^2 + p_2^2}{r} + 2\lambda_1 \left(1 - \frac{R^2}{r}\right) p_2 + \lambda_1^2 \left(r + \frac{R^4}{r}\right) - \lambda\lambda_1 r. \end{aligned} \tag{6}$$

The form of the expressions eqref e13 allows us to conclude that in the case $\lambda \cdot \lambda_1 < 0$ the joint surface of the integrals H and F is compact (diffeomorphic to the sphere), in the case $\lambda \cdot \lambda_1 > 0$ — is non-compact (diffeomorphic to a two-sheeted hyperboloid).

Bifurcation Diagram

Theorem

The bifurcation diagram Σ of the momentum map $\mathcal{F}$ consists of curves

$$\Pi_{1,2}: \begin{cases} h = \dfrac{z^2}{2ar} + \dfrac{1}{2}\lambda_1[\lambda_1 \ln(r - R^2) - \lambda \ln r], \\ f = \dfrac{z^2}{r} + 2\lambda_1 \left(1 - \dfrac{R^2}{r}\right) z + \lambda_1^2 \left(r + \dfrac{R^4}{r}\right) - \lambda\lambda_1 r, \end{cases} \quad r \in (R^2; +\infty).$$

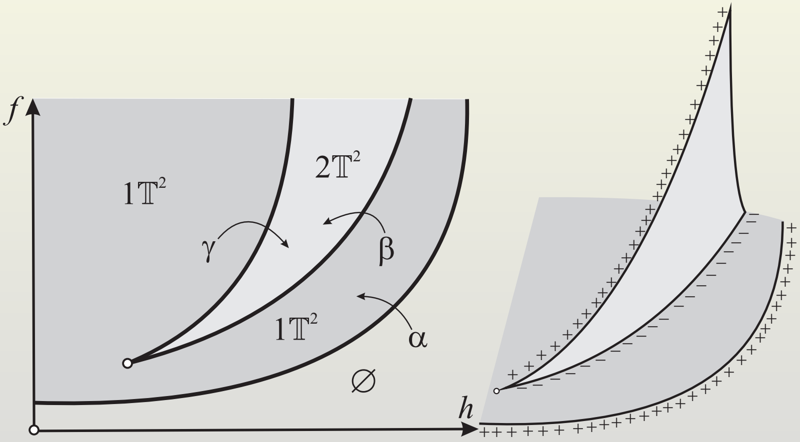
here $z = z_{1,2}(r)$ is a real solution of the equation

$$(r^2 - R^4)z^2 + \{R^4(R^2 - r)\lambda_1 - r[\lambda_1 r - \lambda(r - a)]R^2 + r^2(\lambda - \lambda_1)(a - r)\}z+$$

$$+ar\lambda_1(R^2 - r)[\lambda_1 r + \lambda(R^2 - r)] = 0.$$

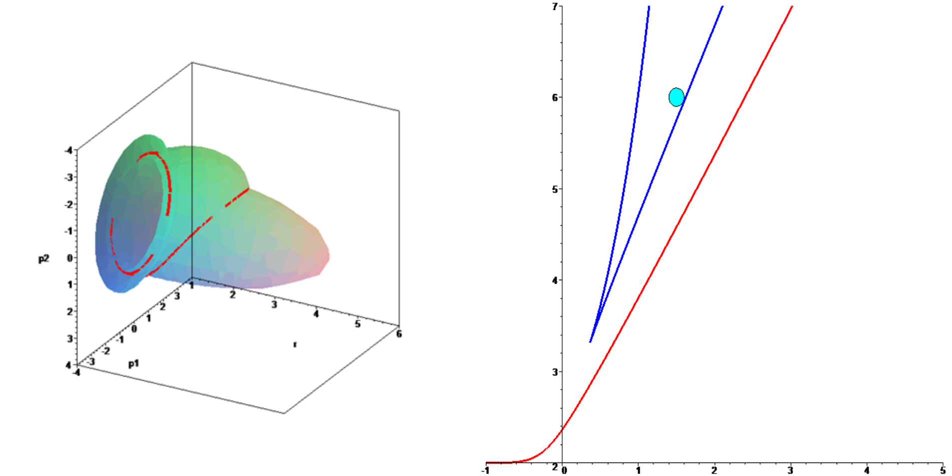
Bifurcation Diagram

Bifurcation diagram and bifurcation complex in a case of compact symplectic leaf



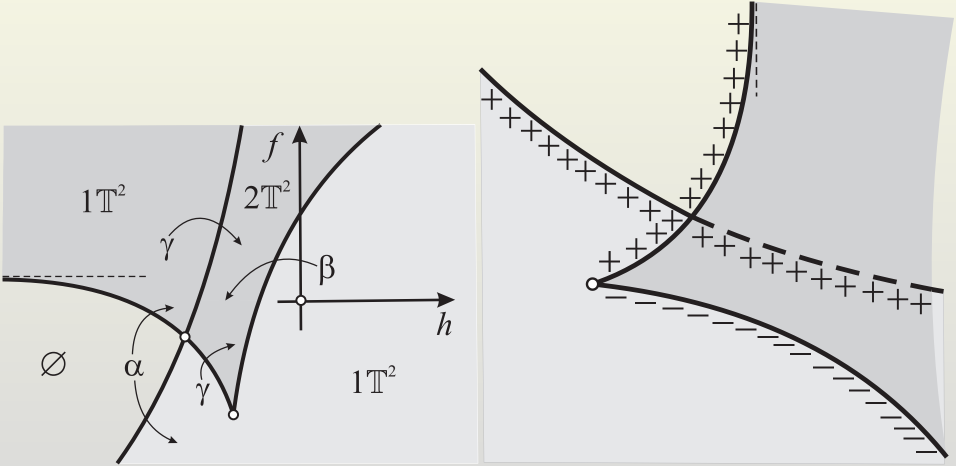
Bifurcation Diagram

Bifurcation diagram and critical set in a case of compact symplectic leaf



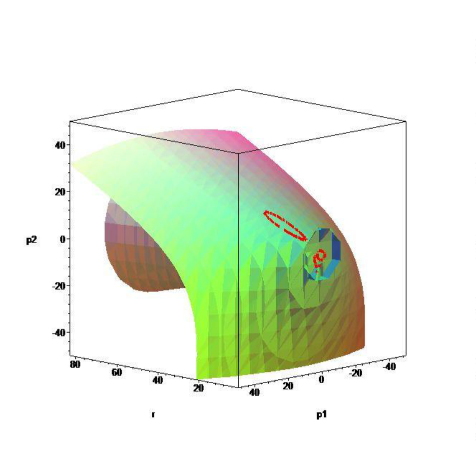
Bifurcation Diagram

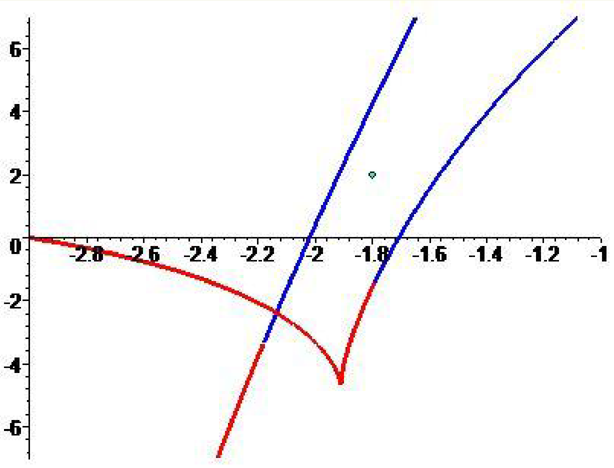
Bifurcation diagram and bifurcation complex in a case of non-compact symplectic leaf



Bifurcation Diagram

Bifurcation diagram and critical set in a case of non-compact symplectic leaf



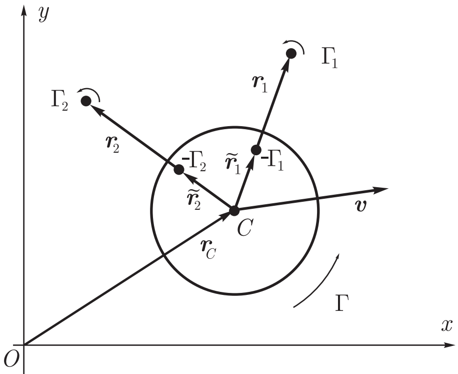


Governing equations

$$
\begin{cases}
\dot{\boldsymbol{r}}_c = \boldsymbol{v} = (v_1,\ v_2), \quad \dot{\boldsymbol{r}}_j = -\boldsymbol{v} + \operatorname{grad} \widetilde{\varphi}_j(\boldsymbol{r}),\ j = 1, 2 \\
a\dot{v}_1 = -\lambda v_2 + \lambda_1(\dot{\widetilde{y}}_1 - \dot{y}_1) + \lambda_2(\dot{\widetilde{y}}_2 - \dot{y}_2), \\
a\dot{v}_2 = \lambda v_1 - \lambda_1(\dot{\widetilde{x}}_1 - \dot{x}_1) - \lambda_2(\dot{\widetilde{x}}_2 - \dot{x}_2),
\end{cases}
\quad \Leftrightarrow \dot{\boldsymbol{\zeta}} = \boldsymbol{f}(\boldsymbol{\zeta}), \quad \boldsymbol{\zeta} = (x_c,\ y_c,\ v_1,\ v_2,\ x_1,\ y_1,\ x_2,\ y_2)
$$

$$
\varphi(\boldsymbol{r}) = -\frac{R^2}{r^2}(\boldsymbol{r},\ \boldsymbol{v}) - \lambda \operatorname{arctg} \frac{y}{x} + \lambda_1 \left(\operatorname{arctg} \left(\frac{y - \widetilde{y}_1}{x - \widetilde{x}_1} \right) - \operatorname{arctg} \left(\frac{y - y_1}{x - x_1} \right) \right) +
$$

$$
+\lambda_2 \left(\operatorname{arctg} \left(\frac{y - \widetilde{y}_2}{x - \widetilde{x}_2} \right) - \operatorname{arctg} \left(\frac{y - y_2}{x - x_2} \right) \right).
$$



The equations of motion $\dot{\boldsymbol{\zeta}} = \boldsymbol{f}(\boldsymbol{\zeta})$, $\boldsymbol{\zeta} = (x_c, y_c, v_1, v_2, x_1, y_1, x_2, y_2)$ admit the following integral (the system's energy)

$$H = \frac{1}{2a}\boldsymbol{v}^2 + \frac{1}{2}\sum_{j=1}^{2}\lambda_j^2 \ln(\boldsymbol{r}_j^2 - R^2) + \frac{1}{2}\lambda_1\lambda_2 \ln \frac{R^4 - 2R^2(\boldsymbol{r}_1, \boldsymbol{r}_2) + \boldsymbol{r}_1^2\boldsymbol{r}_2^2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2}.$$

and the equations can be represented in the Hamiltonian form

$$\dot{\zeta}_i = \{\zeta_i, H\} = \sum_k \{\zeta_i, \zeta_k\}\frac{\partial H}{\partial \zeta_k}$$

where the energy H serves as the Hamiltonian function. The tensor's non-zero components read

$$\{v_1, x_i\} = \frac{1}{a}\frac{r_i^4 - R^2(x_i^2 - y_i^2)}{r_i^4}, \quad \{v_1, y_i\} = -\frac{1}{a}\frac{2R^2 x_i y_i}{r_i^4},$$

$$\{v_2, x_i\} = -\frac{1}{a}\frac{2R^2 x_i y_i}{r_i^4}, \quad \{v_2, y_i\} = \frac{1}{a}\frac{r_i^4 + R^2(x_i^2 - y_i^2)}{r_i^4},$$

$$\{v_1, v_2\} = \frac{\lambda}{a^2} - \sum_i \frac{\lambda_i}{a^2}\frac{r_i^4 - R^4}{r_i^4}, \quad \{x_i, y_i\} = -\frac{1}{\lambda_i}, \quad \{x_c, v_1\} = \{y_c, v_2\} = a^{-1}.$$

This Lie-Poisson bracket satisfies the Jacobi identity.

3. Integrals of motion and reduction.

Besides the energy, the governing equations $\dot{\zeta} = \boldsymbol{f}(\zeta)$ allow two additional integrals due to translational symmetry (conservation of linear momenta)

$$Q = av_2 - \sum \lambda_i(\widetilde{x}_i - x_i), \quad P = av_1 + \sum \lambda_i(\widetilde{y}_i - y_i)$$

and one due to the rotational symmetry

$$I = a(v_1 y_c - v_2 x_c) - \frac{1}{2}\sum_{j=1}^{2} \lambda_j \boldsymbol{r}_j^2 + \frac{1}{2}\sum_{j=1}^{2} \lambda_j \left(\frac{R^2}{\boldsymbol{r}_j^2} - 1\right)(\boldsymbol{r}_j, \boldsymbol{r}_c).$$

The Lie-Poisson brackets of Q, P and I read

$$\{Q, P\} = \lambda, \quad \{I, Q\} = P, \quad \{I, P\} = -Q.$$

Therefore, if $\lambda = 0$ then on the common level $P = Q = 0$ the system's order can be reduced by three units and *thus (on this level) the system of a cylinder and two vortices is Liouville integrable*.

Using the momenta we get rid of the cylinder's velocities v_1, v_2 and obtain four ODEs and two conservation laws (energy and angular momentum)

$$\dot{x}_1 = -\frac{\lambda_1\left(\frac{R^2 y_1}{x_1^2+y_1^2} - y_1\right) + \lambda_2\left(\frac{R^2 y_2}{x_2^2+y_2^2} - y_2\right)}{a} \cdot \left(1 + \frac{R^2(y_1^2 - x_1^2 + 2x_1 y_1)}{(x_1^2+y_1^2)^2}\right) -$$

$$\frac{\lambda_1\left(\frac{R^2 y_1}{x_1^2+y_1^2} - y_1\right)}{\left(\frac{R^2 x_1}{x_1^2+y_1^2} - x_1\right)^2 + \left(\frac{R^2 y_1}{x_1^2+y_1^2} - y_1\right)^2} +$$

$$\lambda_2\left(\frac{y_2 - y_1}{(x_1-x_2)^2 + (y_1-y_2)^2} + \frac{y_1 - \frac{R^2 y_2}{x_2^2+y_2^2}}{\left(\frac{R^2 x_2}{x_2^2+y_2^2} - x_1\right)^2 + \left(\frac{R^2 y_2}{x_2^2+y_2^2} - y_1\right)^2}\right)$$

$$\dot{y}_1 = \dots, \quad \dot{x}_2 = \dots, \quad \dot{y}_2 = \dots$$

Denote it by $\dot{\boldsymbol{x}} = \boldsymbol{g}(\boldsymbol{x})$ where $\boldsymbol{x} = (x_1, y_1, x_2, y_2)$.

$$H = \frac{1}{2a}\left(\sum_{j=1}^{2} \lambda_j(\widetilde{x}_j - x_j)\right)^2 + \frac{1}{2a}\left(\sum_{j=1}^{2} \lambda_i(\widetilde{y}_j - y_j)\right)^2$$

$$+\frac{1}{2}\sum_j \lambda_i^2 \ln(\boldsymbol{r}_i^2 - R^2) + \frac{1}{2}\lambda_1\lambda_2 \ln \frac{R^4 - 2R^2(\boldsymbol{r}_1, \boldsymbol{r}_2) + \boldsymbol{r}_1^2\boldsymbol{r}_2^2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^2}.$$

$$I = \sum_{j=1}^{2} \lambda_j(x_j^2 + y_j^2).$$

What can be done next?

The equations $\dot{x} = g(x)$ are Hamiltonian with the classical bracket $\{x_i, y_i\} = -\frac{1}{\lambda_i}$!!

To reduce the system's order it is customary to take the integrals of the symmetry field $v_I = \{I, \cdot\}$, as our new (attitude or relative) variables, that is,

$$p_1 = x_1^2 + y_1^2, \quad p_2 = x_2^2 + y_2^2, \quad p_3 = x_1 x_2 + y_1 y_2, \quad p_4 = x_1 y_2 - y_1 x_2.$$

Their non-zero brackets are

$$\{p_1, p_3\} = \frac{2p_4}{\lambda_1}, \quad \{p_1, p_4\} = -\frac{2p_3}{\lambda_1}, \quad \{p_2, p_3\} = -\frac{2p_4}{\lambda_2},$$

$$\{p_2, p_4\} = \frac{2p_3}{\lambda_2}, \quad \{p_3, p_4\} = \frac{p_1}{\lambda_2} - \frac{p_2}{\lambda_1}$$

The integrals H and I now read

$$H = \frac{1}{2a}\left(\sum_{j=1}^{2} \frac{\lambda_j^2 (R^2 - p_j)^2}{p_j} + \frac{\lambda_1 \lambda_2 (R^2 - p_1)(R^2 - p_2) p_3}{p_1 p_2}\right) +$$

$$+ \sum_{j=1}^{2} \frac{\lambda_j^2}{2} \ln\left(p_j - R^2\right) + \frac{\lambda_1 \lambda_2}{2} \ln \frac{R^4 - 2R^2 p_3 + p_1 p_2}{p_1 + p_2 - 2p_3}$$

$$I = \lambda_1 p_1 + \lambda_2 p_2.$$

4. Connection between the absolute and relative motion.

Given $p_i(t)$, to be able to find the absolute motion $x_i(t),\ y_i(t))$ one needs an additional quadrature

$$\dot{\alpha} = \frac{R^2 + p_1}{ap_1^2}\left(\lambda_1(R^2 - p_1) + \frac{\lambda_2(R^2 - p_2)p_3}{p_2}\right) + \frac{\lambda_1}{R^2 - p_1} + \frac{\lambda_2(R^2 - p_2)(R^2(p_1 - p_3) - p_1(p_3 - p_2))}{p_1(p_1 + p_2 - 2p_3)(R^4 - 2R^2 p_3 + p_1 p_2)}. \quad (1)$$

Therefore,

$$x_1 = \sqrt{p_1}\cos\alpha,\ y_1 = \sqrt{p_1}\sin\alpha,\ x_2 = \frac{p_3\cos\alpha - p_4\sin\alpha}{\sqrt{p_1}},\ y_2 = \frac{p_4\cos\alpha + p_3\sin\alpha}{\sqrt{p_1}}. \quad (2)$$

5. Stationary configurations.

The time evolution of p_i is governed by $\dot{p}_i = \{p_i,\ H\}$, namely:

$$\dot{p}_1 = 2\lambda_2 p_4(R^2 - p_1)(R^2 - p_2)\left(\frac{1}{ap_1p_2} + \frac{1}{(p_1 + p_2 - 2p_3)(R^4 - 2R^2 p_3 + p_1 p_2)}\right)$$

$$\dot{p}_2 = ..., \dot{p}_3 = ..., \dot{p}_4 = ...$$

The only option for p_i to be const is $p_4 = 0$. There no static or translational equilibrium configurations (no moving Foppl's equilibria). There are stationary rotations: the vortices and the cylinder's center are on the same line and go along concentric circles.

6. The case of $\lambda_1 > 0$, $\lambda_2 > 0$.

Instead of p_1, p_2, p_3, p_4 choose l, e_1, e_2, e_3 where

$$e_1 = \frac{\lambda_1 p_1 - \lambda_2 p_2}{4}, \quad e_2 = \frac{\sqrt{\lambda_1 \lambda_2}}{2} p_3, \quad e_3 = \frac{\sqrt{\lambda_1 \lambda_2}}{2} p_4.$$

Their brackets are

$$\{e_1, e_2\} = e_3, \quad \{e_2, e_3\} = e_1, \quad \{e_3, e_1\} = e_2 \tag{3}$$

The leafs $p_3^2 + p_4^2 = p_1 p_2$, $l = c$ are compact, diffeomorhic to S^2 and are given by

$$l = c, \quad e_1^2 + e_2^2 + e_3^2 = G^2 = \frac{c^2}{16}. \tag{4}$$

In this case the introduction of canonical coordinates l, L is straightforward

$$e_1 = L, \quad e_2 = \sqrt{G^2 - L^2} \sin l, \quad e_3 = -\sqrt{G^2 - L^2} \cos l, \quad \{l, L\} = 1.$$

One-degree-of-freedom system

We have canonical variables l and L, the Hamiltonian function $H(L, l, c)$ of the form

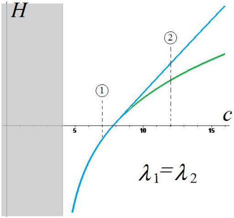
$$H(L,l,c) = \frac{1}{a}\left(\frac{\lambda_1\,\left(R^2\lambda_1 - c/2 - 2\,L\right)^2}{4\,L+c} + \frac{\lambda_2\,\left(R^2\lambda_2 - c/2 + 2\,L\right)^2}{4\,L-c}\right)$$
$$+\frac{\left(2\,R^2\lambda_1 - c - 4\,L\right)\left(2\,R^2\lambda_2 - c + 4\,L\right)\sqrt{\lambda_2\,\lambda_1}\sin l}{2a\sqrt{c^2-16\,L^2}}$$
$$+\frac{\lambda_1^2}{2}\ln\left(-2\,R^2\lambda_1 + c + 4\,L\right) + \frac{\lambda_2^2}{2}\ln\left(c - 4\,L - 2\,R^2\lambda_2\right)$$
$$+\frac{\lambda_1\lambda_2}{2}\ln\left(\frac{4\,R^4\lambda_1\,\lambda_2 - 4\,R^2\sqrt{\lambda_2\,\lambda_1}\sqrt{c^2-16\,L^2}\sin l - 16\,L^2 + c^2}{2\,(c+4\,L)\,\lambda_2 + 2\,\lambda_1\,(c-4\,L) - 4\,\sqrt{\lambda_2\,\lambda_1}\sqrt{c^2-16\,L^2}\sin l}\right)$$

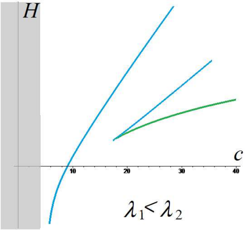
Now

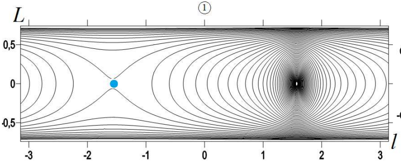
$$\dot{l} = \frac{\partial H(L,l,c)}{\partial L}, \quad \dot{L} = -\frac{\partial H(L,l,c)}{\partial l},$$

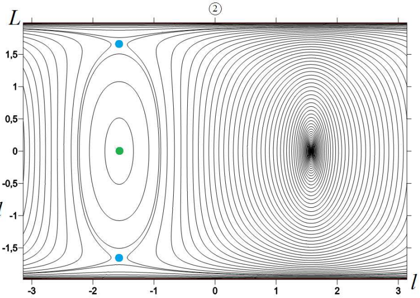
Bifurcation diagram.

The integrals $H = h$ и $I = c$ are sought to be dependent. For each c solve
$\frac{\partial H(L,l,c)}{\partial L} = 0, \quad \frac{\partial H(L,l,c)}{\partial l} = 0 \Rightarrow (L_0,\ l_0) \Rightarrow$ a point on the plane
$(c,\ H) = (c,\ H(L_0, l_0, c))$









I WILL STOP THERE AND

THANK YOU FOR LISTENING