# A model for the cubic connectedness locus

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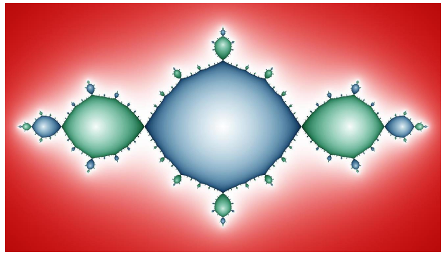
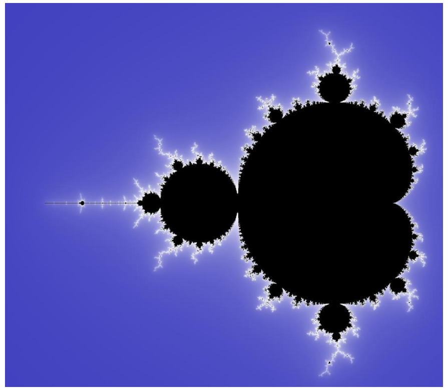
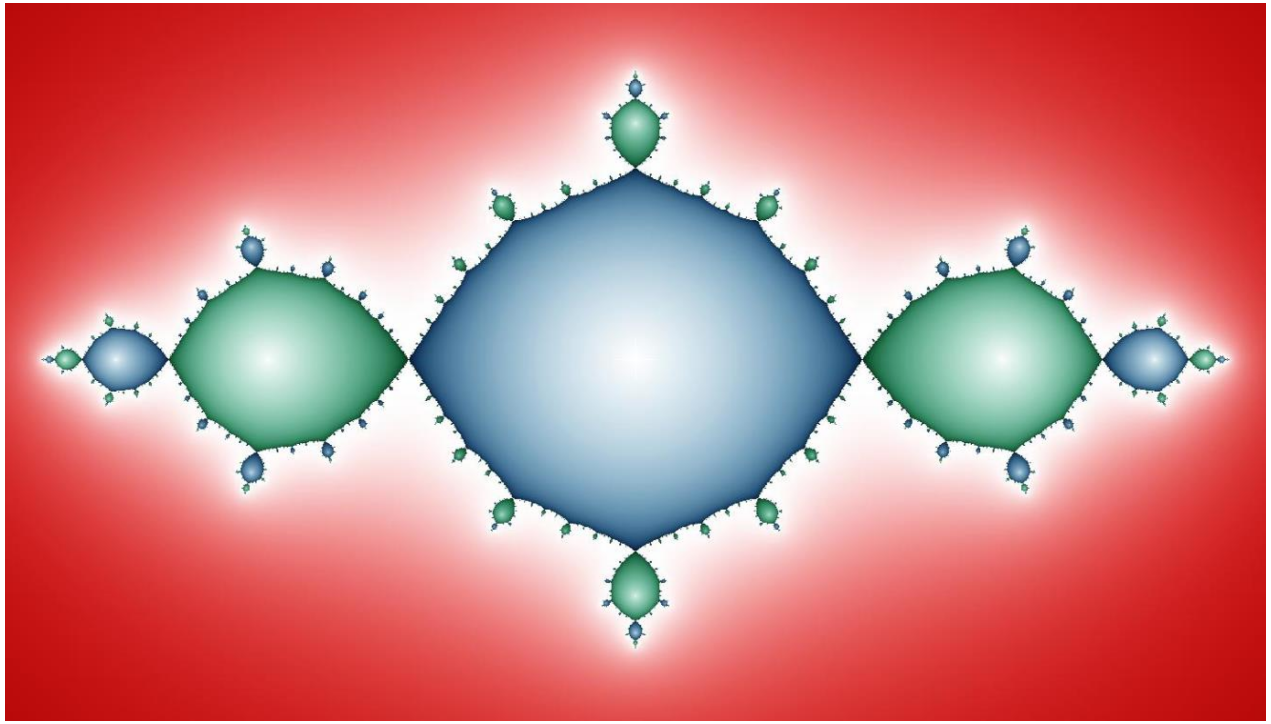
# Polynomial dynamics on $\mathbb{C}$

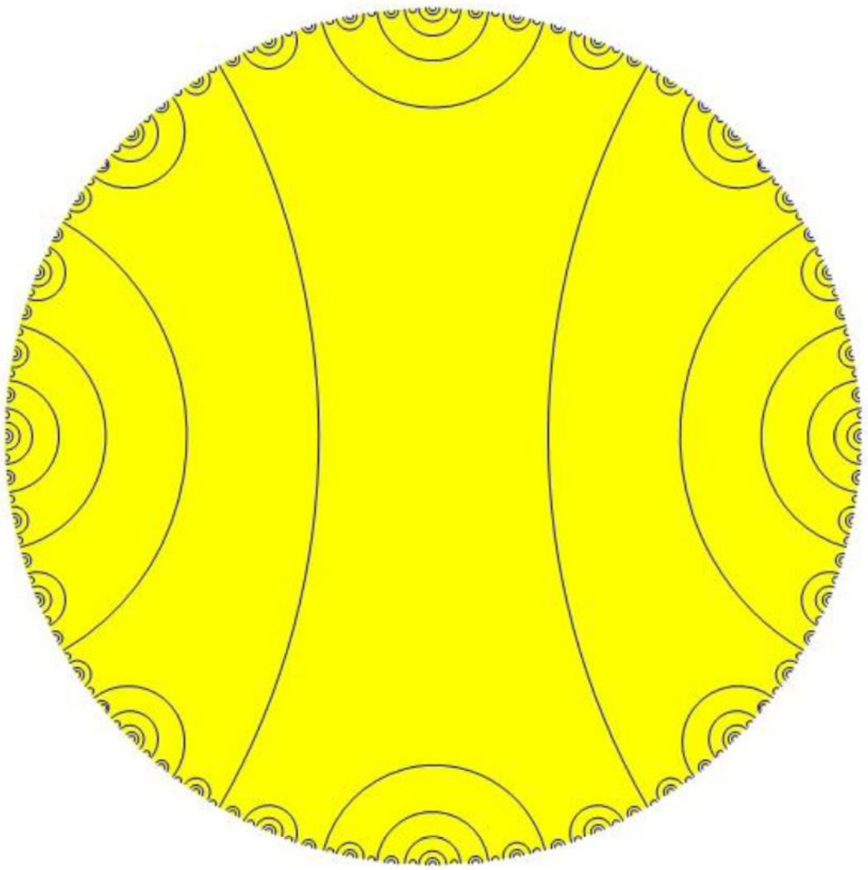
- Consider monic polynomials $P: \mathbb{C} \to \mathbb{C}$ whose filled Julia set $K(P) = \{z \in \mathbb{C} \mid P^n(z) \nrightarrow \infty\}$ is connected.
- The Julia set $J(P) = \partial K(P)$, the Fatou set $\mathbb{C}P^1 \setminus J(P)$.
- All such quadratic polynomials are conjugate to $Q_c(z) = z^2 + c$ with $c \in \mathcal{M}_2$ (quadratic Mandelbrot set).
- The cubic connectedness locus $\mathcal{M}_3$ consists of (affine conjugacy classes of) such cubic polynomials.
- $P$ is hyperbolic if all critical points are in (super)attracting basins.



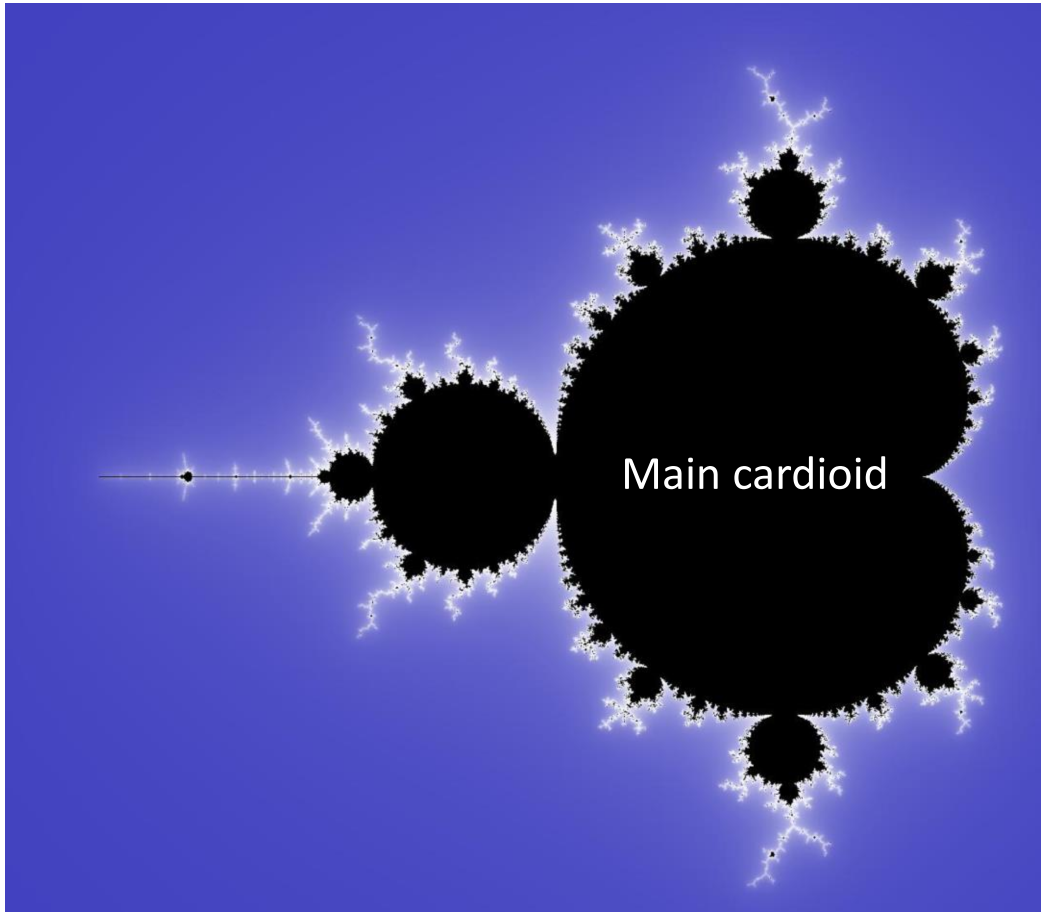


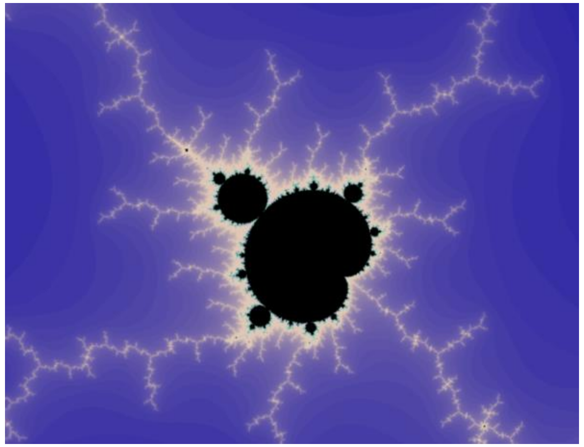
# Julia sets and Thurston invariant laminations

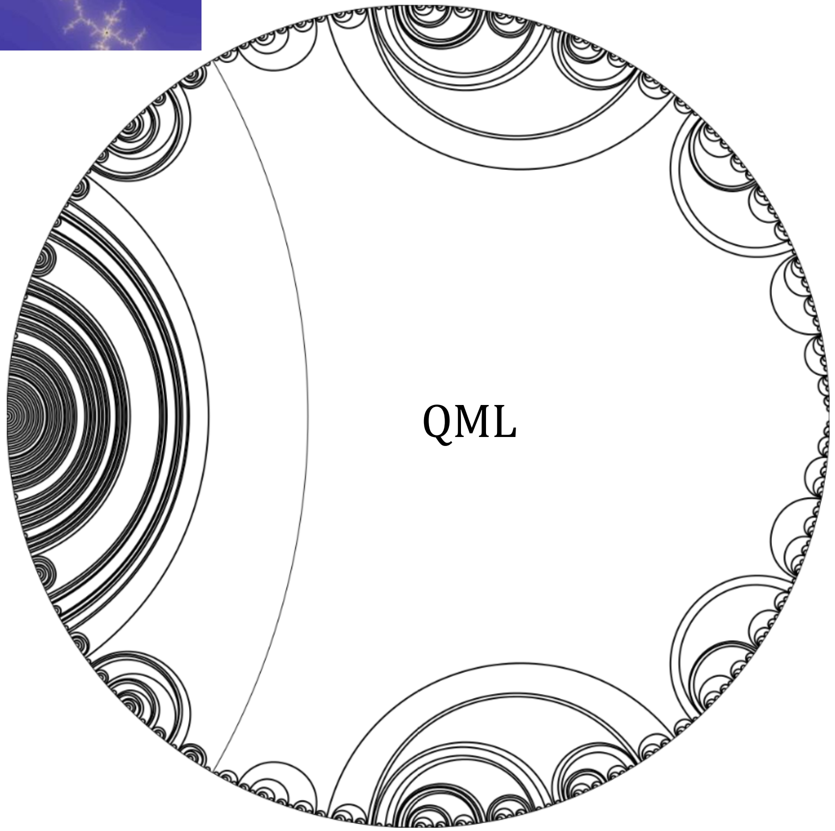




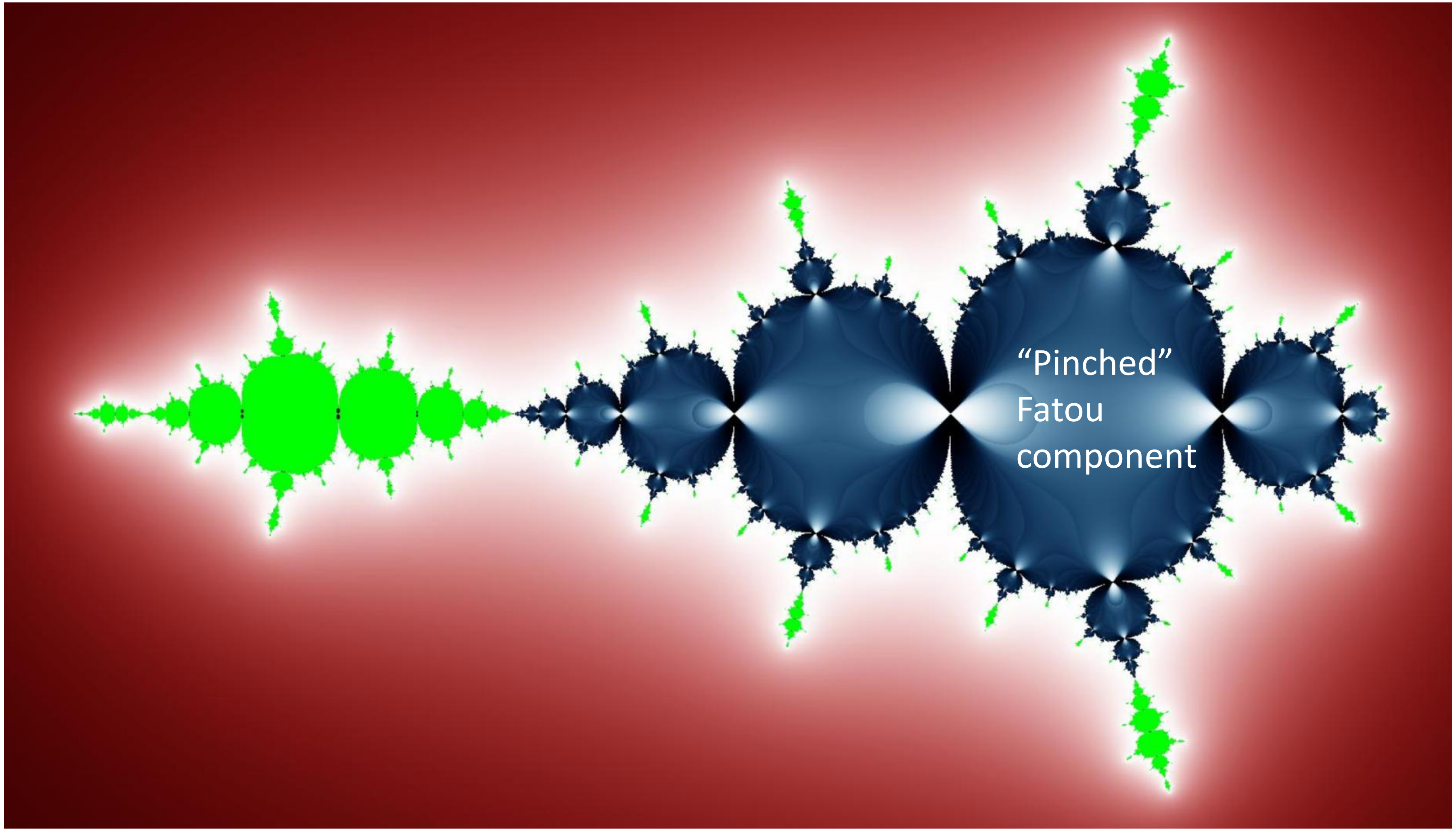
# The Mandelbrot set



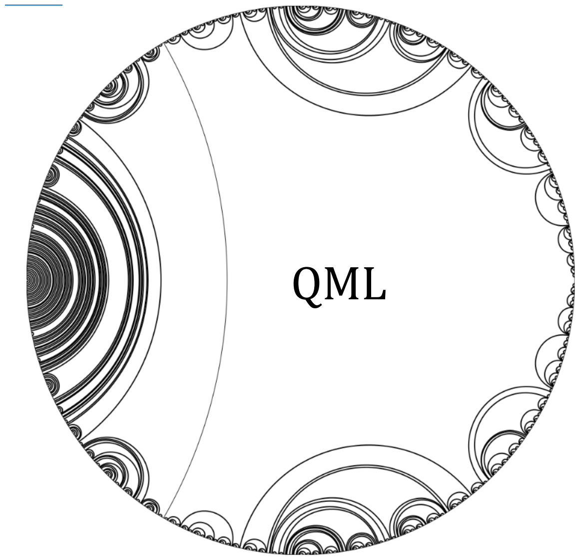


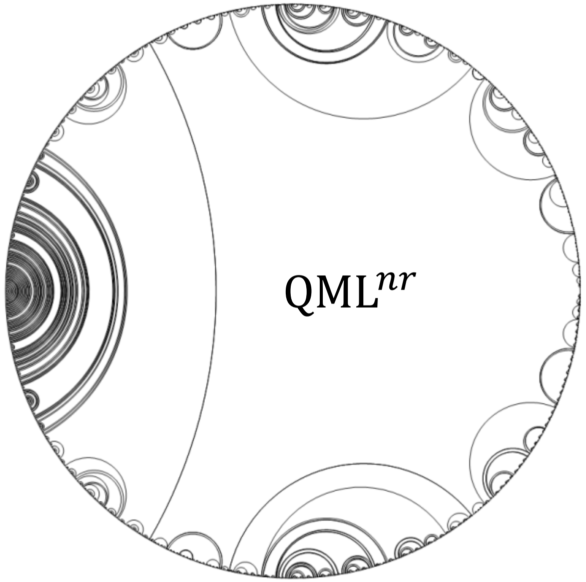
# Tuning and renormalization



# A rough model of $\mathcal{M}_2$

- A baby Mandelbrot set consists of all "tunings" of the same hyperbolic polynomial (not in the main cardioid).
- Collapse the (filled) main cardioid and all baby Mandelbrot sets.
- The result is a dendritic model for $\mathcal{M}_2$.
- We aim at constructing a similar model for $\mathcal{M}_3$.
- In particular, we define a combinatorial upper semi-continuous (UCS) partition of $\mathcal{M}_3$.

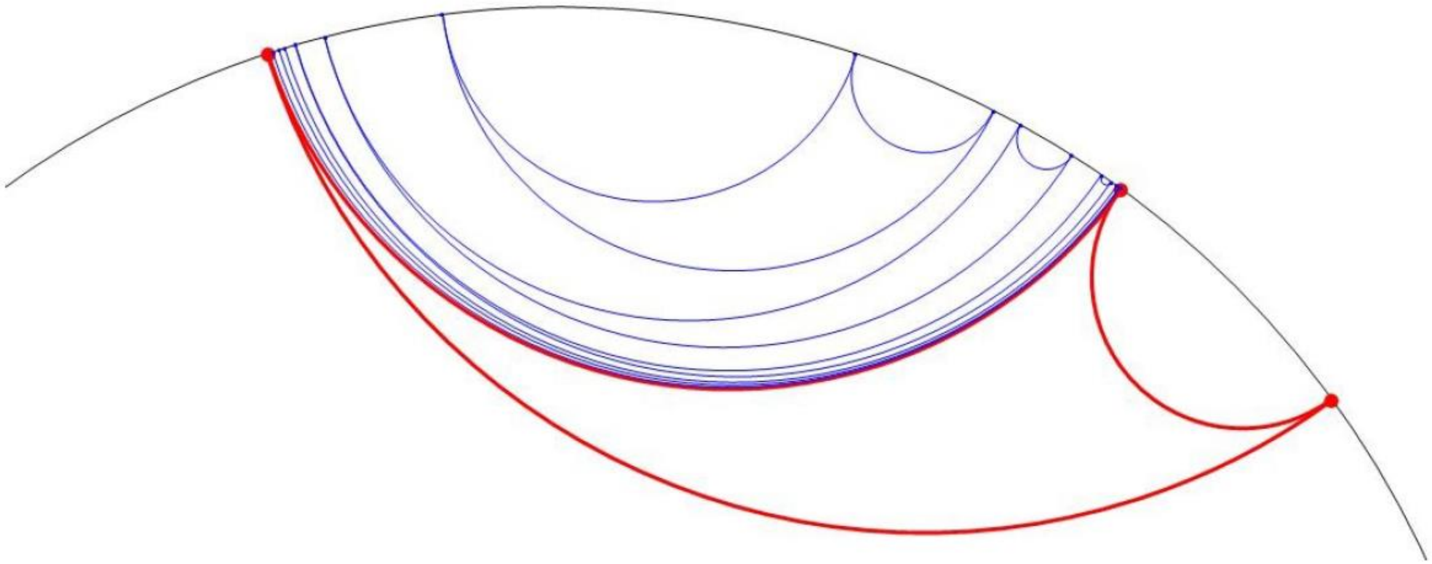


# Upper semi-continuous families

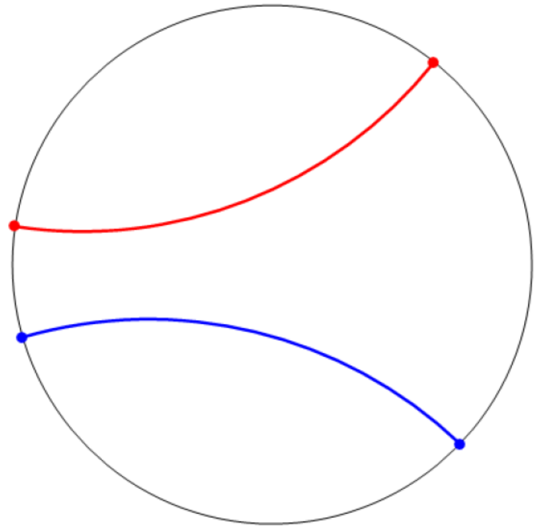
- A collection $\mathcal{F}$ of compact sets (in a topological space) is USC if

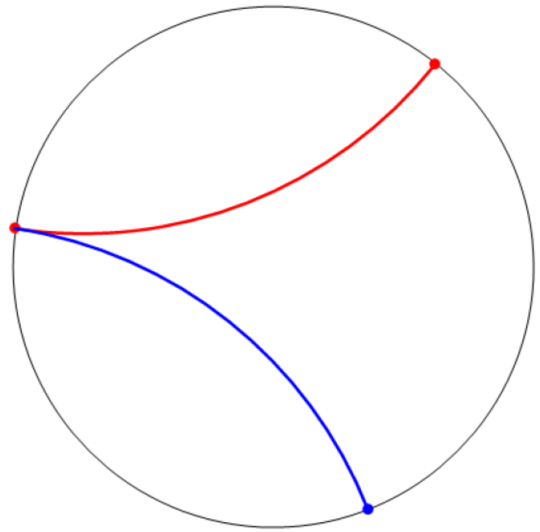
$$(\forall\, Z \in \mathcal{F})(\forall \text{ open } U \supset Z)(\exists \text{ open } V \supset Z)(\forall\, X \in \mathcal{F})$$
$$X \cap V \neq \emptyset \implies X \subset U$$



# Cubic critical portraits

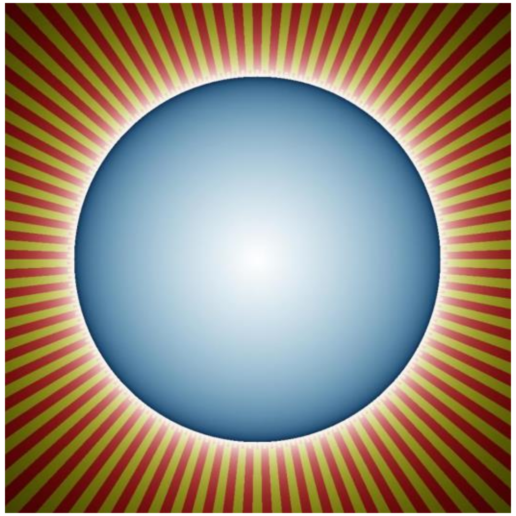
- $\mathbb{S} = \{z \in \mathbb{C} \mid |z| = 1\}$, the map $\sigma_3 : \mathbb{S} \to \mathbb{S}$ is defined as $\sigma_3(z) = z^3$.
- A chord $\ell = ab$ of $\mathbb{S}$ is critical if $\sigma_3(a) = \sigma_3(b)$.
- A critical portrait is an unordered pair of critical chords that are disjoint in $\mathbb{D}$.
- The set of all cubic critical portraits is denoted by $\mathrm{CrP}$; it is homeomorphic to the Möbius band.
- Our model space is a quotient of $\mathrm{CrP}$.

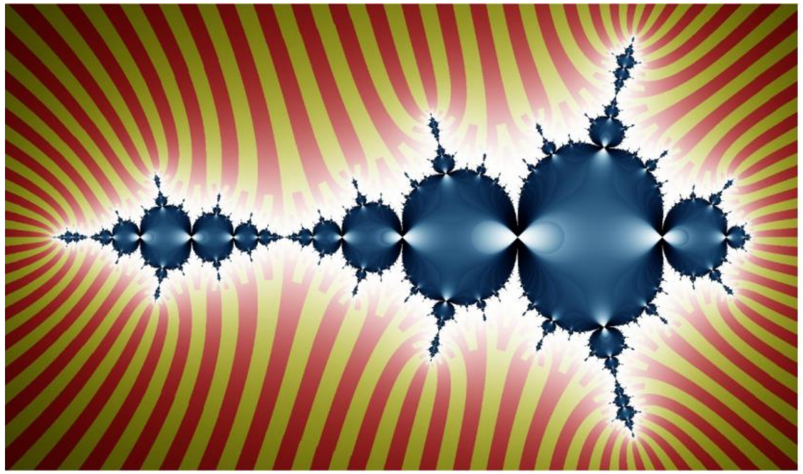




# External rays

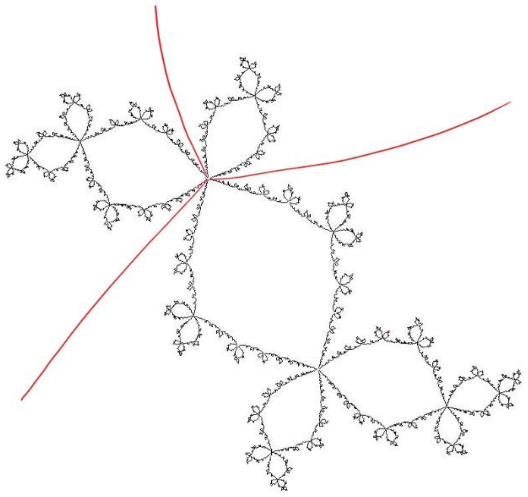
- Suppose $P(z) = z^3 + az + b$ and $K(P)$ is connected.
- The Riemann map $\psi\colon \mathbb{D} \to \mathbb{C}P^1 \setminus K(P)$ with $\psi(0) = \infty$ and $\lim_{z\to 0} z\psi(z) > 0$ conjugates $z \mapsto z^3$ with $P$.
- External rays are $\psi$-images of the radii: $R_P(\theta) = \psi\{te^{2\pi i\theta} \mid 0 < t < 1\}$.
- A repelling periodic point $z$ of $P$ defines one or several external rays $R$ that land at $z$, i.e., $\bar{R} \cap K(P) = \{z\}$.

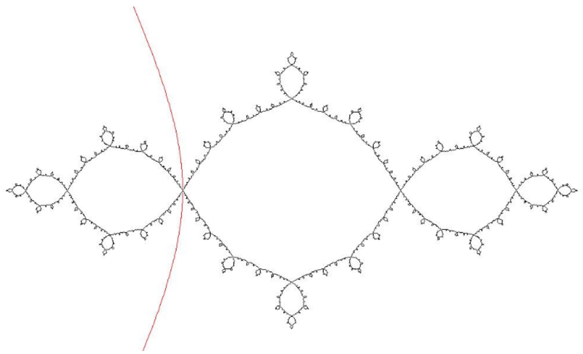




# Combinatorial counterparts

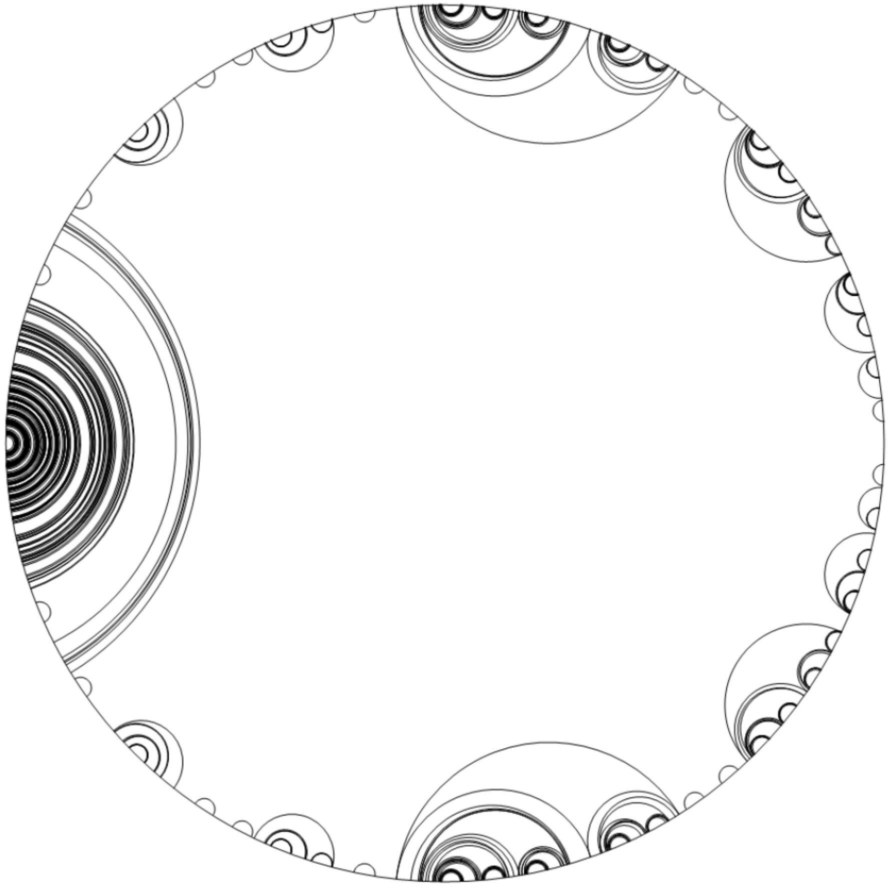
- A point $x \in K(P)$ is legal if $\exists\ n$ s.t. $P^n(x)$ is a repelling periodic point.
- If $R_P(\alpha), R_P(\beta)$ land at $x$, then the pair $\{\alpha, \beta\}$ is legal.
- $\mathcal{Z}_P$ = the set of all legal pairs of angles = the L-set of $P$.
- $P$ is visible if $\mathcal{Z}_P \neq \emptyset$.
- A critical portrait $\{c, d\}$ is compatible with $\mathcal{Z}_P$ if neither $c$ nor $d$ separate a pair $\{e^{2\pi i\alpha}, e^{2\pi i\beta}\}$ with $\{\alpha, \beta\} \in \mathcal{Z}_P$.
- $\mathcal{C}_P$ = the set of $\mathcal{K} \in \mathrm{CrP}$ compatible with $\mathcal{Z}_P$ = the combinatorial counterpart of $P$.





# Alliances: properties

- For every $P$ with connected $K(P)$, we define an alliance $\mathcal{A}_P \subset \mathrm{CrP}$.
- Alliances are closed.
- If $P$ is visible, then $\mathcal{C}_P \subset \mathcal{A}_P$.
- Distinct alliances are disjoint.
- Alliances form a USC partition of $\mathrm{CrP}$.
- One alliance is called prime.
- Other alliances are called regular.
- Regular alliances are combinatorial counterparts.



# Main Theorem

**Main Theorem.** *All alliances form a USC partition* $\{\mathcal{A}_P\}$ *of* $\mathrm{CrP}$. *The union of regular alliances is open and dense in* $\mathrm{CrP}$. *The map* $P \mapsto \mathcal{A}_P$ *is continuous and maps* $\mathcal{M}_3$ *onto the quotient space* $\mathrm{CrP}/\{\mathcal{A}_P\}$.

- Prime alliance corresponds to all invisible polynomials and to many polynomials that are "not sufficiently visible".
- Inou-Kiwi (2012) and Shen-Wang (2020) imply that hyperbolic regular fibers of the map $\mathcal{M}_3 \to \mathrm{CrP}/\{\mathcal{A}_P\}$ are connected.
- Open question: are all fibers connected?

# The prime alliance

- For a critical chord $c$, let $I(c)$ be the smaller open arc bounded by the endpoints of $c$.
- $\{c, d\} \in \mathrm{CrP}$ is weak if $\sigma_3^n(c \cap \mathbb{S}) \cap I(d) = \emptyset$ ($\forall n \geq 0$) and $\sigma_3^n(d \cap$

# Regular alliances

- If $P$ admits no nontrivial renormalization, then $\mathcal{C}_P$ is a regular alliance.
- Then all $\mathcal{K} \in \mathcal{C}_P$ have only strong friends of friends.
- Such $\mathcal{K}$ are called regular.
- Regular $\mathcal{K}$ form an open dense subset of $\mathrm{CrP}$.
- Alliances are disjoint, closed, and form a USC collection.

Thank you!