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Volume preserving diffeomorphisms as Poincare maps for volume preserving flows.

Dynamics in Siberia 2021

Time variable in a dynamical systems can be real (continuous time) or integer (discrete time). We have two parallel theories in these two classes of DS with essentially the same (at least, similar) results, ideas and methods.

For most of particular problems one context is technically more convenient in comparison with another. Hence, there exists a necessity to have a tool which associates with a system from one class "the corresponding system" from another class.

The standard way to go in the direction (continuous $\longrightarrow$ discrete) is the Poincaré map.

The backward direction (discrete $\longrightarrow$ continuous) is less clear and usually needs more work.

Suspension construction changes considerably the phase space and in many cases is unsatisfactory.

Need other approaches ...

Some piece of folklore

Proposition. Let $q : M \hookleftarrow$ be a diffeo of a smooth compact manifold to itself. Suppose q is isotopic to the identity. Then there exists a smooth time dependent vector field v on M such that q coincides with the Poincaré map $\{0\} \times M \hookleftarrow$ for the differential equation on $\mathbb{T} \times M$

$$\dot{x} = v(t, x), \quad \dot{t} = 1.$$

Comment. The vector field v is non-unique. Much freedom. Why not autonomous?

Because if v is time independent then v is a symmetry field for q:

$$Dq\, v = v \circ q.$$

Existence of a nontrivial symmetry field is a very restrictive condition for a diffeomorphism. Periodic points should be degenerate, etc.

Proof of the proposition.

Let $\gamma_s : M \hookleftarrow$, $s \in [0,1]$ be an isotopy which joins q with the identity map: $\gamma_0 = \mathrm{id}$ and $\gamma_1 = q$.

W.l.g. we may assume that

(1) γ_s is smooth in s,
(2) $\gamma_s = \mathrm{id}$ for s near 0 and $\gamma_s = q$ for s near 1.

We put

$$v(x,t) := \Big(\frac{\partial}{\partial t} q_t\Big) \circ q_t^{-1}(x).$$

Then the flow ϕ^t of the differential equation $\dot{t} = 1$, $\dot{x} = v$ on $[0,1] \times M$ satisfies

$$\phi^1(0,x) = (1, q(x)).$$

Since $v(t,x)$ vanishes near $t=0$ and $t=1$, it can be continued smoothly to $\mathbb{T} \times M$. □

This argument is almost trivial. But there are other versions of the problem which need more (sometimes much more) effort. For example,

- q symplectic, need v Hamiltonian,
- q volume-preserving, need v divergence free,
- q real-analytic, need v real-analytic,
- $\dots$

All this is done, more or less ...

New result

Let M, $\dim M = m$ be a smooth compact manifold and let ν be a volume form on it. This means that ν is a differential m-form, nowhere vanishing on M. Then $\omega = dt \wedge \nu$ is a volume form on $\mathbb{T} \times M$.

We define the natural projections

$$\mathbb{T} \times M \ni (t, x) \mapsto \pi_{\mathbb{T}}(t, x) = t, \quad (t, x) \mapsto \pi_M(t, x) = x.$$

Consider the vector field v on $\mathbb{T} \times M$. We assume that

(A) the first component of v is positive: $D\pi_{\mathbb{T}}\, v = v_{\mathbb{T}} > 0$,

(B) v preserves the form ω: $L_v \omega = 0$, where L_v is the Lie derivative.

Let g_v^s be the flow, generated by the vector field v on $\mathbb{T} \times M$. Condition (A) implies that the Poincaré map $P_v : \{0\} \times M \to \{0\} \times M$ is well-defined.

The map P_v preserves the volume form $\lambda = i_v \omega|_{\{t=0\}}$ on $\{0\} \times M$.

Theorem Let $Q : \{0\} \times M \to \{0\} \times M$ be another map which preserves λ. We assume that Q is (smoothly) isotopic to P_v in the group of λ-preserving self maps of $\{0\} \times M$. Then there exists an ω-preserving vector field u on $\mathbb{T} \times M$ such that $D\pi_{\mathbb{T}}\, u > 0$ and $Q = P_u$.

Motivation

In the paper

B.Khesin, S.Kuksin, D.Peralta-Salas, Global, local and dense non-mixing of the 3d Euler equation. Arch. Ration. Mech. Anal. 238 (2020), no. 3, 1087-1112.

The authors prove a non-mixing property of the flow of the 3D Euler equation which has a local nature:

in any neighbourhood of a “typical” steady solution there is a generic set of initial conditions, such that the corresponding Euler flows will never enter a vicinity of the original steady one.

More precisely,

there exist stationary solutions u_0 of the Euler equation on S^3 and divergence-free vector fields v_0 arbitrarily close to u_0, whose (nonsteady) evolution by the Euler flow cannot converge in the C^k Hölder norm ($k > 10$ non-integer) to any stationary state in a small (but fixed a priori) C^k-neighbourhood of u_0.

The set of such initial conditions v_0 is open and dense in the vicinity of u_0.

The authors needed the above theorem as a technical ingredient in the proof.

Sketch of the proof.

(a) Consider the smooth in s family of diffeomorphisms $\mathcal{T}_M(s) : M \to M$ of a manifold M into itself, $s \in \mathbb{R}$. We extend $\mathcal{T}_M$ to a family of diffeomorphisms $\mathcal{T}^s$ of $\mathbb{R} \times M$ putting by definition

$$\mathbb{R} \times M \ni (t, x) \mapsto \mathcal{T}^s(t, x) = \Big(t + s, \mathcal{T}_M(t + s) \circ \mathcal{T}_M^{-1}(t)(x)\Big). \qquad (1)$$

Direct computation shows that $\mathcal{T}^s$ is a flow i.e.,

$$\mathcal{T}^0 = \mathrm{id} \quad \text{and} \quad \mathcal{T}^{s_2} \circ \mathcal{T}^{s_1} = \mathcal{T}^{s_1+s_2} \qquad \text{for any } s_1, s_2 \in \mathbb{R}.$$

The flow $\mathcal{T}^s$ generates the vector field $\mathcal{U}$ on $\mathbb{R} \times M$:

$$\mathcal{U} = \left(\frac{d}{ds}\Big|_{s=0} \mathcal{T}^s\right) \circ \mathcal{T}^{-s}, \qquad D\pi_{\mathbb{R}}(\mathcal{U}) = 1, \qquad (2)$$

where $\pi_{\mathbb{R}} : \mathbb{R} \times M \to \mathbb{R}$ is a natural projection.

(b) We put $\widehat{v} = \dfrac{1}{v_{\mathbb{T}}} v$. Then $D\pi_{\mathbb{T}}\, \widehat{v} = 1$. Let

$$(t, x) \mapsto g^s_{\widehat{v}}(t, x), \qquad (t, x) \in \mathbb{T} \times M$$

be the flow of the vector field $\widehat{v}$. Then $g^s_{\widehat{v}}$ preserves the form $v_{\mathbb{T}}\omega$.

The Poincaré maps P_v and $P_{\widehat{v}}$ coincide. Hence, $g^1_{\widehat{v}}(0, x) = (1, P_v(x))$ for any $x \in M$.

Let G^s be the lift of the flow $g^s_{\widehat{v}}$ to the covering space $\mathbb{R} \times M$. We determine the family $\sigma_s : M \to M$ by

$$G^s(0, x) = (s, \sigma_s(x)), \qquad s \in \mathbb{R}.$$

(c) Let γ_s be a smooth in s isotopy from conditions of the theorem: for any $s \in [0,1]$ the map γ_s is a λ-preserving diffeomorphism of $M \cong \{0\} \times M$, and $\gamma_0 = P_v$, $\gamma_1 = Q$.

Changing smoothly parametrization on γ_s, we can assume that $\gamma_s = P_v$ in a neighborhood of $\{s = 0\}$ and $\gamma_s = Q$ in a neighborhood of $\{s = 1\}$.

We extend γ_s to all the axis $\mathbb{R} = \{s\}$, for example, putting $\gamma_s = P_v$ for $s < 0$ and $\gamma_s = Q$ for $s > 0$.

(d) Consider the family of maps $\mathcal{T}_M(s) : M \to M$,

$$\mathcal{T}_M(s) = \sigma_s \circ P_{\widehat{v}}^{-1} \circ \gamma_s, \qquad s \in \mathbb{R}.$$

Then $\mathcal{T}_M(0) = \mathrm{id}$, $\mathcal{T}_M(1) = Q$, hence $\mathcal{T}_M(1)$ preserves the form λ. Let $\mathcal{T}^s$ be the flow on $\mathbb{R} \times M$ generated by the family $\mathcal{T}_M(s)$ and let $\mathcal{U}$ be the corresponding vector field on $\mathbb{R} \times M$. Then by (1)

$$D\pi_{\mathbb{T}} \mathcal{U} = 1, \quad \mathcal{T}^0 = \mathrm{id}_{\mathbb{T}\times M}, \quad \mathcal{T}^1(0, x) = (1, Q(x)).$$

Near the points $t = 0$ and $t = 1$ we have: $d\gamma_t/dt = 0$. Therefore the vector field $\mathcal{U}$ coincides with $\widehat{v}$. Hence $\mathcal{U}|_{s\in[0,1]}$ can be extended to a periodic vector field $\widehat{\mathcal{U}}$ on $\mathbb{R} \times M$. Let $\widehat{\vartheta}^s$ be the corresponding flow on $\mathbb{R} \times M$. Due to periodicity of $\widehat{\vartheta}^s$ and $\widehat{\mathcal{U}}$ their projections to the flow ϑ^s and the vector field U on $\mathbb{T} \times M$ are well-defined.

(e) Let $\mathbf{1}$ be the vector field on $\mathbb{R} \times M$ determined by the equations

$$D\pi_{\mathbb{T}}\mathbf{1} = 1, \quad D\pi_M\mathbf{1} = 0.$$

The flow $\widehat{\vartheta}^s$ preserves some volume form Ω on $\mathbb{R} \times M$ such that $\imath_{\mathbf{1}}\Omega|_{\{t=0\}} = \lambda$.
Any volume form on $\mathbb{T} \times M$ equals $\widehat{\rho}\omega$. where $\widehat{\rho} : M \to \mathbb{R}$ is a function. Therefore $\Omega = \widehat{\rho}\omega$, where $\imath_{\mathbf{1}}\Omega|_{t=0} = \imath_{\mathbf{1}}\Omega|_{t=1} = \lambda$.
This equation and periodicity of $\widehat{\vartheta}^s$ imply that the function $\widehat{\rho}$ is 1-periodic in t. Hence there exists a function $\rho : \mathbb{T} \times M \to \mathbb{R}$ such that $\widehat{\rho} = \rho \circ \pi_{\mathbb{T}}$. The flow ϑ^s preserves the volume form $\rho\omega$.
The vector field $u = \rho U$ preserves ω. It remains to note that $P_u = P_U = Q$. $\square$