Quasiperiodic version of Gordon's theorem

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Based on joint work with Dmitry Treschev

Consider a smooth Hamiltonian system $\dot{x} = v_H(x)$ with Hamiltonian H on a symplectic manifold (M, Ω). The Hamiltonian vector field is defined by $i_{v_H}\Omega = -dH$. In local symplectic coordinates, $\Omega = dp \wedge dq$, $v_H(q, p) = (H_p, -H_q)$.

Theorem (Gordon)

Suppose all trajectories in a domain $U \subset M$ are periodic, and the fibration of U into periodic orbits is locally trivial. Let T be the period of the orbit through the given point. Then $dT \wedge dH = 0$. If the level sets $U \cap \{H = \mathrm{const}\}$ are connected, then T is a function of H.

This result is called Gordon's theorem[1], although it was known since 19th century. The history discussed in[2]

[1]W.B. Gordon, On the relation between period and energy in periodic dynamical systems, *Journal Math. Mech.* 19 (1969), 111–114

[2]A. Wintner, *Analytical foundations of celestial mechanics*, Princeton University Press, 1941.

A more precise statement can be proved under the same conditions if the symplectic form $\Omega = d\lambda$ is exact.

Theorem

If the level sets $U \cap \{H = E\}$ are connected, then the action $I(\gamma) = \oint_\gamma \lambda$ and the period $T(\gamma)$ of a periodic orbit γ depend only on the energy $E = H|_\gamma$:

$$I(\gamma) = \phi(E), \qquad T(\gamma) = \phi'(E).$$

The classical proof of Grodon's theorem is based on Hamilton's variational principle[3]. Alternative proofs were given by Lewis[4], Gordon, Weinstein[5].

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[3] A. Wintner, *Analytical foundations of celestial mechanics*, Princeton University Press, 1941.

[4] D.C. Lewis, Families of periodic solutions of systems having relatively invariant line integrals, *Proc. Amer.Math.Soc.*, 6 (1955), 181–185.

[5] A. Weinstein, Lagrangian submanifolds and Hamiltonian systems, *Ann. of Math.*, 98 (1973), 377–410.

Examples

- Harmonic oscillator

$$H = \frac{1}{2}(|p|^2 + |q|^2), \quad T = 2\pi, \quad I = \oint p\,dq = 2\pi E.$$

- Kepler problem

$$H = \frac{|p|^2}{2} - \frac{1}{|q|}, \quad T = 2\pi(-2E)^{-3/2}, \quad I = \pi(-2E)^{-1/2}.$$

- Geodesic flows with all geodesics closed, $T = L(2E)^{-1/2}$.

- The assertion of Gordon's theorem holds also when a submanifold $N \subset M$ is fibred into periodic orbits. If the level sets $\{H|_N = E\}$ are connected, then the action of $\gamma \subset \{H|_N = E\}$ is a function of energy only: $I = \phi(E)$.
- If $dH|_N \neq 0$, then the period T also depends on energy only: $T = \phi'(E)$.
- The condition $dE \neq 0$ is necessary.

Example (Hergholz)

Consider a particle in the plane in a Jacobi force field:

$$H(q, p) = \frac{|p|^2}{2} - \frac{1}{2|q|^2}.$$

Then circular periodic orbits $|q| \equiv r > 0$, $|p| \equiv r^{-1}$ have energy $E = 0$ and action $I = \oint p\, dq = 2\pi$, but the period $T = 2\pi r^2$ is non-constant.

Definition of an invariant torus

An invariant n-torus $\Gamma \subset M$ is given by the pair (f, ω) of a smooth embedding $f : \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \to M$ and a frequency vector $\omega \in \mathbb{R}^n$ such that

$$\Gamma = f(\mathbb{T}^n), \quad g_H^t(f(\theta)) = f(\theta + \omega t),$$

where $g_H^t : M \to M$ is the phase flow of the system.
Equivalently,

$$v_H(f(\theta)) = f'(\theta)\omega = \sum_{i=1}^{n} \omega_i \partial_{\theta_i} f(\theta).$$

Quasiperiodic Gordon's theorem

- Suppose the system has a smooth family of invariant tori $\Gamma_z = f_z(\mathbb{T}^n)$, $z \in D$, with frequency $\omega(z)$. $F : D \times \mathbb{T}^n \to M$, $F(z,\theta) = f_z(\theta)$, is a smooth map.
- Suppose the frequencies of all tori are collinear: $\omega(z) = \rho(z)u$, where u is a constant vector, and $\rho(z) > 0$.
- Without loss of generality suppose that u is nonresonant: $u \cdot k = 0$, $k \in \mathbb{Z}^n$, implies $k = 0$. Or else the tori are fibred in lower dimensional nonresonant tori.
- Then trajectories on Γ_z are dense, so Γ_z lies in an energy level $\{H = E(z)\}$.

Theorem

$d\rho(z) \wedge dE(z) = 0$. *If the level sets* $\{z \in D : E(z) = \mathrm{const}\}$ *are connected and* $dE(z) \neq 0$ *in* D, *then* $\rho(z)$ *is a function of energy.*

Corollary

Suppose a submanifold $N \subset M$ is fibred by nonresonant invariant tori, and the monodromy of the fibration is trivial.[a] *Suppose the frequency vectors of all tori $\Gamma \subset N$ are collinear: $\omega = \rho u$, $\rho : N \to \mathbb{R}^+$. If $dE \neq 0$, $E = H|_N$, and the level sets $\{E = \mathrm{const}\}$ are connected, then $\rho = \rho(E)$ is a function of energy only.*

[a] Or else the frequency vector is not globally defined.

As Hergholz's example shows, the condition $dE \neq 0$ is necessary. When N is an open set in M, it is automatically satisfied. Then we say that the system is quasi-isochronous. This is a particular class of super-integrable systems.

Nekhoroshev's generalization of Gordon's theorem

Nekhoroshev proposed another generalization of Gordon's theorem. Suppose the Hamiltonian system has an invariant manifold N such that there are independent Hamiltonian symmetry fields tangent to N which commute on N. If the fibres of the corresponding fibration of N are compact, then the fibres are isotropic invariant tori and the Hamiltonian is a function of the corresponding actions. Our generalization of Gordon's theorem is different: we assume the existence of a family of invariant tori with collinear frequencies and deduce that the frequencies are functions of energy.

Motivation: isochronous systems

- If all trajectories are periodic with constant period, the system is said to be isochronous. Classification of isochronous systems is an old problem going back to Huygens and Abel.
- For systems with $H = p^2/2 + V(q)$, $q \in \mathbb{R}$, final results were obtained by Gorni and Zampieri[6].
- General results for systems in $\mathbb{R}^2$ with an elliptic equilibrium were obtained by Treschev[7].
- The relation between Gordon's theorem and the problem of isochronicity is based on the following trivial observation. If the system with Hamiltonian H satisfies assumptions of Gordon's theorem, then the system with Hamiltonian $\phi(H)$ is isochronous.

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[6] G. Gorni and G. Zampieri, Global isochronous potentials. *Qual. Theory Dyn. Syst.* 12 (2013), 407–416.

[7] D.Treschev, Isochronicity in 1 DOF, Regul. Chaotic Dyn., 27 (2022), 123–131.

Quasi-isochronous systems

A natural generalization of isochronous systems are systems in which all solutions are quasiperiodic with the same nonresonant frequency vector. Such a system may be obtained from a quasi-isochronous system with $T = \phi'(E)$ if we replace H by $\phi(H)$. Unlike the isochronous case, quasi-isochronous systems were not discussed much in mathematical literature. Evident examples of such systems are direct products of isochronous systems. Non-trivial examples are provided e.g. by geodesic flows on surfaces of revolution.

Billiard systems

Discrete quasi-isochronous systems appear as Poincaré maps on the energy level, or as billiard systems. Isochronous billiard systems were considered by Treschev[8] for 2 dimensions, and in [9] for many dimensions.

In these papers an attempt was made to construct billiards with dynamics locally conjugated to linear dynamics, usually to a rigid rotation. Such billiards are quasi-isochronous. Numerical evidence for the convergence of the series representing boundary curves of such billiard tables was given. Recently it was proved that the series are of the Gevrey class[10]. However the problem remains open.

[8] D. Treschev, Billiard map and rigid rotation, *Phys. D*, 255 (2013), 31–34.

[9] D. Treschev, A locally integrable multi-dimensional billiard system, *Discrete Contin. Dyn. Syst.* Ser. A, 37:10 (2017), 5271–5284.

[10] Q.Wang,K.Zhang, Preprint 2022

If the symplectic form is exact, a nonresonant torus Γ is isotropic: $\Omega|_\Gamma = 0$ [11]. Then the action variables are correctly defined:

$$J_i(f) = \oint_{\gamma_i} \lambda, \quad d\lambda = \Omega,$$

where γ_i are basis cycles of $\Gamma = f(\mathbb{T}^n)$. Let

$$\Phi(z) = \sum_{i=1}^{n} u_i I_i(z), \qquad I_i(z) = J_i(f_z),$$

Theorem

$\rho(z)\, d\Phi(z) = dE(z)$. *If the level sets* $\{z \in D : E(z) = \mathrm{const}\}$ *are connected, then* $\Phi(z) = \phi(E(z))$ *is a function of energy only. If moreover* $dE(z) \neq 0$ *in* D*, then the frequency vector is a function of energy only:* $1/\rho(z) = \phi'(E(z))$.

[11] M.Herman, Inegalites a Priori pour des Tores Lagrangiens Invariants par des Diffeomorphismes Symplectiques. *Inst.-Hautes-Etudes-Sci.-Publ.-Math.*, 70 (1990), 47–101.

Percival's variational principle

Suppose for simplicity that the symplectic form $\Omega = d\lambda$ is exact. Percival's functional on $C^\infty(\mathbb{T}^n, M) \times \mathbb{R}^n$ is defined by

$$S(f,\omega) = \int_{\mathbb{T}^n} \Big(\lambda(f'(\theta)\omega) - H(f(\theta))\Big)\, d\theta = J(f)\cdot\omega - E(f).$$

$$E(f) = \int_{\mathbb{T}^n} H(f(\theta))\, d\theta, \quad J_i(f) = \int_{\mathbb{T}^n} \lambda(\partial_{\theta_i} f(\theta))\, d\theta$$

are the average energy and the average action variables.
A calculation similar to the proof of Hamilton's principle gives

$$\delta S(f,\omega) = \int_{\mathbb{T}^n} \Big(\Omega(\delta f(\theta), f'(\theta)\omega) - dH(f(\theta))(\delta f(\theta))\Big)\, d\theta + J(f)\,\delta\omega.$$

If the torus $\Gamma = f(\mathbb{T}^n)$ is invariant, then the first two terms cancel:

$$\Omega(\cdot, f'(\theta)\omega) = -dH(f(\theta)), \quad \delta S(f,\omega) = J(f)\,\delta\omega.$$

For variations with fixed ω we obtain[12]

Theorem (Percival)

The torus $\Gamma = f(\mathbb{T}^n)$ is invariant iff f is a critical point of the functional $S(\cdot, \omega)$.

[12] I.C. Percival, A variational principle for invariant tori of fixed frequency, J. Phys. A, 12 (1979), 57–60.

- There is a Lagrangian version of Percival's functional for Lagrangian systems with a Lagrangian $L : TN \to \mathbb{R}$. The functional is defined on $C^\infty(\mathbb{T}^n, N) \times \mathbb{R}^n$ by

$$S(g,\omega) = \int_{\mathbb{T}^n} L(g(\theta), g'(\theta)\omega)\, d\theta.$$

  If g is a critical point of $S(\cdot,\omega)$, then $g(\omega t)$ is a quasiperiodic solution of the Lagrangian system.
- Mather[13] used this functional to construct Cantor invariant sets of 2D twist maps.
- For positive definite Lagrangian systems with convex in velocity Lagrangians an analog of Percival's functional is the average action functional $\int L\, d\mu$ on the space of invariant measures with given homology class $\omega \in H_1(N, \mathbb{R})$. Gordon's theorem can be extended to this case, but this doesn't seem useful.

[13] J.Mather, Existence of quasiperiodic orbits for twist homeomorphisms of the annulus, Topology, 21 (1982)

Suppose that the symplectic form is exact. Let $(f_z, \omega(z))$, $z \in D$, be a family of invariant tori with collinear frequencies $\omega(z) = \rho(z)u$. Percival's functional

$$P(z) = S(f_z, \omega(z)) = \rho(z)\Phi(z) - E(f_z), \quad \Phi(z) = J(f_z) \cdot u.$$

By Percival's theorem

$$\delta P(z) = J(f_z)\, \delta\omega(z) = \Phi(z)\, \delta\rho(z).$$

We obtain

$$\rho(z)\, \delta\Phi(z) = \delta E(f_z).$$

If the level sets $\{z \in D : E(f_z) = \mathrm{const}\}$ are connected, $\Phi(z) = \phi(E(f_z))$ is a function of average energy only.
If $\delta E(f_z) \neq 0$, then $1/\rho(z) = \phi'(E(f_z))$ is also a function of the average energy. □

Average action

- If the symplectic form is nonexact, then nonresonant invariant tori are not isotropic in general. Hence the action variables of the torus are not well defined. However the average actions are well defined, at least locally.
- Fix some $g_0 \in C^\infty(\mathbb{T}^n, M)$ and let $U \subset C^\infty(\mathbb{T}^n, M)$ be its small neighborhood (in the C^0 topology). Then for any $f \in U$ there exists a homotopy $g_s \in U$, $0 \leq s \leq 1$, such that $g_1 = f$.
- The average actions of the torus $\Gamma = f(\mathbb{T}^n)$ are defined by

$$J_i(f) = \int_{\mathbb{T}^n} \int_0^1 \Omega(\partial_s g_s(\theta), \partial_{\theta_i} g_s(\theta))\, ds\, d\theta \qquad (1)$$

  This does not depend on the homotopy.
- Once the average actions are defined we can extend the generalization of Gordon's theorem to the nonexact case replacing actions by average actions.

Normal form of a quasi-isochronous system

Let $\Gamma_z = F(z, \mathbb{T}^n)$, $z \in D$, be a family of invariant tori with nonresonant collinear frequencies. Suppose for simplicity that D is contractible.

Proposition

$$\hat{\Omega} = F^*\Omega = \sum dA_j(z) \wedge dz_j + \sum dI_i(z) \wedge d\theta_i + \sum c_{ij} d\theta_i \wedge d\theta_j,$$
$$\hat{H} = H \circ F = \phi^{-1}\Big(\sum u_i I_i(z)\Big)$$

where $I_i(z) = J_i(f_z)$ are the average actions of the tori.

A similar normal form was obtained in[14] in a different situation.

[14] F. Fasso, Quasi-periodicity of motions and complete integrability of Hamiltonian systems, *Ergod. Th. & Dynam. Sys.*, 18 (1998), 1349–1362.

Proof

On $N = F(D \times \mathbb{T}^n)$ we have

$$i_{v_H/\rho}\Omega|_N = -d\phi(E), \qquad E = H|_N.$$

The flow of v_H/ρ preserves $\Omega|_N$. Hence $\hat{\Omega}$ is invariant under the flow $(z,\theta) \to (z, \theta + tu)$. Since u is nonresonant, the coefficients of $\hat{\Omega}$ are independent of θ:

$$\hat{\Omega} = \sum a_{ij}(z) dz_i \wedge dz_j + \sum b_{ij}(z) dz_i \wedge d\theta_j + \sum c_{ij}(z) d\theta_i \wedge d\theta_j.$$

Since Ω is closed, $c_{ij} = \mathrm{const}$. Since D is contractible, the forms $\sum_i b_{ij}(z)\, dz_i = dB_j(z)$ are exact. Hence

$$\hat{\Omega} = \sum dA_j(z) \wedge dz_j + \sum dB_i(z) \wedge d\theta_i + \sum c_{ij} d\theta_i \wedge d\theta_j.$$

Since the average actions are symplectic invariants (up to a constant), $I_i(z) = B_i(z) + \mathrm{const}$. □

Normal form in the exact symplectic case

Proposition

Suppose the manifold $N = F(D \times \mathbb{T}^n)$ is exact symplectic. Then in a neighborhood of any torus in N there exist generalized action-angle coordinates x, y, θ, I such that

$$\hat{\Omega} = \sum dy_i \wedge dx_i + \sum dI_i \wedge d\theta_i. \quad (2)$$

This normal form was given by Nekhoroshev[15]. In particular, the system is superintegrable on N. However, there are additional symmetries generated by locally Hamiltonian vector fields with Hamiltonians $u_k \theta_j \circ F^{-1} - u_j \theta_k \circ F^{-1}$.

[15] N.N. Nekhoroshev, Action-angle variables and their generalizations, *Tr. Mosk. Mat. Ob-va*, 26 (1972), 181–198.

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Thank you for your attention!