On rationally integrable projective billiards

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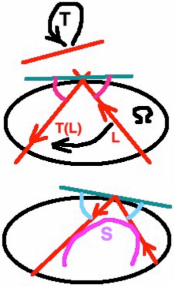
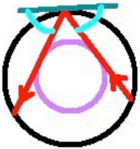
CONVEX PLANAR BILLIARDS



Billiard reflection.
Acts on the cylinder = space of oriented lines.

A curve S is a caustic, if its tangent lines are reflected to its tangent lines



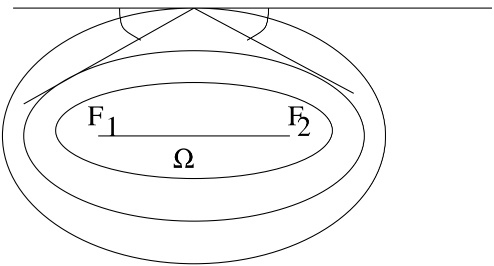
Example: billiard in a disk

Caustics - concentric circles

Family of tangent lines to a caustic = **invariant curve** for billiard map.

Billiard map preserves **area form** $dp \wedge d\phi$, ϕ =azimuth, $p = \pm$dist to O.

Billiard in an ellipse



Caustic:=curve whose tang. lines are reflected from $\partial\Omega$ to its tang. lines.
Confocal elliptic caustics are closed and foliate $\Omega \setminus [F_1, F_2]$**.**

Def. A billiard Ω with smooth strictly convex $\partial\Omega$ is **Birkhoff integrable,** if an inner **neighborhood of** $\partial\Omega$ **in** Ω**,** is foliated by **closed caustics,** and $\partial\Omega$ is a leaf. **Example: ellipse** is Birkhoff integrable.

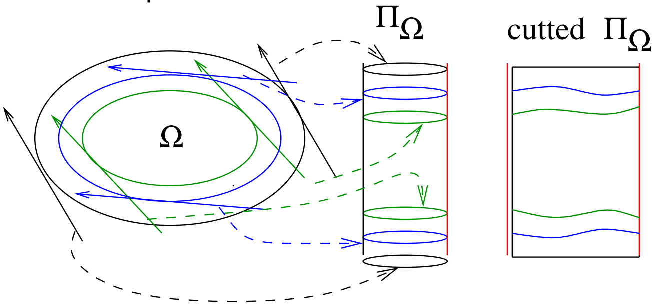
Birkhoff Conjecture. Ω – **Birkhoff integrable.** $==>$ $\partial\Omega$ – **ellipse.**

Foliation is an important condition.

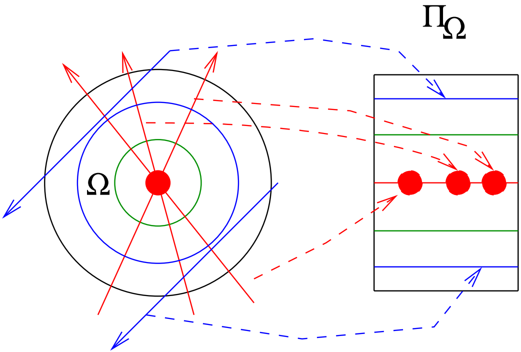
V.Lazutkin (1973) **Every** strictly convex bounded planar billiard has a **Cantor family of closed caustics.** A **KAM**-like theorem.

Billiard map acts on **Phase cylinder** Π_Ω:= {oriented lines intersecting Ω}.

Birkhoff integrable billiard:
caustics $\to$ **closed** T**-invariant curves** in Π_Ω near its boundary.
Two invariant curves per **caustic.**



Case $\partial\Omega =$ **circle:** phase cylinder is **foliated** by invariant **closed curves:**
- families of lines tangent to concentric circles, two curves per each circle;
- and one "central curve": family of oriented lines through the center.



Theorem (M.Bialy) (1993). Let the **phase cylinder** of a billiard Ω be **completely foliated** by invariant **closed curves.** Then Ω is a **disk.**

Case of ellipse. Foliation of phase cylinder: "cutted view".

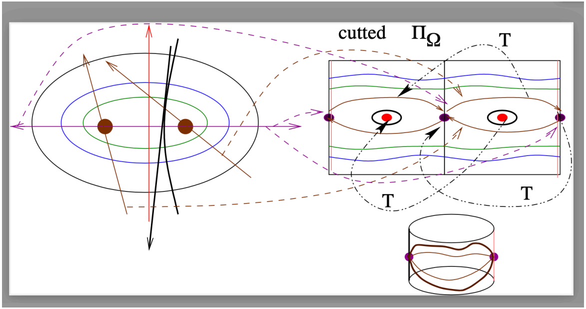
A **singular foliation** by T^2-invariant curves:

4 Brown curves = families of lines through foci

2 Red sing. points = **small** axis with two possible orientations. **Centers.**

2 Violet sing. points = **big** axis with two possible orientations. **Saddles.**

T := the billiard map acting on the phase cylinder.



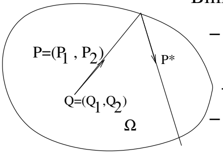
Numerical experience and Conjecture (D.V.Treschev, 2013).
$\exists$ a planar billiard whose squared billiard map T^2 has a **fixed point** where germ of T^2 is conjugated to **rotation** $(r, \phi) \mapsto (r, \phi + 2\pi\theta)$, $\theta \notin \mathbb{Q}$.

Not true for ellipses. **Proof** of existence is **open problem.**

V.Kaloshin, A.Sorrentino (2016): proof of **local Birkhoff Conjecture**: **any integrable deformation of an ellipse is an ellipse.** Ann. Math. 18.

Very recent result: **Mikhail Bialy and Andrei Mironov.**
Proof of Birkhoff conjecture for
- centrally-symmetric billiards, where
- foliation by closed caustics extends to caustic tang. to 4-periodic orbits.

Ann. of Math.



Billiard flow on $TR^2|_\Omega$

– geodesic flow moves (Q,P) until Q reaches the boundary

– reflection: (Q,P) –> (Q,P*)

– geodesic flow moves (Q,P*)

– Trivial first integral: $||P||^2$

Birkhoff integrability $<=>$ existence of a **first integral** indep. with $||P||^2$ on a neighborhood of the unit tangent bundle of $\partial\Omega$ in $T\mathbb{R}^2|_\Omega$.

Def. Ω is **polynom. integrable,**
if the flow has a 1st integral $I(Q,P)$, **polyn. in** P, $I|_{\{||P||=1\}} \not\equiv const$.

Bolotin's Polynomial version of Birkhoff Conjecture (1992).
Now **Thm (M.Bialy, A.Mironov, A.G., '17-'18).**
1) $\partial\Omega$ – convex, C^2. It is **polynomially integrable,** iff $\partial\Omega$ is a **conic.**

2) $\partial\Omega$ is **piecewise** C^2**.** It is **polyn. integr.** $<=>$ **confocal billiard:** $=$

$\partial\Omega = \cup$(conical arcs of confoc. pencil $\mathcal{C}$ + segm. of $\mathcal{C}$-**admissible lines.**)
3) min deg (integral) $\in \{$**2,4**$\}$. Similar results on billiards in S^2, $\mathbb{H}^2$.

S.Bolotin 1992: statement and partial results.

Billiards in $\mathbb{R}^n$, S^n, $\mathbb{H}^n$ bounded by confocal quadrics.
A.P.Veselov (1988): compl. integr. with quadr. integrals in involution.
V.Dragović, M.Radnović: Dynamics and interrelations.

Today's main results. Classification of rationally integrable piecewise smooth non-polygonal **projective billiards.**

New phenomena:

1) $\min \deg$(integral) is realized by **arbitrary even number**.

2) Projective generalization of **confocal billiards**, namely, the so-called **dual pencil type projective billiards** may have integrals of deg. 2, 4, **12.**

Projective billiards = billiards with variable reflection law

Introduced by **S.Tabachnikov, 1997.**

Planar projective billiard: a curve $C \subset \mathbb{R}^2$ with **transversal line field** $\mathcal{N}$.

Reflection transformation acting on **oriented lines** intersecting C:
- Each oriented line L is reflected from C at its last intersection point Q with C by **affine involution** $\mathbf{A}_Q : \mathbb{R}^2 \to \mathbb{R}^2$ **preserving** $T_Q\gamma$ **and** $\mathcal{N}(Q)$**:**
- $\mathbf{A}_Q|_{T_QC} \equiv Id$, $\mathbf{A}_Q|_{\mathcal{N}(Q)}$ – **central symmetry** with respect to Q.

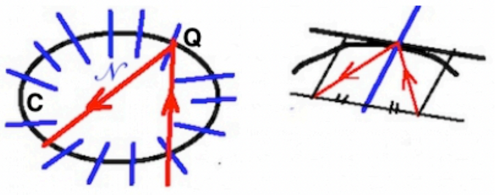


Figure: **Projective billiard and its reflection law.**

Ex: A **usual billiard** is a projective billiard with $\mathcal{N}$ = **normal line field.**

Billiards on S^2 and $\mathbb{H}^2$ viewed as projective billiards

S^2 and hyp. plane $\mathbb{H}^2 \simeq$ surfaces in $(\mathbb{R}^3, < Ax, x >)$, $A = \mathrm{diag}(1,1,\pm 1)$.

$S^2 = \Sigma := \{x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3\}$, $A = Id$.
$\mathbb{H}^2 = \Sigma := \{< Ax, x >= -1 \mid x_3 > 0\} \subset \mathbb{R}^3\}$, $A = \mathrm{diag}(1,1,-1)$.

Geodesics are **sections** of Σ by two-dimensional vector subspaces in $\mathbb{R}^3$.

$\Sigma_+ := \Sigma \cap \{x_3 > 0\}$, Σ_+ is S^2_+ or $\mathbb{H}^2$.
The tautological projection $\pi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{RP}^2_{[x_1:x_2:x_3]}$ sends Σ_+ to $\mathbb{R}^2 = \{x_3 = 1\}$, respectively to $D_1 = \{x_1^2 + x_2^2 < 1\} \subset \mathbb{R}^2$.

A curve $\gamma \subset \Sigma_+$ equipped with the **normal line field** is projected to a curve $C = \pi(\gamma) \subset \mathbb{R}^2$ equipped with a **transversal line field** $\mathcal{N}$

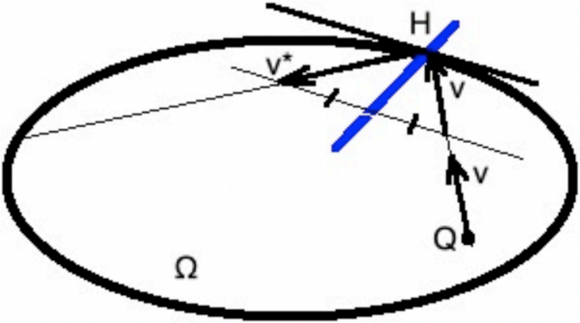
Orbits in the billiard on γ $\mapsto$ orbits of the **projective billiard** on C.
(refl. of geodesics from γ) (reflection of lines from $(C, \mathcal{N})$)

Tabachnikov's Conjecture (a generalization of **Birkhoff Conjecture**). Let $\gamma \subset \mathbb{R}^2$ be a strictly convex closed curve with a transversal line field. Let the corresponding projective billiard have a family of closed caustics foliating a topological annulus adjacent to γ; the curve γ being a leaf.

Then γ is an **ellipse,** and the **foliation** is **dual to a pencil of conics.**

Implies **Birkhoff Conjecture** for billiards on $\mathbb{R}^2$, S^2, $\mathbb{H}^2$.

Projective billiard flow



Birkhoff integrability $<=>$ existence of a

non-constant **0-homogeneous first integral** $I(x,v)$, $I(x,\lambda v)=I(x,v)$, on a neighborhood of the unit tangent bundle of $\partial\Omega$ in $T\mathbb{R}^2|_\Omega$.

Main result. Criterion of existence of **rational** 0-homogeneous integral in velocity for piecewise-smooth non-polygonal projective billiards.

Definition. A projective billiard is **rationally integrable,** if its flow has a first integral that is a **rational 0-homogeneous** function of the velocity.

Prop. For usual billiards **rational integr.** $<=>$ **polynom. integrability.**

Proof of $<=$**.** Let $\exists$ **polyn. int.** $F_x(v) \not\equiv const$ on $\{||v|| = 1\}$, deg $= 2n$

$$=> \quad \frac{F_x(v)}{||v||^{2n}} \quad \text{is a **rational** 0-homogeneous integral.}$$

Remark. General **projective** billiard flow **doesn't** preserve $||v||^2$.

Open problem. Describe all **projective** billiards having **polynomial integrals.**

$\mathcal{M} = \mathcal{M}(x, v) := (-v_2, v_1, x_1 v_2 - x_2 v_1)$. Univ. invariant of geodesic flow.

$=>$ Each rational 0-homog. integral of billiard has form $R(\mathcal{M})$.

Basic example of rationally integrable projective billiard

$\mathcal{A}$, $\mathcal{B}$ – symmetric 3x3-matrices, $\mathcal{B}$ is non-degenerate. $<\,,\,>$ – Euclidean.

$\mathcal{C}^*_\lambda := <(\mathcal{B}-\lambda\mathcal{A})^{-1}x, x> = 0 \subset \mathbb{RP}^3$. $\mathcal{C}^* = (\mathcal{C}^*_\lambda)$ – **dual pencil of conics.**

Let $\alpha, \beta \in \mathcal{C}^*$ - two (nested) conics, β is smaller.

$\exists!$ a projective billiard structure on α for which β - caustic.
Called **dual pencil** type structure (or $\mathcal{C}^*$-**projective billiard structure**.)

Rk. It is **Birkhoff integrable:** each conic $\mathcal{C}^*_\lambda$ inside α is a caustic.

Tabachnikov Conjecture: These are the only Birkhoff integrable examples.

S. Bolotin $=>$ **Thm.** The $\mathcal{C}^*$-projective billiard is **rationally integrable**.

$$\mathcal{M} = \mathcal{M}(x,v) := (-v_2, v_1, x_1v_2 - x_2v_1) \quad - \quad \text{the moment vector.}$$

$$\Psi(x,v) = \frac{<(\mathcal{B}-\lambda_1\mathcal{A})\mathcal{M},\mathcal{M}>}{<(\mathcal{B}-\lambda_2\mathcal{A})\mathcal{M},\mathcal{M}>} \text{ is an } \textbf{integral} \quad \forall\, \lambda_1 \neq \lambda_2.$$

Classif. of rationally integrable smooth connected projective billiards.

Theorem (A.G., 2021). Let $C \subset \mathbb{R}^2_{x_1,x_2}$ – nonlinear C^4-smooth germ of curve equipped with transversal line field $\mathcal{N}$ (projective bill. structure). It is **rationally integrable,** iff C – **conic** and $\mathcal{N}$ is one of following:
1) A **dual pencil type** projective billiard structure, with quadratic integral.
2) Exotic structures: $C = \{c_2 = x_1^2\}$ and $\mathcal{N}$ is directed by one of the following vector fields:
2a) $(\dot{x}_1, \dot{x}_2) = (\rho, 2(\rho-2)x_1), \ \ \rho \in \{2 - \frac{2}{m} \mid m \in \mathbb{N}\}$.
Case 2a1), $\rho = 2 - \frac{2}{2k+1}$**.** Set $\Delta := x_1 v_2 - x_2 v_1$. An integral is

$$\Psi_{2a_1}(x_1, x_2, v_1, v_2) := \frac{(4v_1\Delta - v_2^2)^{2k+1}}{v_1^2 \prod_{j=1}^k (4v_1\Delta - c_j v_2^2)^2}, \ c_j = -\frac{4j(2k+1-j)}{(2k+1-2j)^2}.$$

Case 2a2), $\rho = 2 - \frac{1}{k+1}$**:** an integral is

$$\Psi_{2a_2}(x_1, x_2, v_1, v_2) := \frac{(4v_1\Delta - v_2^2)^{k+1}}{v_1 v_2 \prod_{j=1}^k (4v_1\Delta - c_j v_2^2)^2}, \ c_j = -\frac{j(2k+2-j)}{(k+1-j)^2}.$$

$$\Delta := x_1 v_2 - x_2 v_1.$$

2b1) $\mathcal{N}$: $(\dot{x}_1, \dot{x}_2) = (5x_1 + 3, 2(x_2 - x_1))$. An integral is

$$\Psi_{2b_1} = \frac{(4v_1\Delta - v_2^2)^2}{(4v_1\Delta + 3v_2^2)(2v_1 + v_2)(2\Delta + v_2)}.$$

2b2) $\mathcal{N}$: $(\dot{x}_1, \dot{x}_2) = (3x_1, 2x_2 - 4)$. An integral is

$$\Psi_{2b2}(x_1, x_2, v_1, v_2) = \frac{(4v_1\Delta - v_2^2)^2}{(v_2^2 + 4\Delta^2 + 4v_1\Delta + 4v_1^2)(v_2^2 + 4v_1^2)}.$$

2c1) $\mathcal{N}$: $(\dot{x}_1, \dot{x}_2) = (x_2, x_1 x_2 - 1)$. An integral is

$$\Psi_{2c1}(x_1, x_2, v_1, v_2) = \frac{(4v_1\Delta - v_2^2)^3}{(v_1^3 + \Delta^3 + v_1 v_2 \Delta)^2}.$$

2c2) $\mathcal{N}$: $(\dot{x}_1, \dot{x}_2) = (2x_1 + 1, x_2 - x_1)$. An integral is

$$\Psi_{2c2}(x_1, x_2, v_1, v_2) = \frac{(4v_1\Delta - v_2^2)^3}{(v_2^3 + 2v_2^2 v_1 + (v_1^2 + 2v_2^2 + 5v_1 v_2)\Delta + v_1\Delta^2)^2}.$$

2d) $\mathcal{N} : (\dot{x}_1, \dot{x}_2) = (7x_1 + 4, 2x_2 - 4x_1)$. An integral is $\Psi_{2d}(x_1, x_2, v_1, v_2)$

$$= \frac{(4v_1\Delta - v_2^2)^3}{(v_1\Delta + 2v_2^2)(2v_1 + v_2)(8v_1v_2^2 + 2v_2^3 + (4v_1^2 + 5v_2^2 + 28v_1v_2)\Delta + 16v_1\Delta^2)}.$$

New result: piecewise-smooth case.

Theorem. A projective billiard on **piecewise C^4-smooth** curve with a nonlinear arc is **rationally integrable,** iff it is of one of the following types:

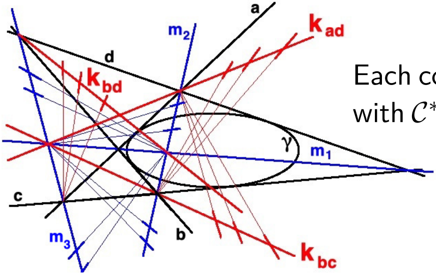
1) A **dual pencil type** billiard: by **definition,** it consists of
- arcs of conics from a dual pencil $\mathcal{C}^*$, with $\mathcal{C}^*$-projective billiard structure;
- maybe some segments of the so-called $\mathcal{C}^*$-**admissible lines**

(with extra conditions on their collection if the pencil is degenerate).
The $\min\deg(\text{integral}) = 2$, 4 or **12.**

2) An **exotic** piecewise smooth billiard: it consists of
- arcs of **just one** conic with an **exotic line field** from previous theorem;
- maybe some segments of so-called **admissible** lines for the **exotic field.**

Dual pencil $\mathcal{C}^*$ of conics tangent to complex lines a, b, c, d.

(non-degenerate dual pencil)



Each conic in $\mathcal{C}^*$ is equipped with $\mathcal{C}^*$-projective structure

Admissible lines:

3 standard lines m_j**:** m_3 with transv. line field through $m_2 \cap m_1$ etc.;

6 skew lines k_{en} **numer. by unordered** $\{e, n\} \subset \{a, b, c, d\}$, $e \neq n$**:**

k_{bc} is equipped with transversal field of lines through $a \cap d$, etc...

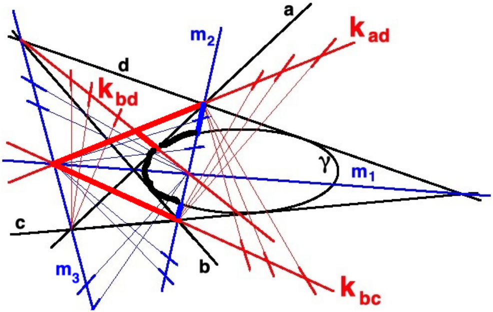
Thm. Each projective billiard consisting of arcs of conics from $\mathcal{C}^*$ and segments of admissible lines is rationally integrable.

min deg(integral) $= 2$, if there are no skew line segments;

min deg(integral) $=$ **12**, if $\exists$ segments of some **neghbor** skew lines k_{en}, $k_{n\ell}$;

min deg(integral) $=$ **4** in any other case.

Two dual pencil type billiards with integral of $\min\deg = 12$**:**
these are the billiards with the boldest boundaries

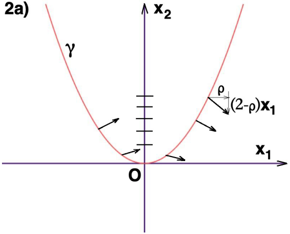


Remark. A dual pencil may induce a projective billiard with integral of $\min\deg =$**12** $<=>$ it consist of conics tangent to **four distinct real lines.**

Example. **Confocal ellipses and hyperbolas** are tangent to four **non-real** complex lines (isotropic lines through foci).
$=>$ For this confocal pencil case of integral of $\min\deg =$ **12** is impossible.

Case deg $=$ **4** is possible: ellipse with $\perp$ line field; vert. line through focus.

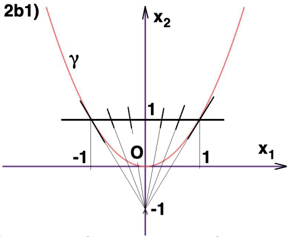
Admissible lines for exotic dual billiards on $\gamma = \{x_2 = x_1^2\}$



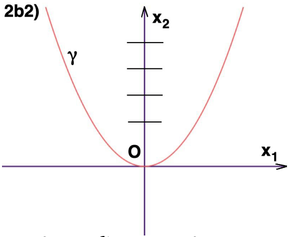
Case 2a): $\mathcal{N}|_\gamma = \mathbb{R}(\rho, 2(\rho - 2))$,
$\rho \in \{2 - \frac{2}{k} \mid k \in \mathbb{N}\}$.

The **only admissible line** is the Ox_2-axis.

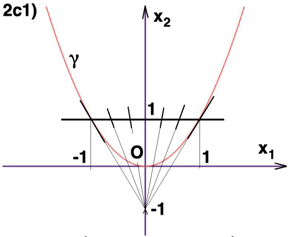
Admissible lines for exotic dual billiard structures



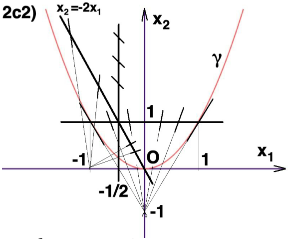
2b1): $\mathcal{N}|_\gamma = \mathbb{R}(5x_1 + 3, 2(x_2 - x_1))$.



2b2): $\mathcal{N}|_\gamma = \mathbb{R}(3x_1, 2x_2 - 4)$.



2c1): $\mathcal{N}|_\gamma = \mathbb{R}(x_2, x_1x_2 - 1)$.

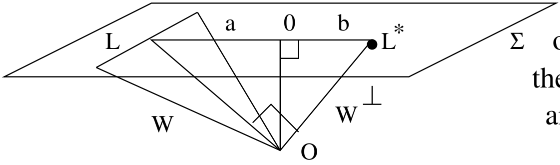


2c2): $\mathcal{N}|_\gamma = \mathbb{R}(2x_1 + 1, x_2 - x_1)$.

Case 2d): $\mathcal{N}|_\gamma = \mathbb{R}(7x_1 + 4, 2x_2 - 4x_1)$. **No admissible lines.**

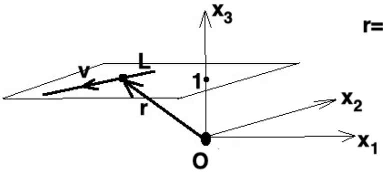
Dual billiards (to projective billiards)

Orthogonal polarity: 2-dimensional subspace $W \subset \mathbb{R}^3_{x_1,x_2,x_3} \mapsto$ line $W^\perp$.
Induces a **projective duality:** $\mathbb{RP}^{2*} = \{\text{lines in } \mathbb{RP}^2\} \mapsto \mathbb{RP}^2 = \{\text{points}\}$,
$L := \pi(W \setminus \{0\}) \mapsto L^* := \pi(W^\perp \setminus \{0\})$. **Preserves incidence relations.**



orthogonal polarity L → L* is the composition of **polar duality** and central symmetry: **ab=1**

The **duality map** $L \mapsto L^*$ is given by the **moment map:**

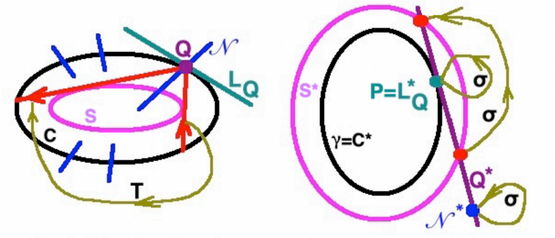


The dual reflection: projective involution σ

Curve $C \subset \mathbb{R}^2 \subset \mathbb{RP}^2$, $\mathcal{N}$ - **transv. line field** on C; $\gamma = C^*$ - dual curve.

The **projective billiard reflection involution** T defined by $\mathcal{N}$, acting on the **space** $\mathbb{RP}^1_Q$ **of lines** through a given point $Q \in C$, is **conjugated via duality** to a **projective involution**

$$\sigma = \sigma_P : Q^* \to Q^* \text{ of the dual line } Q^*; \quad P := L^*_Q.$$



T = billiard reflection of lines through Q

σ=T* is a projective involution Q* —> Q*

The involution $\sigma = \sigma_P : Q^* \to Q^*$ **fixes** P and $\mathcal{N}^* = (\mathcal{N}(Q))^*$, **permutes intersection points** of Q^* with the curve S^* **dual** to caustic S.

A curve $\gamma \subset \mathbb{RP}^2_{[\mathcal{M}_1:\mathcal{M}_2:\mathcal{M}_3]}$, $P \in \gamma$. $L_P :=$ **proj. tangent line** to γ at P.

Definition. A **dual billiard structure** on γ is a **family of projective involutions** $\sigma_P : L_P \to L_P$ fixing P; parametrized by $P \in \gamma$.

A **straightline interval** $J \subset L$ equipped with a projective billiard structure.

Its **dual** is the **point** $Q = L^*$ with a **point dual billiard structure:**$=$ family of **projective** involutions $\ell \to \ell$ of lines ℓ through Q, fixing Q.

Defined for ℓ from an open $U \subset \mathbb{RP}^1$, $U := \{$lines dual to points of $J\}$.

A **dual multibilliard** is a collection of
- nonlinear C^4-smooth curves equipped with dual billiard structures;
- points, called **vertices** equipped with **point** dual billiard structures.

$R(\mathcal{M})$ - integral of proj. billiard flow $<=>$ $R(\mathcal{M})$ is **integral of dual multibilliard:** its restrictions to tangent lines to its curves (to lines through vertices) are invariant under involutions.

Theorem. A proj. billiard formed by conics of real non-degen. dual pencil $\mathcal{C}^*$ and segments of adm. lines incl. k_{ab}, k_{bc} has integral of min deg $=$ **12.**

Proof. Prove the same for the dual multibilliard.

Duality: $\mathcal{C}^* \mapsto$ pencil $\mathcal{C}$ of conics through A, B, C, D.
line $k_{ab} \mapsto$ point K_{AB} with proj. invol. $\sigma_{K_{AB}} : \mathbb{RP}^2 \to \mathbb{RP}^2$.

1) $\forall$ rational integral of the multibilliard is constant on each conic of $\mathcal{C}$.
2) $\sigma_{K_{AB}}$, $\sigma_{K_{BC}}$ permute conics of $\mathcal{C}$: acts on its parameter space $\overline{\mathbb{C}}$.

3) Their actions on $\overline{\mathbb{C}}$ generate **group** $\simeq S_3$**,** since they permute conics $AB \cup CD$, $AD \cup BC$, $AC \cup BD$; permut. gener. S_3.

$=>$ a generic orbit of the group consists of **six conics.**
$=>$ **min deg (int) = 12.**

Main result <=> classif. of rationally integrable dual multibilliards

Goal: 1) show that each curve is a **conic**; 2) describe vertices.

Step 1. Let γ - C^4-curve, dual billiard; $R \circ \sigma_P = R$ on L_P $\forall$ $P \in \gamma$. Then

$R|_\gamma \equiv const$.

Proof. R is an **even** function on L_P: $R \circ \sigma_P = R$, Hence, $(R|_{L_P})'(P) = 0$.
$\Rightarrow$ $\frac{dR}{dv} = 0$ along every vector v tangent to γ $\Rightarrow$ $R|_\gamma \equiv const$.

$\Rightarrow$ γ - **piecewise algebraic**: $\mathbb{C}$-**Zariski closure** $\overline{\gamma}$ – **algebr. curve** in $\mathbb{CP}^2$.

Fix an **irreducible component** $\alpha \subset \overline{\gamma}$. **Goal:** prove that α – **conic.**

Step 2. σ_P extends to a **holom.** family of project. invol. $\sigma_P : L_P \to L_P$,
$P \in \alpha^o := \alpha \setminus \{P_1, \dots, P_\ell\}$, := **sing. holom. dual billiard struct.** on α.

Step 3. Thm A. Let an irred. alg. curve $\alpha \subset \mathbb{CP}^2$ admit a **rationally integrable** sing. holomorphic dual billiard structure. Then α is a **conic.**

Proof. Show that: (i) all the local branches of α are **quadratic**;
(ii) **at most one** singular point is a base point of a **singular branch.**

Theorem B: (i) + (ii) => α is a **conic.**

Step 4. Proj. billiard structure at $\forall$ vertex Q is a **birational involution:**
- either a projective involution $\mathbb{RP}^2 \to \mathbb{RP}^2$ fixing each line through Q;
- or **fixing all points of a regular conic** through Q; Q is **indeterminacy.**

The second type of involutions arises only in **degenerate pencil cases.**

THANK YOU FOR YOUR ATTENTION!

Show that: (i) all the local branches of α are **quadratic**;
(ii) **at most one** singular point is a base point of a **singular branch.**

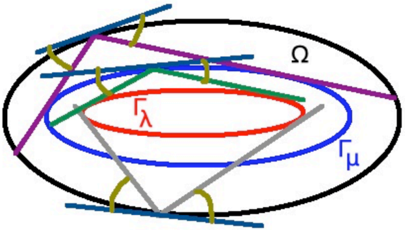
Thm C. Let an irreducible germ b of complex analytic curve at $O \in \mathbb{C}^2$ admit a germ of singular holomorphic dual billiard structure with **meromorphic** first integral R: $R \circ \sigma_P|_{L_P} = R|_{L_P} \ \ \forall P \in \gamma$. Then
1) the germ b is **quadratic;**
2) if b is **singular,** then R is **rational,**
- $R \equiv const$ along the tangent line L_O to α at O.
- $L_O \setminus \{O\}$ is a **regular leaf** of the foliation $R = const$.

Theorem C implies statements (i), (ii).

Birkhoff Conjecture. *Let* Ω – **Birkhoff integrable,** *i.e., a* **neighborhood of** $\partial\Omega$ **in** Ω *is foliated by* **closed caustics.** *Then* $\partial\Omega$ – **ellipse.**

Partial results

H.Poritsky (1950), **E.Amiran** (1988).
Let Ω Birkhoff integrable, Γ_λ – closed caustics, and let each Γ_λ be also a caustic of the **billiard in each bigger** Γ_μ**.** Then Ω is an ellipse.



Open Problem. Extend Poritsky result to **projective billiards**

THANK YOU FOR YOUR ATTENTION!

Thm C. Let an irreducible germ b of complex analytic curve at $O \in \mathbb{C}^2$ admit a germ of singular holomorphic dual billiard structure with **meromorphic** first integral R. Then

1) the germ b is **quadratic;**

2) if b is **singular,** then R is **rational,**

- $R \equiv const$ along the tangent line L_O to α at O.
- $L_O \setminus \{O\}$ is a **regular leaf** of the foliation $R = const$.

Plan of proof of Theorem C.

Idea going back to **S.Tabachnikov's** paper (2008) on outer billiards, used and developed in papers of **M.Bialy** and **A.Mironov** (2016-2017):

work with the **Hessian** of appropriately normalized integral.

$R|_\alpha \equiv const$. Normalize to $R|_\alpha \equiv 0$.

f – **defining polynomial** of α: $\alpha = \{f = 0\}$. $R = f^k g$, $g|_\alpha \not\equiv 0$.

The **normalized integral:**

$$G := R^{\frac{1}{k}} = fg^{\frac{1}{k}}.$$

$R|_\alpha \equiv 0$, $\alpha = \{f = 0\}$, f – **irred. polynomial**. $R = f^k g$, $g|_\alpha \not\equiv 0$.

$$G := R^{\frac{1}{k}} = fg^{\frac{1}{k}}. \qquad G \circ \sigma_P|_{L_P} = G$$

The Hessian:

$$H(G) := \frac{\partial^2 G}{\partial x_1^2}\left(\frac{\partial G}{\partial x_2}\right)^2 - 2\frac{\partial^2 G}{\partial x_1 \partial x_2}\frac{\partial G}{\partial x_2}\frac{\partial G}{\partial x_1} + \frac{\partial^2 G}{\partial x_2^2}\left(\frac{\partial G}{\partial x_1}\right)^2.$$

Key property: $H(G)|_\gamma \neq 0$ outside singular and inflection points of the curve γ and zeros (poles) of the function $g|_\gamma$.

Step 1 of the proof of Thm C. **Differential equation** on $H(G)$ along α.

Fix affine coord. (z, w). In the coord. z on L_P the **invol.** $\sigma_P : L_P \to L_P$ is **conjug.** to $\theta \mapsto -\theta$ by a map $\mathcal{F}_P : \mathbb{C}_\theta \to L_P$; $\mathcal{F}_P : \theta \mapsto z(P) + \frac{\theta}{1+\psi(P)\theta}$.
$G \circ \sigma_P|_{L_P} = G \Rightarrow G(\theta) = G(-\theta) \Rightarrow G(\theta)$ **has no θ^3-term.**

$$\Rightarrow \qquad \frac{dH(G)|_\alpha}{dz}(P) = 6\psi(P)H(G).$$

$$H(G) := \frac{\partial^2 G}{\partial x_1^2}\left(\frac{\partial G}{\partial x_2}\right)^2 - 2\frac{\partial^2 G}{\partial x_1 \partial x_2}\frac{\partial G}{\partial x_2}\frac{\partial G}{\partial x_1} + \frac{\partial^2 G}{\partial x_2^2}\left(\frac{\partial G}{\partial x_1}\right)^2.$$

Fix affine coord. (z, w). Involution $\sigma_P : L_P \to L_P$ acting in coord. z on L_P is **conjugated** to $\theta \mapsto -\theta$ via a map $\mathcal{F}_P : \theta \mapsto z(P) + \frac{\theta}{1+\psi(P)\theta}$.

$$\frac{dH(G)|_\alpha}{dz}(P) = 6\psi(P)H(G).$$

Fix an $O \in \alpha$, a local branch b of α at O. Coord. (z, w) **adapted** to b: centered at O, b is tangent to the z-axis.

Corollary. Let $d \in \mathbb{Q}$ s.t. $H(G)|_b = cz^d(1 + o(1))$, as $z \to 0$. Then

$$\psi(P) = \frac{1}{z(P)}\left(\frac{d}{6} + o(1)\right); \quad \rho := -\frac{d}{3}.$$

=> In coord. $\zeta|_{L_P} := \frac{z}{z(P)}$, $\sigma_P(\zeta) \to \eta_\rho(\zeta) := \frac{(\rho-1)\zeta-(\rho-2)}{\rho\zeta-(\rho-1)}$, as $P \to O$.

$O \in \alpha$, (b, O)-local branch. $H(G)|_b = cz^d(1 + o(1))$.

In coordinate $\zeta := \frac{z}{z(P)}$ on L_P, as $P \to O$,
$\sigma_P(\zeta) \to \eta_\rho(\zeta) := \frac{(\rho-1)\zeta-(\rho-2)}{\rho\zeta-(\rho-1)}$, $\rho = -\frac{d}{3}$, $\rho :=$ the **residue** of σ_P at O.

Param. of b: $t \mapsto (t^{qs}, ct^{ps}(1 + o(1)))$, $1 \leq q < p$, $p, q \in \mathbb{N}$, $(p, q) = 1$.

The **$(p, q; \rho)$-billiard:** the dual billiard structure on $\gamma_{p,q} := \{w^q = z^p\}$ given in the coord. ζ on L_P, $P \in \gamma_{p,q}$ by involutions $\sigma_P(\zeta) := \eta_\rho(\zeta)$.

Theorem 1. Let b be equipped with a singular dual billiard structure having a germ of **meromorphic integral** R. Let $G = R^{\frac{1}{k}}$, $H(G)$, ρ, p, q be as above. Then the $(p, q; \rho)$-billiard is **quasihomogeneously integrable:** has a (p, q)-**quasihomogeneous** rational integral.

Theorem 2. A $(p, q; \rho)$-billiard is **quasihomogeneously integrable,** iff $q = 1$, $p = 2$, **and** $\rho \in \mathcal{M} := \{0, 1, 2, 3, 4\} \cup \{2 \pm \frac{2}{m} \mid m \in \mathbb{N}_{\geq 3}\}$.

Theorem 2. A $(p, q; \rho)$-billiard is **quasihomogeneously integrable,** iff $q = 1$, $p = 2$, **and** $\rho \in \mathcal{M} := \{0, 1, 2, 3, 4\} \cup \{2 \pm \frac{2}{m} \mid m \in \mathbb{N}_{\geq 3}\}$.

Implies **quadraticity** of b and yields a priori possible valies of residue ρ.

Classification of rationally integrable singular dual billiards on conics.

Residue formula: Let a sing. holom. dual billiard on conic have well-defined residues at singularities. Then **the sum or residues $= 4$.**

Rational integrability $=>$ **the residues lie in $\mathcal{M}$** (Thm 2).

$=>$ The only possible residue configurations are:

- $(1, 1, 1, 1)$, $(2, 1, 1)$, $(2, 2)$, $(3, 1)$, $(4, 0)$ (correspond to **pencils of conics**)
- **exotic ones:** $(2 - \frac{2}{m}, 2 + \frac{2}{m})$, $(\frac{3}{2}, \frac{3}{2}, 1)$, $(\frac{4}{3}, \frac{4}{3}, \frac{4}{3})$, $(\frac{4}{3}, \frac{5}{3}, 1)$.

Miracle. All these residue configurations correspond to **complex rationally integrable** dual billiards!