ON THE CRITERIA FOR THE FORMATION OF SINGULARITIES OF A SMOOTH SOLUTION FOR QUASILINEAR HYPERBOLIC SYSTEMS

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Dynamics is Siberia

One quasilinear equation

$$u_t + a(u)u_x = 0, \quad a(u) \in C^1(\mathbb{R}), \quad u\Big|_{t=0} = u_0(x) \in C^2(\mathbb{R})$$

Equations for characteristics:

$$\dot{x} = a(u), \quad \dot{u} = 0, \quad x(0) = x_0, \quad u(0) = u_0(x_0).$$

Extended system for $p = u_x$:

$$p_t + a(u)p_x = -a'(u)p_x^2,$$

$$\dot{p} = -a'(u)p^2,$$

$$p(t) = \frac{u_x(x_0)}{1 + a'(u_0(x_0)u_x(x_0)t}.$$

If $a'(u_0(x_0)u_x(x_0) \geq 0$ for all $x_0 \in \mathbb{R}$, then the solution remains smooth for all $t > 0$.

Hyperbolic system of 2 equations, homogeneous

System

$$U_t + A(U)U_x = 0, \quad U = (u, v)^T,$$

can be written in the Riemann invariants

$$R_1 = R_1(u, v), \quad R_2 = R_2(u, v),$$

$$(R_k)_t + \xi_k(R_1, R_2)(R_k)_x = 0, \quad k = 1, 2.$$

Every Riemann invariant conserves along its characteristic direction,

$$\left(\frac{d(R_k)}{dt}\right)_k = 0, \quad k = 1, 2,$$

Denote $p_k = (R_k)_x$, $k = 1, 2$, and differentiate wrt to x:

$$(p_1)_t + \xi_1(R_1, R_2)(p_1)_x = -\frac{\partial \xi_1(R_1, R_2)}{\partial R_1} p_1^2 - \frac{\partial \xi_1(R_1, R_2)}{\partial R_2} p_1 p_2$$

$$(p_2)_t + \xi_2(R_1, R_2)(p_2)_x = -\frac{\partial \xi_2(R_1, R_2)}{\partial R_1} p_1 p_2 - \frac{\partial \xi_2(R_1, R_2)}{\partial R_2} p_2^2.$$

Introduce

$$\phi_1(R_1, R_2) = \exp\left(-\int_0^{R_2} \frac{\partial \xi_1(R_1, R_2)}{\partial R_2} \frac{1}{\xi_2(R_1, R_2) - \xi_1(R_1, R_2)} dR_2 \right) > 0,$$

$$\phi_2(R_1, R_2) = \exp\left(\int_0^{R_1} \frac{\partial \xi_2(R_1, R_2)}{\partial R_1} \frac{1}{\xi_2(R_1, R_2) - \xi_1(R_1, R_2)} dR_1 \right) > 0.$$

$$\left(\frac{d\,(\phi_1 p_1)}{dt}\right)_1 = (\phi_1 p_1)_t + \xi_1(R_1, R_2)(\phi_1 p_1)_x = -\psi_1(R_1, R_2)(\phi_1 p_1)^2$$

$$\left(\frac{d\,(\phi_2 p_2)}{dt}\right)_2 = (\phi_2 p_2)_t + \xi_2(R_1, R_2)(\phi_2 p_2)_x = -\psi_2(R_1, R_2)(\phi_2 p_2)^2,$$

where $\psi_k = \frac{1}{\phi_k} \frac{\partial \xi_k}{\partial R_k}$.

Theorem

Assume ξ_k to be smooth,

$$\frac{\partial \xi_k}{\partial R_k} > \delta_k > 0, \quad k = 1, 2.$$

Then the solution to the Cauchy problem

$$(R_1(t,x), R_2(t,x))|_{t=0} = (R_1^0(x), R_2^0(x)) \in C^2(\mathbb{R}) \cap B(\mathbb{R}),$$

keeps smoothness for $t > 0$ iff

$$\min_{x\in\mathbb{R}} \min_{k} (R_k^0(x))' \geq 0.$$

Otherwise, the derivatives become unbounded for some finite time.

Non-strictly hyperbolic system

$\mathbf{V} = (V_1, V_2, \dots, V_n)$, $V_i = V_i(t,x)$, $t \geq 0$, $x \in \mathbb{R}$

$$\frac{\partial \mathbf{V}}{\partial t} + A(\mathbf{V}) \frac{\partial \mathbf{V}}{\partial x} = Q\mathbf{V},$$

where $A(\mathbf{V}) = V_1 \mathbb{E}$, $\mathbb{E}$ is $n \times n$ unit matrix,
$Q = Q_{ij}$ is $n \times n$ constant matrix.

Eigenvalues: $\lambda_i(\mathbf{V}) = V_1$,
Eigenvectors: $\mathbf{V}_i = (0, ..., \underbrace{1}_{i-th\, place}, ..., 0)$,
$i = 1, \dots, n$.

The Cauchy problem:

$$\mathbf{V}(0,x) = \mathbf{V}_0(x).$$

The Euler-Poisson equations

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\mathbf{V}) = 0, \quad \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = k\nabla\Phi, \quad \Delta\Phi = n - n_0,$$

n (density),
Φ (a force potential),
$\mathbf{V}$ (velocity)
depend on the time t and the point $x \in \mathbb{R}^{\mathbf{d}}$, $\mathbf{d} \geq 1$,
$n_0 > 0$ is the density background.
Positive or negative k corresponds to the repulsive and attractive force.

The equations of hydrodynamics of "cold" or electron plasma in the non-relativistic approximation in dimensionless quantities:

$$\frac{\partial n}{\partial t} + \operatorname{div}(n\mathbf{V}) = 0, \quad \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\,\mathbf{V} = -\mathbf{E} - [\mathbf{V}\times\mathbf{B}],$$
$$\frac{\partial \mathbf{E}}{\partial t} = n\mathbf{V} + \mathrm{rot}\,\mathbf{B}, \qquad \frac{\partial \mathbf{B}}{\partial t} = -\mathrm{rot}\,\mathbf{E}, \qquad \operatorname{div}\mathbf{B} = 0,$$

n and $\mathbf{V} = (V_1, V_2, V_3)$ are the density and velocity of electrons, $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$ are vectors of electric and magnetic fields. All components of solution depends on $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^3$.

A class of solutions depending only on the radius-vector of point $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, i.e.

$$\mathbf{V} = F(t,r)\mathbf{r}, \quad \mathbf{E} = G(t,r)\mathbf{r}, \quad \mathbf{B} = Q(t,r)\mathbf{r}, \quad n = n(t,r),$$

where $\mathbf{r} = (x_1, x_2, x_3)$.
It implies $\mathbf{B} \equiv 0$, $\mathrm{rot}\,\mathbf{E} = 0$.

Under the assumption that the solution is sufficiently smooth and that the steady-state density n_0 is equal to 1, we get

$$n = 1 - \operatorname{div} \mathbf{E},$$

therefore n can be removed from the system.

The resulting system is

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\, \mathbf{V} = -\mathbf{E}, \quad \frac{\partial \mathbf{E}}{\partial t} + \mathbf{V} \mathrm{div}\, \mathbf{E} = \mathbf{V}. \tag{1}$$

If we introduce the potential Φ such that $\nabla \Phi = -\mathbf{E}$, we can rewrite the system as the Euler-Poisson equations with $n_0 = 1$. Can be considered in any space dimensions **d**.

$$\frac{dV}{dt} = -E, \quad \frac{dE}{dt} = V, \quad \frac{dx}{dt} = V.$$

It imlpies $V^2 + E^2 = V^2(0) + E^2(0) = \mathrm{const}$ along each of the characteristics $x = x(t)$.
For $v = V_x$, $e = E_x$ we get

$$\frac{dv}{dt} = -v^2 - e, \quad \frac{de}{dt} = (1-e)v, \quad e < 1,$$

$$v^2 + 2e - 1 = C(e-1)^2,$$

a second-order curve, its type depends on the sign of

$$\Delta = v^2 + 2e - 1.$$

if $\Delta(0) < 0$, then the phase curves is ellipse, the derivatives remain bounded for $t > 0$. Otherwise, the phase curve is a parabola for $\Delta(0) = 0$ or a hyperbola for $\Delta(0) > 0$, the derivatives become infinite.

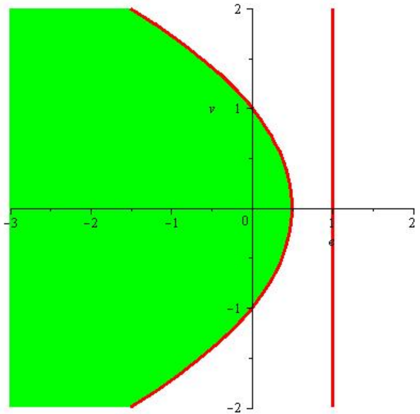


Figure: Criterion of smoothness, $\Delta(0) < 0$

Criterion of a singularity formation

Theorem

For the existence and uniqueness of continuously differentiable $2\pi-$ periodic in time solution (V, E) of

$$\begin{aligned} \frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} &= -E, \quad \frac{\partial E}{\partial t} + V\frac{\partial E}{\partial x} = V, \\ (V, E)|_{t=0} &= (V_0, E_0) \in C^2(\mathbb{R}) \end{aligned}$$

it is necessary and sufficient that inequality

$$\left(V_0'(x)\right)^2 + 2\,E_0'(x) - 1 < 0$$

holds at each point $x \in \mathbb{R}$.
If there exists at least one point x_0 for which the opposite inequality holds, then the derivatives of the solution become infinite in a finite time.

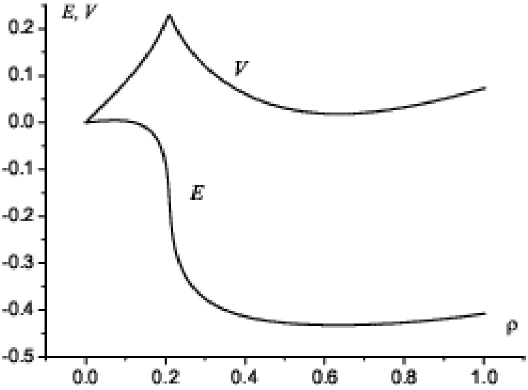


Figure: Spatial distribution of velocity and electric field near the moment of formation of singularity (by E.V.Chizhonkov).

Consider the initial data

$$(\mathbf{V}, \mathbf{E})|_{t=0} = (F_0(r)\mathbf{r}, G_0(r)\mathbf{r}), \quad (F_0(r), G_0(r)) \in C^2(\bar{\mathbb{R}}_+), \quad (2)$$

where $\mathbf{r} = (x_1, \dots, x_\mathbf{d})$, $\quad r = |\mathbf{r}|$, with the physically natural condition $n|_{t=0} > 0$.

Definition

Solution $(\mathbf{V}, \mathbf{E})$ is called an affine solution if it has the form $\mathbf{V} = \mathfrak{V}(t)\mathbf{r}$, $\mathbf{E} = \mathfrak{E}(t)\mathbf{r}$, where $\mathfrak{V}$ and $\mathfrak{E}$ are $(\mathbf{d} \times \mathbf{d})$ matrices.

Definition

Solution $(\mathbf{V}, \mathbf{E})$ is called a simple wave if it has the form $\mathbf{V} = F(t, r)\mathbf{r}$, $\mathbf{E} = G(t, r)\mathbf{r}$, where $F(t, r)$ and $G(t, r)$ are functionally dependent.

Theorem

The solution of the Cauchy problem (1), (2) *for* $\mathbf{d} \geq 2$, $\mathbf{d} \neq 4$, *blows up in a finite time for all initial data, possibly except for the data, corresponding to simple waves.*

If the solution is globally smooth in time, then it is either affine or tends in the C^1*-norm to an affine solution as* $t \to \infty$.

Explicit solutions along characteristics

F and G satisfy the following Cauchy problem:

$$\frac{\partial G}{\partial t} + Fr\frac{\partial G}{\partial r} = F - \mathbf{d}FG, \quad \frac{\partial F}{\partial t} + Fr\frac{\partial F}{\partial r} = -F^2 - G,$$

$$(F(0,r), G(0,r)) = (F_0(r), G_0(r)), \quad (F_0(r), G_0(r)) \in C^2(\bar{\mathbb{R}}_+).$$

Along the characteristic

$$\dot{r} = Fr,$$

starting from the point $r_0 \in [0, \infty)$ system (3) takes the form

$$\dot{G} = F - \mathbf{d}FG, \qquad \dot{F} = -F^2 - G, \tag{3}$$

Thus,

$$\frac{1}{2}\frac{dF^2}{dG} = -\frac{F^2 + G}{1 - \mathbf{d}G},$$

which is linear with respect to F^2 and can be explicitly integrated.

For $\mathbf{d} = 2$

$$2F^2 = (2G-1)\ln|1-2G| + C_2(2G-1) - 1,$$
$$C_2 = \frac{1+2F^2(0,r_0)}{2G(0,r_0)-1} - \ln|1-2G(0,r_0)|,$$

for $\mathbf{d} = 1$ and $\mathbf{d} \geq 3$

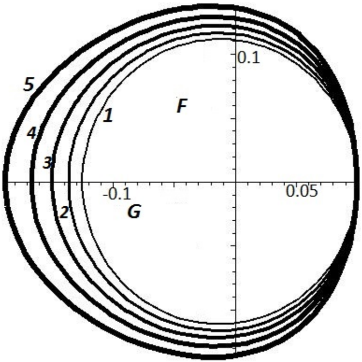
$$F^2 = \frac{2G-1}{\mathbf{d}-2} + C_{\mathbf{d}}|1-\mathbf{d}G|^{\frac{2}{\mathbf{d}}},$$
$$C_{\mathbf{d}} = \frac{1-2G(0,r_0)+(\mathbf{d}-2)F^2(0,r_0)}{(\mathbf{d}-2)|1-\mathbf{d}G(0,r_0)|^{\frac{2}{\mathbf{d}}}}.$$

Lemma

The period of revolution on the phase curve depends on $\mathbf{d}$ *and the starting point of trajectory, except for* $\mathbf{d} = 1$ *and* $\mathbf{d} = 4$, *where* $T = 2\pi$. *In the other cases the following asymptotics holds for the deviation of order* ε *from the origin:*

$$T = 2\pi\left(1 + \frac{1}{24}(\mathbf{d} - 1)(\mathbf{d} - 4)\varepsilon^2 + o(\varepsilon^2)\right), \quad \varepsilon \to 0,$$

i.e. for $\mathbf{d} \in (1, 4)$ *the period is less that* 2π, *for* $\mathbf{d} > 4$ *the period is greater that* 2π.



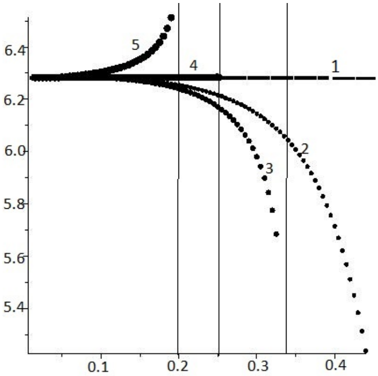


Figure: Phase portrait on the plane (F, G) for $G_+ = 0.1$ (left) and the dependence of the period on G_+ (right) for $\mathbf{d} = 1, 2, 3, 4, 5$.

Denote $\mathcal{D} = \operatorname{div} \mathbf{V}$, $\lambda = \operatorname{div} \mathbf{E}$, $J_{ij} = \partial_{x_i} V_i \, \partial_{x_j} V_j - \partial_{x_j} V_i \, \partial_{x_i} V_j$, $i, j = 1, \dots, \mathbf{d}$.
System (1) implies

$$\frac{\partial \mathcal{D}}{\partial t} + (\mathbf{V} \cdot \nabla \mathcal{D}) = -\mathcal{D}^2 + 2J - \lambda, \qquad \frac{\partial \lambda}{\partial t} + (\mathbf{V} \cdot \nabla \lambda) = \mathcal{D}(1 - \lambda),$$

along the characteristic curve,

$$\dot{\mathcal{D}} = -\mathcal{D}^2 + 2\,(\mathbf{d} - 1)\,F\,\mathcal{D} - \lambda - (\mathbf{d} - 1)\,\mathbf{d}\,F^2, \qquad \dot{\lambda} = \mathcal{D}\,(1 - \lambda).$$

New variables: $u = \mathcal{D} - \mathbf{d}\,F$, $v = \lambda - \mathbf{d}\,G$:

$$\dot{u} = -u^2 - 2\,F\,u - v, \quad \dot{v} = -u\,v + (1 - \mathbf{d}\,G)\,u - \mathbf{d}\,F\,v. \qquad (4)$$

Linearization and the Radon lemma

Theorem (The Radon lemma)

A matrix Riccati equation

$$\dot{W} = M_{21}(t) + M_{22}(t)W - WM_{11}(t) - WM_{12}(t)W,$$

is equivalent to the homogeneous linear matrix equation

$$\dot{Y} = M(t)Y, \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

Let on some interval $\mathcal{J} \in \mathbb{R}$ *the matrix-function* $Y(t) = \begin{pmatrix} \mathfrak{Q}(t) \\ \mathfrak{P}(t) \end{pmatrix}$ *be a solution with the initial data*

$$Y(0) = \begin{pmatrix} I \\ W_0 \end{pmatrix}$$

Then $W(t) = \mathfrak{P}(t)\mathfrak{Q}^{-1}(t)$ *is the solution with* $W(0) = W_0$ *on* $\mathcal{J}$.

System (4) can be written as a matrix Riccati equation

$$W = \begin{pmatrix} u \\ v \end{pmatrix}, \quad M_{11} = \begin{pmatrix} 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 1 & 0 \end{pmatrix},$$

$$M_{21} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} -2\,F & -1 \\ 1 - d\,G & -d\,F \end{pmatrix}.$$

Thus, we obtain the Cauchy problem

$$\begin{pmatrix} \dot q \\ \dot p_1 \\ \dot p_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -2\,F & -1 \\ 0 & 1 - d\,G & -d\,F \end{pmatrix} \begin{pmatrix} q \\ p_1 \\ p_2 \end{pmatrix}, \qquad \begin{pmatrix} q \\ p_1 \\ p_2 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ u_0 \\ v_0 \end{pmatrix},$$

with periodical coefficients, known from (3).

The standard change of the variable $p_1(t) = P(t)\, e^{-\frac{\mathbf{d}+2}{2}\int\limits_0^t F(\tau)\,d\tau}$ reduces to

$$\ddot{P} + QP = 0, \quad Q = 1 - \frac{\mathbf{d}+2}{2} G - \frac{1}{4}(\mathbf{d}-2)(\mathbf{d}-4)F^2, \qquad (5)$$

$$q(t) = 1 + \int\limits_0^t p_1(\tau)\,d\tau = 1 + \int\limits_0^t P(\xi)\, e^{-\frac{d+2}{2}\int\limits_0^\xi F(\tau)\,d\tau}\, d\xi. \qquad (6)$$

Solution of (4) blows up if and only if $q(t)$ vanishes at some point t_*, $0 < t_* < \infty$.

1. $Q(t)$ is periodic with period T, $Q(t) = Q(-t)$, therefore (5) has solutions

$$e^{\mu t}\mathcal{P}(t), \quad e^{-\mu t}\mathcal{P}(-t),$$

$\mathcal{P}$ is T-periodic, which can be taken as a fundamental system provided that μ is real.

2. Suppose $z(t)$ is a solution of (5) with initial conditions $z(0) = 1$, $z'(0) = 0$. Then

$$z(t) = \frac{1}{2\mathcal{P}(0)} \left(e^{\mu t}\mathcal{P}(t) + e^{-\mu t}\mathcal{P}(-t) \right),$$

Thus,

$$\cosh \mu T = z(T). \tag{7}$$

Unboundedness takes place for $|\cosh \mu T| > 1$.

If $z(T) > 1$, then $\mu \in \mathbb{R}$, and the general solution of (5) has the form

$$P = C_+ e^{\mu t}\mathcal{P}(t) + C_- e^{-\mu t}\mathcal{P}(-t).$$

Thus, for an arbitrary choice of the data P is unbounded and q oscillates with a growing amplitude.

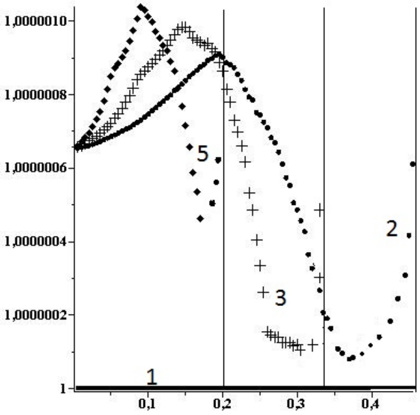


Figure: Dependence of $e^{\mu T}$ on G_+ for $\mathbf{d} = 1$ (and $\mathbf{d} = 4$ till $G_+ = 0.25$), solid line, $\mathbf{d} = 2$, solid circles, $\mathbf{d} = 3$, crosses, $\mathbf{d} = 5$, solid diamonds.

Simple waves

Theorem predicts the existence of non-affine solutions with special initial data (2), which are globally smooth and tends to an affine solution as $t \to \infty$.

To construct them, we look for simple waves $F = F(G)$. System (3) reduces to one equation

$$\frac{\partial G}{\partial t} + F(G) r \frac{\partial G}{\partial r} = F(G)(1 - \mathbf{d} G),$$

with $F(G)$ can be found from previous formulas, the periods of oscillations are equal for all characteristics. If we fix $C_{\mathbf{d}}$, we obtain the relation between G and F in this special kind of solution, and the corresponding initial data.

If G_r does not blow up, it tends to zero as $t \to \infty$ (E, V tend to the affine solution).

Thank you!