On quadratic conservation laws for the Newton equations of motion in Euclidean space

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Fri Mar 10 12:00	Leonid Rybnikov	Bethe subalgebras and Kirillov-Reshetikhin crystals	
Fri Mar 17 12:00	Grigori Olshanski	TBA	
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We will consider equations of motion in the phase space $T^*\mathbb{R}^n$

$$\dot{q}_i = p_i\,, \qquad \dot{p}_i = -\frac{\partial}{\partial q_i} V(q)$$

defined by Hamilton function which is polynomial of second order in momenta

$$H_A = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V_A(q)\,.$$

Linear integrals of motion I_k can be easily found using modern computer software or Noether's symmetry theory.

Thus, first really "hard" problem is a search of polynomial of second order in momenta

$$H_B = \sum_{ij} B^{ij}(q)\, p_i p_j + V_B(q)$$

in the involution $\{H_A, H_B\} = 0$ and with $V_{A,B} \neq 0$.

Let A and B be non-degenerate symmetric second-order tensor fields on Euclidean space $\mathbb{R}^n$. If the Schouten bracket between them is zero

$$[\![A, B]\!] = 0$$

and the eigenvalue problem

$$(A - \lambda B)\psi = 0$$

has n simple real eigenvalues and normal eigenvectors, then A and B generate a n-dimensional linear space of second-order tensor fields, all in involution and with the common eigenvectors.
It allows us to calculate n independent functions on the cotangent bundle $T^*\mathbb{R}^n$

$$T_1 = \sum_{ij} A^{ij} p_i p_j\,, \quad T_2 = \sum_{ij} B^{ij} p_i p_j\,, \quad T_3 = \sum_{ij} K_3^{ij} p_i p_j, \quad \ldots, \quad T_n = \sum_{ij} K_n^{ij} p_i p_j$$

in the involution

$$\{T_i, T_j\} = 0\,.$$

By adding suitable potentials (without changing T_i!!!)

$$H_1=T_1+V_1(q_1,\ldots,q_n)\,,\quad H_2=T_2+V_2(q_1,\ldots,q_n),\quad \ldots,\quad H_n=T_n+V_n(q_1,\ldots,q_n)$$

we obtain the n-dimensional space of first integrals in involution.

Thus, two second-order tensors A and B define the completely integrable system, if they satisfy a set of conditions in $\mathbb{R}^n$ which can be verified without an explicit calculation of all the integrals of motion.

Levi-Civita, “Sul le trasformazioni del le equazioni dinamiche”,Ann. di Mat., serie 2a, 24 (1896), 255-300.

L. P. Eisenhart, “Separable systems of Stäckel”, Ann. Math., 35:2 (1934), 284-305.

J. Haantjes, “On X_m-forming sets of eigenvectors”, Indag. Mathematicae, 17 (1955), 158-162.

Horwood J., McLenaghan R., Smirnov R., ”Invariant classification of orthogonally separable Hamiltonian systems in Euclidean space”, Commun. Math. Phys., 259, (2005), 679-709.

If we take two second order polynomials in momenta

$$H_1 = T_1 + V_1(q_1, \dots, q_n), \qquad H_2 = T_2 + V_2(q_1, \dots, q_n)$$

we can divide one equation

$$\{H_1, H_2\} = 0$$

on two equations

$$\{T_1, T_2\} = 0, \qquad \{T_1, V_2\} = \{T_2, V_1\}$$

using Euler's theorem on homogeneous functions.

Following to Levi-Chivita, Darboux, Liouville and Stäckel we can divide the problem on two independent parts:

- study integrable geodesic flow;
- add suitable potentials.

In fact "can" was replaced by "must" after these classical works.

Main proposition

In order to obtain new quadratic conservation laws for integrable Newton's equations of motion in Euclidean space

$$\ddot{q}_i = F(q_1, \dots, q_n)\,,$$

we need to abandon this very convenient, but already completely studied, sequence of calculations.

In 2015-2022 we directly solved a system of "indivisible" equations

$$\{T_1, T_2\} = 0\,, \qquad \{T_1, V_2\} = \{T_2, V_1\}$$

and found a number of new integrable and superintegrable systems in T^*R^n, $n > 3$, see references in

Tsiganov A.V., On integrable systems outside Nijenhuis and Haantjes geometry, J. Geom. Phys, v.178, 104571, 2022.

Today we want to discuss second-order tensors A and B in the Euclidean space $\mathbb{R}^n$, corresponding to quadratic conservation laws that arise in the study of Hamiltonian integrable systems associated with a hierarchy of multicomponent nonlinear Schrödinger equations (NLS) and symmetric space theory.

In this case, the corresponding spectral problem

$$(A - \lambda B)\psi = 0$$

does not have the necessary set of simple real eigenvalues and normal eigenvectors, which does not prevent integrability by Liouville theorem at all.

Integrability vs separation of variables

In fact the study of this spectral problem is necessary not for integrability, but only for the separation of variables by means of a narrow simplest class of point canonical transformations.

Although a number of explicit expressions for the Hamiltonians

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(q_1, \dots, q_n)$$

corresponding to hermitian symmetric spaces of the type A.III, BD.I, C.I, and D.III in Cartan's classification, have been reproduced in various textbooks

Perelomov A.M., Integrable systems of classical mechanics and Lie algebras, Springer Basel AG, 1989.

Trofimov, V. V., Fomenko, A. T., Geometric and algebraic mechanisms of the integrability of Hamiltonian systems on homogeneous spaces and Lie algebras, In: Dynamical Systems VII (Eds.: V. I. Arnold, S. P. Novikov), Springer, 1994.

Reyman A.G., Semenov-Tian-Shansky M.A., Integrable Systems, RCD, Moscow-Izhevsk, 2003.

The corresponding polynomial integrals of motion have not been studied et al.

Newton's equations of motion

$$\ddot{q}^{\alpha} = \sum_{\beta,\gamma,\delta} \mathcal{R}^{\alpha}_{\beta,\gamma,-\delta} q^{\beta} q^{\gamma} q_{\delta} - \omega_{\alpha} q_{\alpha}, \qquad \alpha,\beta,\gamma,\delta = 1,\dots,N$$

and Hamiltonian

$$H = \frac{1}{2}\sum_{\alpha} \mathrm{g}^{\alpha,-\alpha} p_{\alpha}^2 - \frac{1}{4}\sum_{\alpha,\beta,\gamma,\delta} \mathcal{R}_{-\alpha,\beta,\gamma,-\delta} q^{\alpha} q^{\beta} q^{\gamma} q^{\delta} + \frac{1}{2}\sum_{\alpha} \omega_{\alpha} (q^{\alpha})^2$$

was studied by Fordy, Kulish (1983), Fordy, Woiciechowski, Marshall (1986) and Reiman (1986).

- $q^{\alpha} = q_i$ - Cartesian coordinates in $\mathbb{R}^N$;
- $p_{\alpha} = p_i$ - momenta in $T^*\mathbb{R}^N$;
- $\mathrm{g}^{\alpha,-\alpha}$ - constant metric in $\mathbb{R}^N$;
- $R^{\alpha}_{\beta,\gamma,-\delta}$ - constant curvature tensor on symm. space;
- ω_{α} - parameters ("frequencies").

In this case, $A = \mathtt{g}$ is metric in Euclidean space and $B = K$ is a Killing tensor, which satisfies the Killing equation

$$\nabla_i K^{jk} + \nabla_j K^{ki} + \nabla_k K^{ij} = 0,$$

where ∇ is the Levi-Civita connection of $\mathtt{g}$.

In Euclidean space, the generic Killing tensor of valency two is given by

$$K = \sum_{i,j} a_{ij} X_i \circ X_j + \sum_{i,j,k} b_{ijk} X_i \circ X_{j,k} + \sum_{i,j,k,m} c_{ijkm} X_{i,j} \circ X_{k,m}\,,$$

where

$$X_i = \partial_i \qquad X_{i,j} = q_i X_j - q_j X_i\,, \qquad \partial_k = \frac{\partial}{\partial q_k}$$

is a basis of translations and rotations and $\circ$ denotes symmetric product.

We can find all the Killing tensors of valency two related to Hamiltonian $H = T + V$ solving the equation

$$d\,(KdV) = 0\,, \qquad (*)$$

which means that 1-form KdV is an exact

$$(KdV)_\alpha = g_{\alpha,\beta} K^{\beta,\gamma} \partial_\gamma V\,.$$

Substituting generic solution K and potential

$$V = \frac{1}{4} \sum_{\alpha,\beta,\gamma,\delta} \mathcal{R}_{-\alpha,\beta,\gamma,-\delta}\, q^\alpha q^\beta q^\gamma q^\delta - \frac{1}{2} \sum_\alpha \omega_\alpha \,(q^\alpha)^2$$

into (*) we obtain a linear system of equations for coefficients a_{ij}, b_{ijk} and c_{ijkm} solvable on a computer.

Substituting obtained K and unknown V into (*) one gets generalization of potential

$$V_g = c_4 V + c_3 V_3 + c_2 V_2 + C_1 V_1\,.$$

Relation of these Hamiltonian systems with the generalised multicomponent NLS hierarchy gives the Lax matrix

$$L(\lambda) =\lambda^2\mathcal{A} + \lambda\sum_\alpha q^\alpha(e_\alpha - e_{-\alpha}) - \frac{1}{a}\sum_\alpha g^{\alpha,-\alpha}p_\alpha(e_\alpha + e_{-\alpha})$$
$$+\frac{1}{a}\sum_{\alpha,\beta} q_\alpha q_\beta[e_\alpha, e_{-\beta}] + \Lambda\,.$$

- $\mathcal{A}$ is an element of the Lie algebra $\mathfrak{g}$ defining Cartan involution σ and decomposition

$$\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}_-\oplus\mathfrak{m}_-\,,\quad [\mathcal{A},X]=0\,,\qquad [\mathcal{A},X]=\pm aX$$

- e_α and $e_{-\alpha}$ are Weil generators in $\mathfrak{m}_+$ and $\mathfrak{m}_-$;
- Λ - constant matrix defined by "frequencies" ω_α;
- metric and curvature tensor

$$g^{\alpha,\beta} = \langle e_\alpha, e_\beta\rangle\,,\quad \mathcal{R}_{\alpha,\beta,\gamma,\delta} = \langle[e_\alpha,e_\beta],[e_\gamma,e_\delta]\rangle.$$

There are Hamiltonian in so-called natural form

$$H = \frac{1}{4}\mathrm{tr}\, L^2(\lambda)\Big|_{\lambda=0} = T + V$$
$$= \frac{1}{2}\sum_{\alpha} \mathrm{g}^{\alpha,-\alpha} p_{\alpha}^2 - \frac{1}{4}\sum_{\alpha,\beta,\gamma,\delta} \mathcal{R}_{-\alpha,\beta,\gamma,-\delta}\, q^{\alpha} q^{\beta} q^{\gamma} q^{\delta} + \frac{1}{2}\sum_{\alpha} \omega_{\alpha}\,(q^{\alpha})^2$$

and integral of motion

$$G = \mathrm{tr}\ L^4(\lambda)\Big|_{\lambda=0} = \sum_{\alpha,\beta,\gamma,\delta} \mathcal{R}_{-\alpha,\beta,\gamma,-\delta}\, p^{\alpha} p^{\beta} p^{\gamma} p^{\delta}$$
$$+ \sum_{\alpha,\beta} S^{\alpha,\beta}(q) p_{\alpha} p_{\beta} + W(q)\,,$$

which is polynomial of fourth order in momenta, and other independent polynomials of order two, four, six, eight, etc.

We consider only second order conservation laws.

Symmetric spaces of A.III type

Consider Newton's equations of motion in Euclidean space $\mathbb{R}^{mn}$ associated with the Riemannian pair

$$SU(m+n)/S\big(U(m)\times U(n)\big)\,,\qquad 1<m\le n\,,\qquad n+m\ge 4\,.$$

The typical representation of $su(m+n)$ is a set of $(m+n)\times(m+n)$ matrices with an obvious block-matrix structure related to Cartan decomposition

$$\mathfrak{g}\equiv\mathfrak{k}\oplus\mathfrak{m},\qquad \mathfrak{k}=s\big(u(m)\oplus u(n)\big)\,.$$

Here $\mathfrak{k}$ consists of block-diagonal matrices, while the linear space $\mathfrak{m}$ is spanned by block-off-diagonal matrices:

$$\mathfrak{k}\simeq\left(\begin{array}{cc} u(m) & 0\\ 0 & u(n)\end{array}\right),\qquad \mathfrak{m}\simeq\left(\begin{array}{cc} 0 & M\\ M^* & 0\end{array}\right).$$

Lax matrix reads as

$$L(\lambda)=\begin{pmatrix} -2\lambda^2 I_m+QQ^T+a & 0 \\ 0 & 2\lambda^2 I_n-Q^TQ+b \end{pmatrix}+\begin{pmatrix} 0 & P-2\mathrm{i}\lambda Q \\ P^T+2\mathrm{i}\lambda Q^T & 0 \end{pmatrix},$$

where I_m and I_n are the $m\times m$ and $n\times n$ unit matrices, a and b are diagonal matrices depending on m real numbers a_k and n real numbers parameters b_i

$$a=\mathrm{diag}_m(a_1,\ldots,a_m)\,,\qquad b=\mathrm{diag}_n(b_1,\ldots,b_n)\,,\qquad a_i,b_i\in\mathbb{R}\,,$$

and T means matrix transposition, $\mathrm{i}=\sqrt{-1}$, and $m\times n$ matrices Q and P are

$$Q=\begin{pmatrix} q_1 & q_2 & \cdots & q_n \\ q_{n+1} & q_{n+2} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n(m-1)+1} & q_{n(m-1)+2} & \cdots & q_{mn} \end{pmatrix}\qquad P=\begin{pmatrix} p_1 & p_2 & \cdots & p_n \\ p_{n+1} & p_{n+2} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n(m-1)+1} & p_{n(m-1)+2} & \cdots & p_{mn} \end{pmatrix}$$

When $a_i = 0$ and $b_i = 0$ characteristic polynomial of the Lax matrix

$$\tau(z, \lambda) = \det\left(z\, I - L(\lambda)\right),$$

contains less than mn independent integrals of motion in involution. Thus, similar to the full Toda lattice we have to use other tensor invariants of the Lax matrix to prove integrability by Liouville theorem.

When $a_i \neq 0$ and $b_i \neq 0$, there are two basic sets of integrals of motion obtained from the characteristic polynomial of the Lax matrix which are associated with $so(m)$ and $so(n)$, respectively.

Because

$$\{\tau(x, \lambda), \tau(y, \mu)\} = 0,$$

all these integrals of motion are in the involution for each other.

First basis of integrals of motion

The m residues of the function

$$\Delta_1(z,\lambda)=\frac{\tau(z,\lambda)}{\prod_{i=1}^m(z-a_i+2\lambda^2)}$$

at $z=a_i-2\lambda^2$ generate mn independent integrals of motion $h_i^{(2\ell)}$

$$\mathrm{Res}\,\Delta_1(z,\lambda)\big|_{z=a_i-2\lambda^2}=\sum_{k=0}^{n-1}\lambda^{2k}h_i^{\left(2(n-k)\right)}\,,\qquad i=1,\ldots,m,$$

which are polynomials of degree at most $2m$ since we take $m\leq n$.

- m quadratic polynomials in momenta $h_1^{(2)},\ldots,h_m^{(2)}$;
- m quartic polynomials in momenta $h_1^{(4)},\ldots,h_m^{(4)}$;
- m sextic polynomials in momenta $h_1^{(6)},\ldots,h_m^{(6)}$;
- ....
- m polynomials of $2m$-order in momenta $h_1^{(2m)},\ldots,h_m^{(2m)}$

and $m(n-m)$ remaining polynomials of $2m$-order in momenta.

Second basis of integrals of motion

The n residues of the function

$$\Delta_2(z,\lambda)=\frac{\tau(z,\lambda)}{\prod_{i=1}^n(z-b_i-2\lambda^2)}$$

at $z=b_i+2\lambda^2$ generate mn independent integrals of motion $H_i^{(2\ell)}$

$$\operatorname{Res}\Delta_2(z,\lambda)\big|_{z=b_i+2\lambda^2}=\sum_{k=0}^{m-1}\lambda^{2k}H_i^{\bigl(2(m-k)\bigr)}\,,\qquad i=1,\ldots,n$$

which are polynomials of order 2ℓ in momenta. So, there are

- n quadratic polynomials in momenta $H_1^{(2)},\ldots,H_n^{(2)}$;
- n quartic polynomials in momenta $H_1^{(4)},\ldots,H_n^{(4)}$;
- n sextic polynomials in momenta $H_1^{(6)},\ldots,H_n^{(6)}$;
- ....
- n polynomials of $2m$-order in momenta $H_1^{(2m)},\ldots,H_n^{(2m)}$.

First set of quadratic integrals of motion

Polynomials of the second order in momenta have the following form

$$h_i^{(2)} = \sum_{k\neq i}^{m} \frac{M_{ik}^2}{a_i - a_k} + t_i(p) + v_i(q)\,,$$

where functions

$$M_{ik} = \sum^{n} J_{j\ell}\,, \qquad J_{j\ell} = q_j p_\ell - q_\ell p_j\,,$$

constitute realization of Lie algebra $so^*(m)$ associated with compositions of n simple rotations in $\mathbb{R}^{mn}$.

Functions $t_i(p)$ correspond to compositions of the n translations

$$t_i(p) = \sum^{n} p_\ell^2\,,$$

and $v_i(q)$ are polynomials of the fourth order in coordinates q_i.

Second set of quadratic integrals of motion

Polynomials of the second order in momenta have the following form

$$H_i^{(2)} = \sum_{k\neq i}^{n} \frac{N_{ik}^2}{b_i - b_k} + T_i(p) + U_i(q)\,,$$

where functions

$$N_{ik} = \sum^{m} J_{j\ell}\,, \qquad J_{j\ell} = q_j p_\ell - q_\ell p_j\,,$$

form realization of $so^*(n)$ via compositions of m simple rotations in $\mathbb{R}^{mn}$.

Functions $T_i(p)$ correspond to compositions of the m translations

$$T_i(p) = \sum_{\ell}^{m} p_\ell^2\,,$$

and $U_i(q)$ are polynomials of the fourth order in coordinates.

Summing up, we have $n+m-1$ quadratic integrals of motion

$$h_1^{(2)}+\cdots+h_m^{(2)}=2H=H_1^{(2)}+\cdots+H_n^{(2)}\,,$$

associated with the linear combinations of rotations, which realise $so^*(m)$ and $so^*(n)$, and with the linear combinations of translations.

Proposition

Associated with A.III hermitian symmetric space Newton's equations of motion

$$\ddot{q}^\alpha=\sum_{\beta,\gamma,\delta}\mathcal{R}^\alpha_{\beta,\gamma,-\delta}q^\beta q^\gamma q_\delta-\omega_\alpha q_\alpha,\qquad \alpha,\beta,\gamma,\delta=1,\ldots,N$$

in Euclidean space R^{mn} have only $n+m-1$ independent quadratic integrals of motion in involution.

We have not a proof, only examples in Porubov, Tsiganov, "Second order Killing tensors related to symmetric spaces" arXiv:2301.02774

Euclidean space $\mathbb{R}^4$

Lax matrix

$$L(\lambda)=\begin{pmatrix} q_1^2+q_2^2+a_1-2\lambda^2 & q_1q_3+q_2q_4 & p_1-2i\lambda q_1 & p_2-2i\lambda q_2\\ q_1q_3+q_2q_4 & q_3^2+q_4^2+a_2-2\lambda^2 & p_3-2i\lambda q_3 & p_4-2i\lambda q_4\\ p_1-2i\lambda q_1 & p_3-2i\lambda q_3 & b_1-q_1^2-q_3^2+2\lambda^2 & -q_1q_2-q_3q_4\\ p_2-2i\lambda q_2 & p_4-2i\lambda q_4 & -q_1q_2-q_3q_4 & b_2-q_2^2-q_4^2+2\lambda^2 \end{pmatrix}.$$

Hamiltonian

$$H=\sum_{i=1}^{4}\frac{p_i^2}{2}++\frac{1}{2}(q_1^2+q_2^2)^2+\frac{1}{2}(q_3^2+q_4^2)^2+(q_1q_3+q_2q_4)^2$$

$$+\frac{a_1-b_1}{2}\,q_1^2+\frac{a_1-b_2}{2}\,q_2^2+\frac{a_2-b_1}{2}\,q_3^2+\frac{a_2-b_2}{2}\,q_4^2\,,$$

Spectral curve of the Lax matrix $L(\lambda)$ is a non-hyperelliptic curve, its genus is equal to five $g=5$.

Second order integrals of motion

$$f_1 = -\frac{M_{12}^2}{a_1 - a_2} + p_1^2 + p_2^2 + v_1 \qquad f_2 = \frac{M_{12}^2}{a_1 - a_2} + p_3^2 + p_4^2 + v_2,$$

$$F_1 = \quad \frac{N_{12}^2}{b_1 - b_2} + p_1^2 + p_3^2 + V_1 \qquad F_2 = -\frac{N_{12}^2}{b_1 - b_2} + p_2^2 + p_4^2 + V_2 .$$

Here functions

$$M_{12} = J_{1,3} + J_{2,4} = (q_1 p_3 - q_3 p_1) + (q_2 p_4 - q_4 p_2) .$$

$$N_{12} = J_{1,2} + J_{3,4} = (q_1 p_2 - q_2 p_1) + (q_3 p_4 - p_3 q_4) ,$$

describe two independent double rotations in $\mathbb{R}^4$

$$\{M_{12}, N_{12}\} = 0 .$$

In $\mathbb{R}^4$ there are double isoclinic or equiangular rotations or Clifford displacements, which can be associated with the left- and right-multiplication of quaternion. They are classical objects in the geometry of the fourth-dimensional Euclidean space and Clifford algebras.

For integrable systems associated with $SU(4)/S(U(2)\times U(2))$

$$f_1 + f_2 = 2H = F_1 + F_2$$

we have quadratic integrals of motion associated with these double rotations and one integral of fourth order

$$G = \mathrm{tr} L^4(\lambda = 0) = \sum_{\alpha,\beta,\gamma,\delta} \mathcal{R}_{-\alpha,\beta,\gamma,-\delta}\, p^\alpha p^\beta p^\gamma p^\delta + \cdots$$

How to get this integral of motion in the framework of Euclidean geometry or the Clifford algebras theory?

Euclidean space $\mathbb{R}^9$

For the symmetric space $SU(6)/S(U(3)\times U(3))$ second order integrals of motion have the following form

$$
\begin{aligned}
f_1 &= \frac{M_{12}^2}{a_1-a_2}+\frac{M_{13}^2}{a_1-a_3}-p_1^2-p_2^2-p_3^2+v_1\\
f_2 &= \frac{M_{21}^2}{a_2-a_1}+\frac{M_{23}^2}{a_2-a_3}-p_4^2-p_5^2-p_6^2+v_2\\
f_3 &= \frac{M_{31}^2}{a_3-a_1}+\frac{M_{32}^2}{a_3-a_2}-p_7^2-p_8^2-p_9^2+v_3\\
F_1 &= -\frac{N_{12}^2}{b_1-b_2}-\frac{N_{13}^2}{b_1-b_3}-p_1^2-p_4^2-p_7^2+V_1\\
F_2 &= -\frac{N_{21}^2}{b_2-b_1}-\frac{N_{23}^2}{b_2-b_3}-p_2^2-p_5^2-p_8^2+V_2\\
F_3 &= -\frac{N_{31}^2}{b_3-b_1}-\frac{N_{32}^2}{b_3-b_2}-p_3^2-p_6^2-p_9^2+V_3
\end{aligned}
$$

Associated with triple rotations functions are

$$\begin{aligned}
M_{12} =& (q_1 p_4 - p_1 q_4) + (q_2 p_5 - p_2 q_5) + (q_3 p_6 - p_3 q_6)\,,\\
M_{13} =& (q_1 p_7 - p_1 q_7) + (q_2 p_8 - p_2 q_8) + (q_3 p_9 - p_3 q_9)\,,\\
M_{23} =& (q_4 p_7 - p_4 q_7) + (q_5 p_8 - p_5 q_8) + (q_6 p_9 - p_6 q_9)\,,\\
N_{12} =& (q_1 p_2 - p_1 q_2) + (q_4 p_5 - p_4 q_5) + (q_7 p_8 - p_7 q_8)\,,\\
N_{13} =& (q_1 p_3 - p_1 q_3) + (q_4 p_6 - p_4 q_6) + (q_7 p_9 - p_7 q_9)\,,\\
N_{23} =& (q_2 p_3 - p_2 q_3) + (q_5 p_6 - p_5 q_6) + (q_8 p_9 - p_8 q_9)\,.
\end{aligned}$$

Two independent realisations of $so^*(3)$ with Poisson brackets

$$\{M_{12}, M_{13}\} = M_{23}\,,\qquad \{M_{13}, M_{23}\} = M_{12}\,,\quad \{M_{23}, M_{12}\} = M_{13}\,,$$

$$\{N_{12}, N_{13}\} = N_{23}\,,\qquad \{N_{13}, N_{23}\} = N_{12}\,,\quad \{N_{23}, N_{12}\} = N_{13}\,,$$

and

$$\{N_{ij}, M_{kl}\} = 0\,.$$

Symmetric spaces C.I and D.III type

Because

$$\frac{Sp(n)}{U(n)} \subset \frac{SU(2n)}{S\left(U(n)\times U(n)\right)}$$

there reduction of the A.III Lax matrices to C.I case

$$L(\lambda)=\begin{pmatrix} -2\lambda^2 I_m+QQ^T+a & 0\\ 0 & 2\lambda^2 I_n-Q^TQ+b\end{pmatrix}+\begin{pmatrix} 0 & P-2\mathrm{i}\lambda Q\\ P^T+2\mathrm{i}\lambda Q^T & 0\end{pmatrix},$$

Roughly speaking we have to put $m=n$ and make $n\times n$ matrices Q and P symmetric.

Another reduction of the A.III to D.III case

$$\frac{SO(2n)}{U(n)} \subset \frac{SU(2n)}{S\left(U(n)\times U(n)\right)}$$

Roughly speaking we have to put $m=n$ and make $n\times n$ matrices Q and P antisymmetric.

Symmetric spaces BD.I type

Symmetric space

$$\frac{SO(m+n)}{SO(m)\times SO(n)}$$

is only Hermitian when $m=2$ since in general $so(m)+so(n)$ has no centre.

When $m=2$ the $so(2)$ subalgebra is the centre and depending upon whether q is odd or even this symmetric space is associated with either $B_{(n+1)/2}$ or $D_{(n+2)/2}$ root systems.

Let us consider representation of the Lie algebra $so(2n+1)$ by $(2n+1)\times(2n+1)$ matrices X, which satisfy

$$X+SX^TS^{-1}=0\,,\qquad S=\sum_{k=1}^{2n+1}(-1)^{k+1}E_{k,2n+2-k}\,,$$

where E_{ij} are matrices whose only non-zero entry is a unit in row i and column j.

In this case Lax matrix has the following block structure

$$L(\lambda) = \begin{pmatrix} 2\lambda^2 & \vec{x}^T & 0 \\ \vec{y} & 0 & s\cdot\vec{x} \\ 0 & \vec{y}^T\cdot s & -2\lambda^2 \end{pmatrix} + C + \Lambda\,,$$

with $(2n-1)\times(2n-1)$ block of zeroes,

$$\vec{x}_i = p_i - 2\mathrm{i}q_i\,,\quad \vec{y}_i = p_i + 2\mathrm{i}q_i\,,\quad i = 1,\ldots,2n-1\,,$$

and s is $(2n-1)\times(2n-1)$ matrix

$$s = \sum_{k=1}^{2n-1}(-1)^k E_{k,2n-k}\,.$$

Matrix Λ is a non diagonal matrix which satisfies $\Lambda + S\Lambda^T S^{-1} = 0$.

Example - algebra $so(5)$, config. space $\mathbb{R}^5$

The spectral curve of 7×7 Lax matrix

$$\begin{aligned} z^7 &- 4(\lambda^4 - 2\lambda^2 a_1 + a_1^2 + a_2^2 + a_3^2 + 2a_4^2 + H/2)z^5 \\ &+ \Big(16(a_2^2 + a_3^2 + 2a_4^2)\lambda^4 + F_1\lambda^2 + G_1\Big)z^3 \\ &- \Big(64a_2^2(a_3^2 + 2a_4^2)\lambda^4 + F_2\lambda^2 + G_2\Big)z = 0 \end{aligned}$$

contains Hamiltonian

$$\begin{aligned} H = \sum_{k=1}^{5} p_k^2 + 4\left(\sum_{k=1}^{5} q_k^2\right)^2 - 2(2q_1q_5 - 2q_2q_4 + q_3^2)^2 + (a_1 - a_2)q_1^2 \\ + (a_1 - a_3)q_2^2 + q_3(a_1q_3 - 2a_4q_2 - 2a_4q_4) + (a_1 + a_3)q_4^2 + (a_2 + a_1)q_5^2\,, \end{aligned}$$

two second order integrals of motion $F_{1,2}$ and two fourth order integrals of motion $G_{1,2}$, one of which is

$$2(G_1 + H^2) = -\frac{1}{4}\sum_{\alpha,\beta,\gamma,\delta} \mathcal{R}_{-\alpha,\beta,\gamma,-\delta}\, p^\alpha p^\beta p^\gamma p^\delta + \cdots$$

N-wave hierarchy

In similar manner we can study integrable systems associated with the multicomponent Fokas-Lenells, Derivative NLS, massive Thirring-like model, etc.

We can also consider reductive homogeneous spaces with non-zero torsion, for instance associated with the N-wave hierarchy

$$\frac{SU(N)}{S(U(n_1)\times U(1)\times U(n_k))}\,,\qquad \sum n_k = N\,.$$

In this case we obtain integrable quadratic-linear "magnetic" Hamiltonians

$$H_A = T_A + V_A(q) + \sum c_i p_i\,,$$

and nonhomogeneous cubic, quartic, etc, integrals of motion. We can get also new Lax matrices for the integrable systems with polynomial and rational potentials obtained by brute force method in our previous works.

Conclusion

The question about the existence of quadratic integrals of motion for Hamiltonians of natural form

$$H = \sum_{ij} g^{ij} p_i p_j + \sum a_j V_j(q)$$

has been discussed for quite a long time, starting with the works of Levy-Civita, Darboux, Stäckel and up to the present time.

Most of the classical and modern works first study the question of the existence of integrable geodesic flows at $a_j = 0$. Then the class of potentials $V_j(q)$ is described which can be added to the given geodesic flow while preserving integrability.

Experimental fact

If we abandon consideration of geodesics, it is possible to construct quadratic conservation laws for a sufficiently broad class of Hamiltonians describing motion in Euclidean space.

Quadratic integrals of motion

$$F_i = \sum_{k\neq i}^{m} \frac{M_{ik}^2}{a_i - a_k} + t_i(p) + v_i(q, a_1, \dots, a_n)\,,$$

consist of linear combinations of basis rotations and translations.

For example, in four-dimensional Euclidean space, right and left isoclinic rotations (Clifford displacements), which are classical objects in Euclidean geometry and Clifford algebra theory, are used to construct integrals of motion.

Open questions remain:

- rigorous definition of this class of Killing tensors in $\mathbb{R}^N$;
- construction of the corresponding integrals of motion of higher degrees on momenta within the classical Euclidean geometry, i.e. without using Lax matrices;
- generalization on Riemannian and pseudo-Riemannian spaces.