Representation and approximation of solution to nonlocal balance equation

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The talk is concerned with the properties of the nonlocal balance equation

$$\partial_t m(t) + \operatorname{div}(f(t,x,m(t))m(t)) = g(t,x,m(t))m(t) \tag{1}$$

on the time interval $[0,T]$ that is endowed with the initial condition

$$m(0) = m_0.$$

Notice that solution of the balance equation is considered in the space of (nonnegative) measures. The latter is endowed with the following distance defined for a given constant $b > 0$ by the rule:

$$\mathcal{W}_{1,b}(m_1,m_2) \triangleq \inf\{b\|m_1 - \hat{m} + b\|m_2 - \hat{m}_2\| + \|\hat{m}_1\| W_1(\|\hat{m}_1\|^{-1}\hat{m}_1, \|\hat{m}_2\|^{-1}\hat{m}_2) : \\ \hat{m}_1 \leqq m_1, \hat{m}_2 \leqq m_2, \|\hat{m}_1\| = \|\hat{m}_2\|\}.$$

Here W_1 denotes the standard first Kantorovich (Wasserstein) distance. We assume that the functions f are continuous, bounded and Lipschitz continuous w.r.t. x and m.

The first result of the talk is concerned with the superposition principle. To introduce it, we consider the space of weighted curves on $[0,T]$

$$\Gamma_T^C \triangleq \{(x(\cdot), w(\cdot)) \in C([0,T];\mathbb{R}^{d+1}) : \ w(t) \in [0,C], \ t \in [0,T]\}.$$

Each element of Γ_T^C can be interpreted as a trajectory of a particle with varying weight. To evaluate a distribution of these particles we use the operator $\lfloor \eta \rfloor_t$ that assigns to a measure $\eta \in \mathcal{P}(\Gamma_T^C)$ a measure on $\mathbb{R}^d$ defined by the rule: for $\phi \in C_b(\mathbb{R}^d)$,

$$\int_{\mathcal{K}} \phi(x) \lfloor \eta \rfloor_t (dx) \triangleq \int_{\Gamma} \phi(x(t)) w(t) \eta(d(x(\cdot), w(\cdot))). \tag{2}$$

The superposition principle is formulated as follows.

Theorem 1. *Given m_0, for sufficiently large C, there exists a unique equilibrium distribution of curves with weights, i.e., a measure $\eta \in \mathcal{P}(\Gamma_T^C)$ such that η is concentrated on the set of solutions of the ODE*

$$\frac{d}{dt}x(t) = f(t, x(t), \lfloor \eta \rfloor_t),$$
$$\frac{d}{dt}w(t) = g(t, x(t), \lfloor \eta \rfloor_t) w(t).$$

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whilst $\lfloor\eta\rfloor_0 = m_0$. *If* $\eta \in \Gamma_T^C$ *is an equilibrium distribution of weighted curves, then* $m(t) \triangleq \lfloor\eta\rfloor_t$ *is a weak solution of nonlocal balance equation* (1) *on* $[0,T]$. *Conversely, if* $m(\cdot)$ *is a weak solution of* (1) *satisfying the initial condition* $m(0) = m_0$, *then there exists an equilibrium distribution of weighted curves* η *such that* $\lfloor\eta\rfloor_t = m(t)$. *In particular, there exists a unique solution of the initial value problem for balance equation* (1).

The second result of the talk is concerned with the approximation of the solution of (1) by solutions of finite-dimensional ODEs. In this case, we assume that there exists a compact set $\mathcal{K}$ such that

$$f(t,x,m) = 0$$

while $x \notin K$ and $\operatorname{supp}(m) \subset \mathcal{K}$. If additionally, the initial distribution is concentrated on $\mathcal{K}$, then solution of (1) is supported on this set. In the following we consider a finite set $\mathcal{S} \subset \mathcal{K}$. Each measure on $\mathcal{S}$ is determined by a sequence $\beta_{\mathcal{S}} = (\beta_{\bar{x}})_{\bar{x}\in\mathcal{S}}) \subset \mathbb{R}$, where $\beta_{\bar{x}} \geq 0$. Given $\beta_{\mathcal{S}}$, then

$$\beta_{\mathcal{S}} \mapsto \mathscr{I}(\beta_{\mathcal{S}}) \triangleq \sum_{\bar{x}\in\mathcal{S}} \beta_{\bar{x}}\delta_{\bar{x}}$$

is a measure on $\mathcal{S}$.

Let $Q(t,\beta_{\mathcal{S}}) = (Q_{\bar{x},\bar{y}}(t,\beta_{\mathcal{S}}))_{\bar{x},\bar{y}\in\mathcal{S}}$ be a Kolmogorov matrix, i.e., for each $\bar{x} \in \mathcal{S}$, $Q_{\bar{x},\bar{y}}(t,\beta_{\mathcal{S}}) \geq 0$ when $\bar{x} \neq \bar{y}$, whereas

$$\sum_{\bar{y}\in\mathcal{S}} Q_{\bar{x},\bar{y}}(t,\beta_{\mathcal{S}}) = 0.$$

The approximating system takes the following form:

$$\frac{d}{dt}\beta_{\bar{y}}(t) = \sum_{\bar{x}\in\mathcal{S}} \beta_{\bar{x}}(t) Q_{\bar{x},\bar{y}}(t,\beta_{\mathcal{S}}(t)) + \beta_{\bar{y}}(t)\hat{g}_{\bar{y}}(t,\beta_{\mathcal{S}}(t)). \tag{3}$$

We will assume the following approximation conditions on $\mathcal{S}$ and Q: there exists $\varepsilon > 0$ such that

(QS1) for each $x \in \mathcal{K}$,

$$\min_{\bar{y}\in\mathcal{S}} \|x - \bar{y}\| \leq \varepsilon;$$

(QS2) for every $t \in [0,T]$, $\bar{x} \in \mathcal{S}$, $\beta_{\mathcal{S}} \in \ell_1^+(\mathcal{S})$,

$$\left\| f(t,\bar{x},\mathscr{I}(\beta_{\mathcal{S}})) - \sum_{\bar{y}\in\mathcal{S}} (\bar{y} - \bar{x}) Q_{\bar{x},\bar{y}}(t,\beta_{\mathcal{S}}) \right\| \leq \varepsilon;$$

(QS3) for every $t \in [0,T]$, $\bar{x} \in \mathcal{S}$, $\beta_{\mathcal{S}} \in \ell_1^+(\mathcal{S})$,

$$\sum_{\bar{y}\in\mathcal{S}} \|x - y\|^2 Q_{\bar{x},\bar{y}}(t,\beta_{\mathcal{S}}) \leq \varepsilon^2.$$

Theorem 2. *Assume that conditions (QS1)–(QS3) are in force. Given* $c > 0$, *there exists* $\widehat{C}$ *depending on* f, g *and* c *such that, if*

- $\beta_{\mathcal{S}}(\cdot) : [0,T] \to \ell_1^+(\mathcal{S})$ *solves* (3) *with the initial condition* $\beta_{\mathcal{S}}(0) = \beta_0$ *such that* $\|\beta_0\|_1 \leq c$*;*

- $m(\cdot) : [0,T] \to \mathcal{M}(\mathcal{K})$ *satisfies* (1) *and the initial condition* $m(0) = m_0$ *such that* $\|m_0\| \leq c$*;*

- $\mathcal{W}_{1,b}(m_0, \mathscr{I}(\beta_0)) \leq \varepsilon$,

then

$$\mathcal{W}_{1,b}(m(t), \mathscr{I}(\beta_{\mathcal{S}}(t))) \leq \widehat{C}(\varepsilon + \mathcal{W}_{1,b}(m_0, \mathscr{I}(\beta_0)).$$