On topological obstructions to the existence of non-periodic Wannier bases

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Let X be a proper metric measure space, that is, X is a set, which is equipped with a metric d and a measure m defined on the Borel σ-algebra defined by the topology on X induced by the metric, and all balls are compact. We say that a discrete subspace $D \subset X$ is uniformly discrete, if $\inf_{g,h\in D, g\neq h} d(g,h) > 0$, and has bounded geometry, if, for any $R > 0$, the number of points of D in each ball of radius R is uniformly bounded.

We will discuss the following question:

Question. Given a subspace $H \subset L^2(X)$, does it admit a *D-compactly supported Wannier basis*, that is, an *orthonormal* basis $\{\phi_x : x \in D\}$ in H such that $\operatorname{supp} \phi_x \subset B_R(x)$ for any $x \in D$, where $R > 0$ is independent of x and $B_R(x)$ denotes the ball of radius R centered at x?

The answer is, in general, negative. There are topological obstructions to the existence of D-finite Wannier bases. We say that a metric space X has bounded geometry if there is an $r > 0$ such that for any $R > 0$ there is a natural N such that any ball of radius R can be covered by at most N balls of radius r. If X is a proper metric space of bounded geometry with measure and the subspace H admits a D-finite Wannier basis with some uniformly discrete subspace of bounded geometry D, then the orthogonal projector p_H in the space $L^2(X,m)$ on H belongs to some C^*-algebra of bounded operators in the space $L^2(X,m)$ — the so-called Roe algebra $C^*(X)$. Moreover, its class $[p_H]$ in the K-theory $K_0(C^*(X))$ of the Roe algebra $C^*(X)$ is trivial: $[p_H] = 0$.

Under the assumption of polynomial growth of X, the case of Wannier functions of rapid decay can be reduced to the case of compactly supported ones.

This is joint work with V. Manuilov.