Dynamics of slow-fast Hamiltonian systems: the saddle-focus case

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Slow-fast Hamiltonian system

- Consider a Hamiltonian system

$$\dot q = \partial_p H, \quad \dot p = -\partial_q H, \qquad \dot x = \varepsilon \partial_y H, \quad \dot y = -\varepsilon \partial_x H$$

  Hamiltonian $H(q, p, x, y)$; symplectic form

$$\omega_\varepsilon = dp \wedge dq + \varepsilon^{-1} dy \wedge dx, \qquad 0 \le \varepsilon \ll 1$$

  $z = (q, p)$ – fast variables, $w = (x, y)$ – slow variables.
- For $\varepsilon = 0$ we obtain the frozen system

$$\dot q = \partial_p H_w, \quad \dot p = -\partial_q H_w, \qquad H_w(q, p) = H(q, p, w)$$

  depending on a constant parameter $w = (x, y)$.
- We are interested in the evolution of the slow variable w.

Adiabatic invariant

- Suppose the frozen system has one DOF and level curves $\gamma_{w,E} = \{H_w = E\}$ are closed.

$$A(w,E) = A(\gamma_{w,E}) = \oint_{\gamma_{w,E}} p\,dq, \quad \tau(w,E) = \partial_E A(w,E)$$

– the action and the period of $\gamma_{w,E}$.

- Fix energy $H = E$. The averaging principle implies that $I(t) = A(w(t),E)$ is an adiabatic invariant:

$$|I(t) - I(0)| \le C\varepsilon, \quad |t| \le \varepsilon^{-1} T$$

Averaged system

The slow variable $w = (x, y)$ shadows trajectories of the averaged system

$$\dot{x} = -\varepsilon \frac{\partial_y A(w, E)}{\tau(w, E)}, \quad \dot{y} = \varepsilon \frac{\partial_x A(w, E)}{\tau(w, E)}$$

Let Φ^s be the flow of

$$w' = -\frac{J \partial_w A(w, E)}{\tau(w, E)}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Then

$$|w(t) - \Phi^{\varepsilon t}(w_0)| \leq C\varepsilon, \quad |t| \leq \varepsilon^{-1} T$$

Destruction of adiabatic invariants

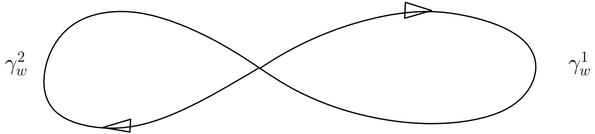
- Averaging doesn't work for trajectories passing near equilibria of the frozen system: then the fast variable is slow.
- Destruction of the adiabatic invariant near the separatrix was studied by Anatoly Neishtadt[1], [2]

[1]A. Neishtadt, Passage through a separatrix in a resonance problem with a slowly-varying parameter. J. Appl. Math. Mech. 39 (1975).

[2]A. Neishtadt, On the change in the adiabatic invariant on crossing a separatrix in systems with two degrees of freedom. Applied Math. and Mech., 51 (1987).

Adiabatic invariant near the separatrix

- Suppose the frozen system with one DOF has a hyperbolic equilibrium $z_0(w)$ with eigenvalues $\pm\lambda(w)$.
- Separatrix $\gamma_w^1 \cup \gamma_w^2 = \{H_w = h(w)\}$, $h(w) = H(z_0(w), w)$



- In each component of the complement of the separatrix there is an adiabatic invariant $A_k(w, E)$.
- For $E \to h(w)$ inside γ_w^k,

$$A_k(w, E) = P_k(w) + \frac{(h(w) - E)\ln|h(w) - E|}{\lambda(w)} + O(|h(w) - E|)$$

- Poincaré function $P_k(w) = A(\gamma_w^k)$ – the action of γ_w^k.

Neishtadt's results

- While the fast variable stays away from the separatrix, the slow variable follows a level curve of the adiabatic invariant $A_k(w, E)$.
- When the fast variable approaches the separatrix, the slow variable approaches the critical curve $Z_E = \{h = E\}$.
- At a crossing of Z_E the adiabatic invariant has a quasi-random jump of order ε.
- Then the slow variable shadows a level curve of another adiabatic invariant till it crosses Z_E again.
- Dynamics is quasirandom.

Gelfreich-Turayev theorem

- Suppose the frozen system with ≥ 2 DOF has hyperbolic chaotic invariant sets $\Lambda_{w,E} \subset \{H_w = E\}$. Let $\gamma^k_{w,E} \subset \Lambda_{w,E}$ be hyperbolic periodic orbits joined by transverse heteroclinics.
- For small ε there exist trajectories shadowing chains of heteroclinics and spending a long time near $\gamma^{k_i}_{w,E}$.
- For trajectories staying near $\gamma^k_{w,E}$, there is a local adiabatic invariant $A_k(w,E) = A(\gamma^k_{w,E})$.
- The slow variable shadows a curve composed by pieces of trajectories of the systems[3],[4]

$$w' = -\frac{J\partial A_k(w,E)}{\tau_k(w,E)}$$

[3] V. Gelfreich and D. Turaev, Unbounded energy growth in Hamiltonian systems with a slowly varying parameter. Comm. Math. Phys. 283 (2008).

[4] N. Brannstrom, E. de Simone, V. Gelfreich, Geometric shadowing in slow-fast Hamiltonian systems, Nonlinearity, 23 (2010).

- We study evolution of the slow variable for trajectories coming close to the slow manifold formed by equilibria of the frozen system.
- Then the results of Gelfreich and Turayev do not apply.
- We obtain partial extension of Neishtadt's results for systems with many DOF [5], [6]

[5] S. Bolotin, Separatrix maps in slow-fast Hamiltonian systems. Proc. Steklov Inst. Math., 322 (2023).

[6] S. Bolotin, Dynamics of slow-fast Hamiltonian systems: the saddle-focus case. Regular and Chaotic Dynamics, 30 (2025).

Leading eigenvalues

An eigenvalue λ of the equilibrium $z_0(w)$ is called leading if

$$|\mathrm{Re}\,\lambda| = \min\{|\mathrm{Re}\,\lambda| : \lambda \text{ eigenvalue}\}$$

There are two generic cases:

- Real simple leading eigenvalues $\pm\alpha(w)$ (saddle).
- Complex simple leading eigenvalues $\pm\alpha(w) \pm i\beta(w)$ (saddle-focus).

Examples

- Natural systems

$$H_w(q,p) = \frac{1}{2}\|p\|^2 + V_w(q), \quad \omega = dp \wedge dq$$

  Hyperbolic equilibria correspond to nondegenerate maxima of the potential energy V_w. All eigenvalues are real.

- Systems with gyroscopic forces:

$$H_w(q,p) = \frac{1}{2}\|p\|^2 + \langle u(q,w), p\rangle + V_w(q)$$

  Hyperbolic equilibria with complex eigenvalues may appear.

- Example: restricted circular 3 body problem.

Leading homoclinics

- $W^{s,u}(w)$ – stable and unstable manifolds of $z_0(w)$.
- $W^{s,u}_{\mathrm{strong}}(w) \subset W^{s,u}(w)$ – strong stable and unstable manifolds corresponding to nonleading eigenvalues.
- A homoclinic orbit $\gamma : \mathbb{R} \to W^s(w) \cap W^u(w)$ of the frozen system is called leading if $\gamma(\mathbb{R}) \not\subset W^s_{\mathrm{strong}}(w) \cup W^u_{\mathrm{strong}}(w)$.
- Generic homoclinics are leading and transverse.

Saddle case

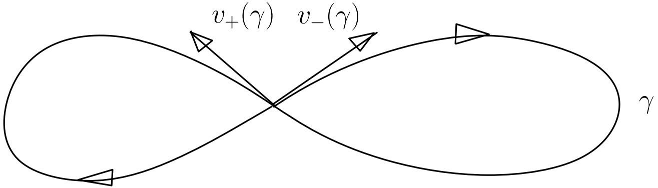
A leading homoclinic γ has the asymptotic directions

$$v_\pm(\gamma) = \alpha^{-1} \lim_{t\to\pm\infty} e^{-\alpha|t|}\dot\gamma(t) \neq 0$$

We call γ positive (negative) if

$$\iota(\gamma) = \omega(v_+(\gamma), v_-(\gamma)) > 0 \qquad (< 0), \qquad \omega = dp \wedge dq$$

For one DOF homoclinics are positive

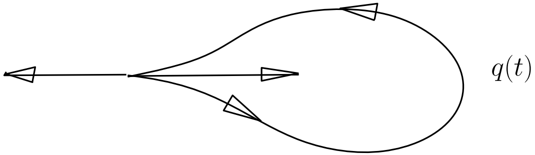


Natural systems

$$H_w(q,p) = \frac{1}{2}\|p\|^2 + V_w(q)$$

A leading homoclinic $\gamma(t) = (q(t), p(t))$ is positive (negative) if

$$\lim_{t\to+\infty} \frac{\dot q(t)}{\|\dot q(t)\|} = \mp \lim_{t\to-\infty} \frac{\dot q(t)}{\|\dot q(t)\|}$$



Separatrix map in the saddle case

- Let γ_w be a positive (negative) homoclinic. $P(w) = A(\gamma_w)$ – Poincaré function.
- Let Π be a section transversely crossing the homoclinic set $\Gamma = \cup_w(\gamma_w(\mathbb{R}), w)$.
- Fix energy E and let $\Pi_E = \Pi \cap \{H = E\}$. The separatrix map is the first return map $S : \Pi_E \to \Pi_E$.

Theorem

For small $\varepsilon > 0$ and $w_0 \in D_\pm(E) = \{0 < \pm(h - E) \leq \delta\}$, the separatrix map has an orbit (z_i, w_i) such that $w_i \in D_\pm(E)$ and

$$w_{i+1} = w_i - \varepsilon J\partial_w P(w_i) - \varepsilon T(w_i, E)\partial h(w_i) + o(\varepsilon)$$

The times of motion from (z_i, w_i) to (z_{i+1}, w_{i+1}) are

$$t_{i+1} - t_i = T(w_i, E) + o(1), \quad T(w, E) = \frac{\ln|E - h(w)|}{\alpha(w)} + \mu(w)$$

Shadowing a positive homoclinic

Let Φ^s be the flow of

$$w' = -\frac{J\partial_w A(w,E)}{\partial_E A(w,E)}$$

$$A(w,E) = P(w) + \frac{h(w)-E}{\alpha(w)}\left(\ln\frac{|h(w)-E|}{\alpha(w)} + \mu(w)\right)$$

Proposition

Suppose $\hat{w}(s) = \Phi^s(w_0) \in D_\pm(E)$ for $|s| \le T$. There exists a trajectory $(z(t), w(t)) \in \{H = E\}$ such that $w(t)$ shadows $\hat{w}(\varepsilon t)$ for $|t| \le \varepsilon^{-1} T$.

- $A_k(w,E)$ – local adiabatic invariant.
- The length of the time interval is determined by the condition $\hat{w}(s) \in D_\pm(E)$ for $|s| \le T$.
- If the Poisson bracket $\{h, P\} > 0$, the trajectory will cross Z_E after time $|t| \sim \varepsilon^{-1}\delta|\ln\delta|$.

Crossing Z_E

- Suppose γ_w^1 is a positive homoclinic, γ_w^2 a negative homoclinic, P_k – Poincaré functions; Φ_k^s – corresponding flows. Suppose $\{h, P_k\} > 0$.
- For $w_0 \in D_+(E)$ there exists $\tau > 0$ such that $w_1 = \Phi_1^\tau(w_0) \in Z_E$.
- Let $\hat{w}(s) = \Phi_1^s(w_0) \in D_+(E)$ for $0 \le s \le \tau$; $\hat{w}(s) = \Phi_2^{s-\tau}(w_1) \in D_-(E)$ for $\tau \le s \le T$.

Proposition

There exists a trajectory $(z(t), w(t)) \in \{H = E\}$ such that $w(t)$ shadows $\hat{w}(\varepsilon t)$, $0 \le t \le \varepsilon^{-1} T$.

If $\{h, P_1\} > 0$, $\{h, P_2\} < 0$, there exist trajectories captured into a neigborhood of the critical set Z_E.

Saddle-focus case

- Suppose the leading eigenvalues $\pm\alpha(w) \pm i\beta(w)$ are complex.
- Let γ_w be a leading transverse homoclinic. Then the frozen system has a chaotic invariant set on $\{H_w = h(w)\}$. This was proved by Devaney for two DOF [7].
- Then the results of Gelfreich and Turayev may be used to construct trajectories with quasirandom drift of the slow variable.
- However more precise results may be obtained using the separatrix map.

[7] R.L. Devaney, Homoclinic orbits in Hamiltonian systems. J. Diff. Equations, 21 (1976).

Separatrix map in the saddle-focus case

Let $K > 0$ be large, $\delta > 0$ small and $K < k_i \lesssim |\ln \delta|$.

Theorem

For small $\varepsilon > 0$ and $w_0 \in D(E) = \{|h - E| < \delta\}$, the separatrix map $S : \Pi_E \to \Pi_E$ has a trajectory (z_i, w_i) such that $w_i \in D(E)$ and

$$w_{i+1} - w_i = -\varepsilon J\partial P(w) - \varepsilon \left(\frac{2\pi k}{\beta(w)} + \mu(w) \right) \partial h(w) + o(\varepsilon)$$

The time of motion between (z_i, w_i) and (z_{i+1}, w_{i+1}) is

$$t_{i+1} - t_i = \frac{2\pi k}{\beta(w)} + \mu(w) + o(1)$$

Approximate adiabatic invariant

Let Φ_k^s be the flow defined by

$$A_k(w,E) = P(w) + \left(\frac{2\pi k}{\beta(w)} + \mu(w)\right)(h(w) - E), \quad K \le \delta \lesssim |\ln\delta|$$

Corollary

Suppose $\hat{w}(s) = \Phi_k^s(w_0) \in D(E)$ for $|s| \le T$. For small $\varepsilon > 0$ there exists a trajectory $(z(t), w(t)) \in \{H = E\}$ such that $w(t)$ shadows $\hat{w}(\varepsilon t)$ for $|t| \le \varepsilon^{-1} T$.

If we have a sequence $K < k_i \lesssim |\ln\delta|$, there exists a trajectory shadowing a concatenation of trajectories of the flows of $\Phi_{k_i}^s$.

Slowly time dependent system

$$\dot{z} = J\partial_z H(z,\tau), \quad z = (q,p), \quad \tau = \varepsilon t$$

- This is a slow-fast Hamiltonian system:

$$\hat{H}(z,\tau,h) = H(z,\tau) - h, \quad \hat{\omega} = dp \wedge dq - \varepsilon^{-1} dh \wedge d\tau$$

- We are interested in the evolution of energy H.
- Suppose the frozen system has a hyperbolic equilibrium $z_0(\tau)$ with simple complex leading eigenvalues $\pm\alpha(\tau) \pm i\beta(\tau)$.
- Suppose for $a \leq \tau \leq b$ there exists a leading transverse homoclinic γ_τ to $z_0(\tau)$. Let $P(\tau) = A(\gamma_\tau)$.

Evolution of energy

Corollary

Let $\delta > 0$ be sufficiently small and $K > 0$ sufficiently large. For $K < k \lesssim |\ln \delta|$ and small $\varepsilon > 0$ there exists a trajectory $z(t)$ such that while $a \le \varepsilon t \le b$ and the energy $h(t) = H(z(t), \varepsilon t)$ satisfies

$$|h(t) - h_0(\varepsilon t)| < \delta, \quad h_0(\tau) = H(z_0(\tau), \tau)$$

we have

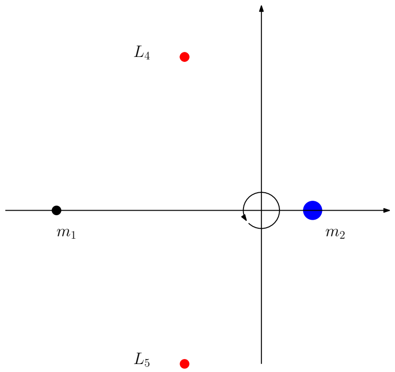
$$h(t) = h_0(\varepsilon t) - \frac{\varepsilon}{2\pi k} \int_0^{\varepsilon t} \beta(\tau) P'(\tau)\, d\tau + o(\varepsilon/k)$$

Restricted 3 body problem

- Jupiter and Sun with masses $m_1 = \mu$ and $m_2 = 1 - \mu$ move along circular orbits with angular velocity Ω.
- In the rotating frame the Hamiltonian of the Asteroid is

$$H(q, p, \Omega) = \frac{1}{2}|p|^2 - \Omega(q_1 p_2 - q_2 p_1) - \frac{1-\mu}{|q - Q_1|} - \frac{\mu}{|q - Q_2|}$$

$Q_{1,2}(\Omega)$ – positions of $m_{1,2}$.



Libration points

- If $\mu(1-\mu) > 1/27$, triangular libration points L_4, L_5 are hyperbolic equilibria with complex eigenvalues $\pm\alpha \pm i\beta$,

$$\alpha = \frac{\Omega}{2}\sqrt{(27\mu(1-\mu))^{1/2} - 1}, \quad \beta = \frac{\Omega}{2}\sqrt{(27\mu(1-\mu))^{1/2} + 1}$$

- The critical value of H is

$$h_0(\Omega) = -\frac{1}{2}\Omega^{2/3}(-\mu^2 + \mu + 3)$$

- For certain values of μ, e.g. $\mu = 1/2$, there exist transverse homoclinic orbits[8] to L_4, L_5 with action $P(\Omega) = \Omega^{-1/3}P(1)$.

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[8] M.J. Capinski, S. Kepley, J.M. James, Computer assisted proofs for transverse collision and near collision orbits in the restricted three body problem, J. of Diff. Equations, 366 (2023).

Evolution of Jacobi's constant

Suppose the radius of Jupiter's orbit, and hence $\Omega(\tau)$, $\tau = \varepsilon t$, change slowly (not a reasonable assumption).

Corollary

Let $K < k \lesssim |\ln \delta|$. There exist trajectories such that while $|H - h_0(\Omega(\varepsilon t))| < \delta$, the energy (Jacobi constant) evolves as

$$H = h_0(\Omega(\varepsilon t)) + \frac{\varepsilon P(1)}{8\pi k}\Omega^{2/3}(\varepsilon t)\sqrt{(27\mu(1-\mu))^{1/2} + 1} + o(\varepsilon/k)$$

Thank you for your attention!