Лагранжевы вложения и суперсимметричная квантовая механика на многообразиях флагов

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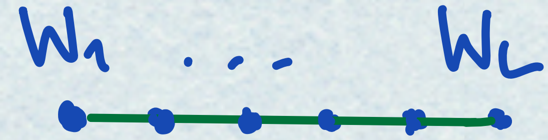
Кратко об истории вопроса



- Спиновые цепочки, интегрируемые методом анзаца Бете

Bethe '1931
Baxter '1970
Yang
Faddeev '1980
Kulish
Sklyanin
Takhtajan
...

- 'Типичные' su_2 спиновые цепочки (старших спинов) НЕ интегрируемые

Haldane '1983

Спектр  $\longleftrightarrow$ Флуктуации W_k

Haldane '1983

Nonlinear Field Theory of Large-Spin Heisenberg Antiferromagnets: Semiclassically Quantized Solitons of the One-Dimensional Easy-Axis Néel State

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The continuum field theory describing the low-energy dynamics of the large-spin one-dimensional Heisenberg antiferromagnet is found to be the O(3) nonlinear sigma model. When weak easy-axis anisotropy is present, soliton solutions of the equations of motion are obtained and semiclassically quantized. Integer and half-integer spin systems are distinguished.

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Nonlinear excitations in one-dimensional magnetic systems have received much theoretical attention in recent years, primarily ferromagnetic, easy-plane, or $S=\frac{1}{2}$ systems.[1] In this Letter, I describe a nonlinear field-theory approach to weakly uniaxially anisotropic *easy-axis antiferromagnets* with *large* spin. Classically, these have a doubly degenerate ground state with axially aligned Néel order; topological soliton excitations corresponding to movable domain walls separating the two possible ground-state configurations are described, and semiclassically quantized. The methods used also reveal the field theory describing the semiclassical *isotropic* Heisenberg antiferromagnet, providing an alternative derivation of the recent identification[2] (based on a quantum action-angle representation of spins) of this model with the O(3) nonlinear sigma model with coupling $g=2/\hbar S$ as $S\to\infty$. The quantization of magnetization carried by the easy-axis-model solutions also shows up an intrinsic difference between integer-spin and half-integer-spin systems, leading to quite different instabilities of the ordered ground state as the anisotropy vanishes, consistent with the predictions[2] of quite different low-energy physics of the isotropic ground state in the two cases.

I will consider the easy-axis model

$$H=|J|\sum_n[\vec{S}_n\cdot\vec{S}_{n+1}+\lambda S_n^zS_{n+1}^z+\mu(S_n^z)^2], \tag{1}$$

with $S_n^2=\hbar^2S(S+1)$, and $\lambda>\mu$ so that the classical ground state is given by $\vec{S}_n=\hbar S(-1)^n\hat{u}$, $\hat{u}=\pm\hat{z}$. In the classical limit, the equations of motion have small-amplitude spin-wave solutions with the frequency-wave-number relation

$$\omega^2(q)=\omega_0^2+[\omega_1\sin(qa)]^2,\quad |q|<\tfrac{1}{2}\pi/a, \tag{2}$$

where a is the lattice spacing, $\omega_1=2J\hbar S$, and $\omega_0=\omega_1(\lambda-\mu)^{1/2}(2+\lambda-\mu)^{1/2}$. I will specialize to the case of *weak* anisotropy $\omega_0/\omega_1\ll1$, when long-wavelength properties may be studied in the continuum limit $a\to0$, $\omega_1\to\infty$, $\omega_1a=c$; the dispersion relation (2) then develops Lorentz invariance with limiting velocity c. The elementary collective excitations (magnons carrying $S^z=\pm\hbar$) are obtained by a semiclassical quantization of the spin waves (e.g., by a linearized Holstein-Primakoff approach); for (crystal) momentum $|P|\ll\frac{1}{2}\pi\hbar/a$, the magnon dispersion is

$$\epsilon(P)=[(\hbar\omega_0)^2+c^2P^2]^{1/2},\quad 0<(\lambda-\mu)^{1/2}\ll1. \tag{3}$$

To study the soliton excitations, a fully nonlinear treatment of (1) is needed. Following Mikeska,[3] I use the classical angle-variable representation

$$\vec{S}_n=(-1)^n\hbar S(\sin\theta_n\cos\varphi_n,\sin\theta_n\sin\varphi_n,\cos\theta_n).$$

The classical equations of motion are easily obtained from (1) in terms of these variables by using the Poisson-bracket algebra $\{\varphi_n,S_{n'}^z\}=\delta_{nn'}$, $\dot{\varphi}_n=\{\varphi_n,H\}$, etc.:

$$\dot{\theta}_n=-\tfrac{1}{2}\omega_1(-1)^n\sum_\pm[\sin\theta_{n+1}\sin(\varphi_{n+1}-\varphi_n)], \tag{4a}$$

$$\dot{\varphi}_n=-\tfrac{1}{2}\omega_1(-1)^n\sum_\pm[(1+\lambda)\cos\theta_{n\pm1}-\mu\cos\theta_n-\cot\theta_n\sin\theta_{n\pm1}\cos(\varphi_{n\pm1}-\varphi_n)]. \tag{4b}$$

To make progress with these equations, I assume as in Ref. 3 that θ_n and φ_n vary *slowly* with n, with a *small* superimposed staggered-fluctuation component; this should be valid at low energies and weak anisotropy $\omega_0\ll c/a$:

$$\theta_n=\theta(x)+a(-1)^n\alpha(x),\quad \varphi_n=\varphi(x)+a(-1)^n\beta(x),\quad x=na. \tag{5}$$

$\theta(x)$ and $\varphi(x)$ are slowly varying angle fields, while $\alpha(x)$ and $\beta(x)$ are small staggered-fluctuation fields, chosen to have dimensions of density. The variables on neighboring sites can be expressed through a

gradient expansion about $x=na$. Full nonlinearity in the angle fields must be maintained, but (4a) and (4b) can be approximated by an expansion up to quadratic order in α, β, and ∇ (or alternatively, by taking the limit $a\to0$ with α, β, ω_0, and c fixed). There is a second (dimensionless) anisotropy parameter $\gamma_1=\frac{1}{2}(\mu+\lambda)$ in addition to $\lambda-\mu\simeq\frac{1}{2}(\omega_0a/c)^2$; for simplicity, I first specialize to the neighborhood of the isotropic model with $\gamma_1\simeq0$, and drop terms involving γ_1. Each equation (4a) and (4b) yields *two* independent equations of motion (for the uniform and staggered parts), essentially Eqs. (2.11)–(2.14) of Ref. 3, but with minor corrections[3]:

$$\dot{\theta}/c=2\beta\sin\theta;\quad \dot{\varphi}/c=-2\alpha/\sin\theta;\quad (\dot{\alpha}\sin\theta)/c=-\tfrac{1}{2}\nabla(\sin^2\theta\,\nabla\varphi)-\alpha\beta\sin2\theta;$$
$$(\dot{\beta}\sin\theta)/c=\tfrac{1}{2}\nabla^2\theta-\tfrac{1}{4}\sin2\theta[(\omega_0/c)^2+(\nabla\varphi)^2-(2\alpha/\sin\theta)^2+4\beta^2]. \tag{6}$$

These simplify considerably when I introduce new density fields $L(x)=-2g^{-1}\alpha\sin\theta$, $\Pi_\theta(x)=2g^{-1}\beta\sin\theta$, where g is the coupling constant $2/\hbar S$ [more accurately,[2] $g=2/\hbar\{S(S+1)\}^{1/2}$]; $L(x)$ is the azimuthal spin density:

$$S^z\equiv\hbar S\sum_n(-1)^n\cos\theta_n\xrightarrow{a\to0}\int dx\,L. \tag{7}$$

The equations (6) become

$$\dot{\theta}=gc\Pi_\theta;\quad \dot{\varphi}=gcL/\sin^2\theta;\quad g\dot{L}/c=\nabla(\sin^2\theta\,\nabla\varphi);\quad g\dot{\Pi}_\theta/c=\nabla^2\theta-\tfrac{1}{2}\sin2\theta[(\omega_0/c)^2+(\nabla\varphi)^2-(gL/\sin^2\theta)^2]. \tag{8}$$

These equations may now be recognized as deriving from the Hamiltonian

$$H=\tfrac{1}{2}c\int dx\left(g(\Pi_\theta^2+L^2/\sin^2\theta)+g^{-1}\{(\nabla\theta)^2+[(\nabla\varphi)^2+(\omega_0/c)^2]\sin^2\theta\}\right), \tag{9}$$

expressed in terms of independent canonical classical fields $\{\varphi(x),L(x')\}=\{\theta(x),\Pi_\theta(x')\}=\delta(x-x')$. The corresponding Lagrangian density is easily obtained; in terms of the unit-vector field $\vec{\Omega}(x,t)=(\sin\theta\times\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$,

$$\mathcal{L}=\tfrac{1}{2}g^{-1}[c^{-1}|\partial_t\vec{\Omega}|^2-c|\nabla\vec{\Omega}|^2-(\omega_0^2/c^2)\Omega_\perp^2],\quad |\vec{\Omega}|^2=1, \tag{10}$$

where $\Omega_\perp^2\equiv(\Omega^x)^2+(\Omega^y)^2$. This is just the Lorentz-invariant *O(3) nonlinear sigma model*,[4] with additional easy-axis anisotropy.

If the analysis is repeated keeping the additional anisotropy term γ_1, the Hamiltonian (9) gains an extra term,

$$H'=\tfrac{1}{2}\gamma_1c\int dx[gL^2+g^{-1}(\nabla\cos\theta)^2]. \tag{11}$$

This term breaks Lorentz invariance, causing a change in the collective-mode "light velocity" from c to $(1+\gamma_1)^{1/2}c$ if the system becomes *easy plane* [i.e., if ω_0^2 in (10) becomes negative]; stability requires $\gamma_1>-1$. This term will not be considered further here.

I now obtain *topological soliton* solutions of the equations of motion (8) by the device of minimizing the Hamiltonian (9) with respect to the fields at fixed values of the conserved quantities S^z (7) and momentum $P=\int dx(\Pi_\theta\nabla\theta+L\nabla\varphi)$, with the boundary condition $\cos\theta\to\pm\,\mathrm{sgn}(x)$ as $|x|\to\infty$. Lorentz invariance allows the moving finite-P solution to be obtained by a boost of the static $P=0$ soliton. I obtain $L=g^{-1}(\omega/c)\sin^2\theta$, $\varphi=\omega\tau$, $\theta=\theta(s)$, where $(s,\tau)=(1-v^2/c^2)^{-1/2}(x-vt,\ t-vx/c^2)$, and

$$\theta''=\tfrac{1}{2}[(\omega_0^2-\omega^2)/c^2]\sin2\theta. \tag{12}$$

For $\omega^2<\omega_0^2$, (12) has soliton solutions $\cos\theta=\pm\tanh[(s-s_0)/R]$, where $c^2/R^2=\omega_0^2-\omega^2$. The total azimuthal spin carried by the soliton is found to be given by $\frac{1}{2}gS^z=\omega/(\omega_0^2-\omega^2)^{1/2}$. This variational procedure leads to the single-soliton solution, but cannot produce multisoliton solutions. When $\omega_0^2=0$, the equations of motion (8) derived from (9) or (10) are known to be *integrable*.[4] It is tempting to speculate that this may remain true when $\omega_0^2\neq0$, allowing multisoliton generalizations of the single-soliton solution obtained here to be found. Somewhat fortuitously (as he studied a system with $\gamma_1\neq0$, but omitted[3] certain terms involving γ_1), Mikeska[3] has previously obtained (12) with $\omega=0$ and its solution from an *Ansatz* for the $S^z=0$ soliton.

At this point, it is useful to introduce the *semiclassical quantization* of the allowed values of the internal precession frequency of the soliton: This simply means that S^z is quantized in *integer steps* $S^z=m\hbar$. The quantum number m can be used to parametrize the internal state of the soliton. The soliton energy-momentum relation obtained from (9) is then given by

$$E_m(P)=[(m^2+S^2)(\hbar\omega_0)^2+c^2P^2]^{1/2}\geq S\hbar\omega_0. \tag{13}$$

- Метод Халдейна DB 2011-2012

  ~ симплектическая геометрия

  (лагранжеви вложения)

  SU(n) обобщение

  Affleck et.al. '2017
  Tanizaki-Sulejmanpasic
  Seiberg et.al.

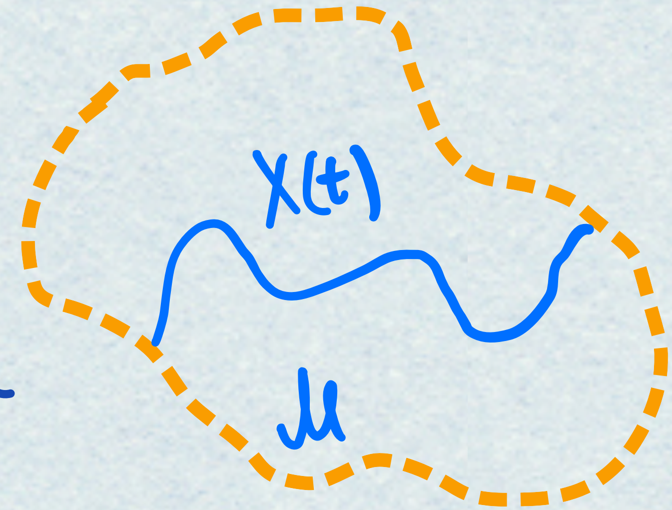
- Метод математически строгий

  $\leftrightarrow$ 'короткие' спиновые цепочки

Классическая механика: 1$^{\underline{я}}$ система

$$(\mathcal{M} \mid G, A)$$

метрика — G, калибровочное поле — A



$$L = \int dt\, \tfrac{1}{2} G_{IJ}(x)\, \dot{x}^I \dot{x}^J - \int dt\, A_I(x)\, \dot{x}^I$$

м.б. топологическим

$$0 \neq [dA] \in H^2(\mathcal{M}, \mathbb{R})$$

" 1D Сигма-модель "

(Магнитные) геодезические на $\mathcal{M}$

Novikov, Schmelzer '1981

Kordyukov, Taimanov '2020

Интегрируемость?

компактная полупростая

M = Однородное пр-во G/H

- Симметрическое пр-во: ДА

  Геодезическая $g(t) = e^{At} g(0)$, $A \in \mathrm{Lie}(G)$.

- Несимметрическое: ДА в случае нормальной метрики

  $\mathrm{Lie}(G) = \mathfrak{h} \oplus \mathfrak{m}$, $\mathfrak{m} = \mathfrak{h}^{\perp}$

  $g \in G \Rightarrow ds^2 = -\langle (g^{-1} dg)_{\mathfrak{m}}, (g^{-1} dg)_{\mathfrak{m}} \rangle$

- В общем случае открытый вопрос

Manakov '1976

Miščenko, Fomenko '1978

Thimm '1981

Bolsinov Jovanović '2004

Arvanitoyeorgos Souris '2016

Таргет - пространство

$$M = \frac{SU(n)}{H}$$

симплектич.
компл.
кэлерово

(Ко)присоединённая орбита

$$g \Lambda g^{-1} \qquad \Lambda \in su_n,\ g \in SU(n)$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_1 & & & & & \\ & & & \lambda_2 & & & & \\ & & & & \ddots & & & \\ & & & & & \lambda_2 & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_m \\ & & & & & & & & \ddots \\ & & & & & & & & & \lambda_m \end{pmatrix}$$

$n_1,\ n_2,\ \ldots,\ n_m$

$$\boxed{\frac{SU(n)}{S(U(n_1)\times\cdots\times U(n_m))}}$$

Многообразие флагов

Пример: мн-е полных флагов

$$F_n := \frac{SU(n)}{S(U(1)^n)}$$

$$u_1, \dots, u_n \qquad \overline{u_j} \cdot u_k = \delta_{jk}$$

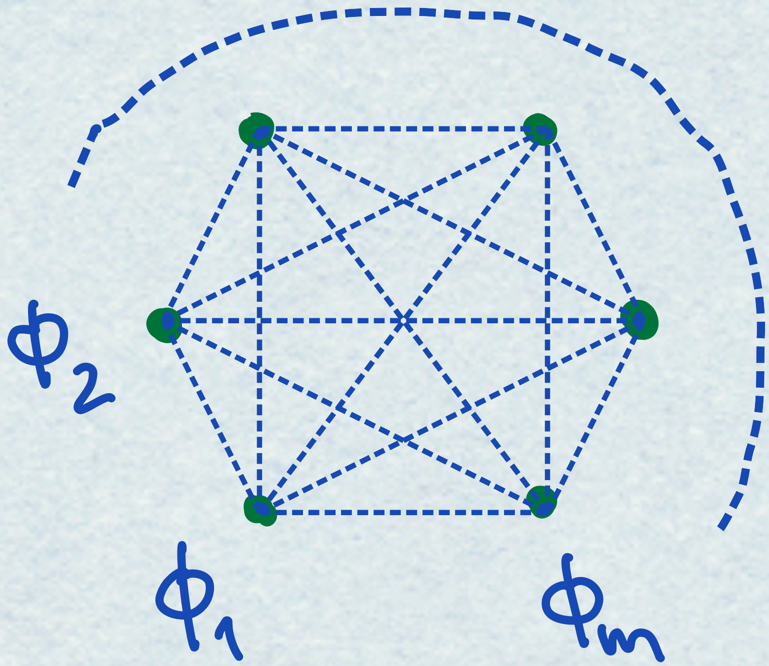
$$\mathbb{C}^n \ni u_1, \quad \mathbb{C}^n \ni u_n \qquad u_k \sim e^{i\alpha_k} u_k$$

$SU(n)$ - инв. метрика: $$ds^2 = \sum_{j<k} \beta_{jk}^2 \left| \overline{u_j} \cdot du_k \right|^2$$

Калибр. поле: $$dA = i \sum_{j=1}^{n} \gamma_j \, \overline{du_j} \wedge du_j$$

$$\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix} \bmod \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \qquad \in H^2(F_n, \mathbb{R}) \simeq \mathbb{R}^{n-1}$$

2ᵃ система: "Классическая спиновая цепочка"



Фазовое простр-во

$$= \Phi_1 \times \ldots \times \Phi_m$$

$$\Phi_k \simeq Gr(n_k, n)$$

$$\sum n_k = n$$

Симпл. форма $\Omega = \sum_{k=1}^{m} s_k \, \Omega_{\Phi_k}$ Fubini-Study

Отображение момента $\mu_k : \Phi_k \mapsto su_n$

Гамильтониан $H = \sum_{j<k} \alpha_{jk}^2 \langle \mu_j, \mu_k \rangle$

Система 1 (магнитные подушки)

и

Система 2 (с компактным фазовым пространством a.k.a. спиновая цепочка)

"ПОЧТИ ЭКВИВАЛЕНТНЫ" DB '2024 A. Kuzovchikov

$$\beta_{jk}^2 = \frac{1}{d_{jk}^2} \qquad \gamma_k = S_k$$

SU(2)-случай $\quad \mathcal{F}_2 = \dfrac{SU(2)}{S(U(1)^2)} = \mathbb{C}P^1$ Система 1

Рассмотрим $N_q := (T^*\mathbb{C}P^1, \; \Omega_q)$

$$\pi: \; T^*\mathbb{C}P^1 \mapsto \mathbb{C}P^1$$

$$\Omega_q := \sum dp_i \wedge dx_i + q\, \pi^*(\Omega^{FS})$$

Гамильтониан геодез. потока

$$H_{geod} = \sum_{i,j} g_{ij}(x)\, p_i p_j$$

'Спиновая цепочка'

$$\mathbb{C}P^1 \times \mathbb{C}P^1$$

$$\Downarrow \qquad \Downarrow$$

$$\vec{n}_1 \qquad \vec{n}_2$$

$$\vec{n}_1^{\,2} = 1\,, \quad \vec{n}_2^{\,2} = 1$$

Скобки Пуассона

$$\{n_A^i, n_B^j\} = \frac{1}{s_A}\,\varepsilon^{ijk}\,\delta_{AB}\,n_A^k$$

$$A, B = 1, 2$$

$$su_2 \simeq \mathbb{R}^3, \quad \vec{\mu}_1 = \vec{n}_1\,, \quad \vec{\mu}_2 = \vec{n}_2$$

Гамильтониан $\quad H = s_1 s_2\,\frac{1}{2}\left(1 + \vec{n}_1\vec{n}_2\right)$

Ур-я движения: $\quad \dot{\vec{n}}_1 = s_2\,\vec{n}_1 \times \vec{n}_2\,, \quad \dot{\vec{n}}_2 = s_1\,\vec{n}_2 \times \vec{n}_1$

Равновесия

↑↑ $\vec{n}_1 = \vec{n}_2$

$(\mathbb{CP}^1)_{max} \cong D$

дивизор

или

↑↓ $\vec{n}_1 = -\vec{n}_2$

$(\mathbb{CP}^1)_{min}$

Утв. $\left(\mathcal{X} := \mathbb{CP}^1 \times \mathbb{CP}^1 \setminus D,\ s_1 \Omega_{FS}^{(1)} + s_2 \Omega_{FS}^{(2)}\right)$

$\simeq$ открытому подмн-ву в N_q

симплектоморфизм

$$s_1 \equiv s, \quad s_2 \equiv s + q$$

$$\lim_{s \to \infty} (\dots) \simeq T^*\mathbb{CP}^1, \qquad \boxed{H\big|_{\mathcal{X}} = H_{geod}}$$

Явное отображение: $\mathbb{C}P^1 \times \mathbb{C}P^1 \setminus D \longmapsto T^*\mathbb{C}P^1$

$$\cup\!\!|$$

$$(\vec{n}_1, \vec{n}_2) \longmapsto \{\vec{n}^2 = 1,\ \vec{p}\vec{n} = 0\} \subset \mathbb{R}^6$$

$$\Omega = d\vec{p} \wedge d\vec{n}\,\big| + q\,\Omega^{FS}$$

$$\boxed{\begin{gathered} \vec{n} = \frac{1}{\sqrt{2(1-\vec{n}_1\vec{n}_2)}}(\vec{n}_1 - \vec{n}_2);\quad \vec{p} = \frac{\sqrt{s_1 s_2}}{\sqrt{2(1-\vec{n}_1\vec{n}_2)}}\,\vec{n}_1 \times \vec{n}_2 \\ H = \vec{p}^{\,2} = s_1 s_2 \frac{1+\vec{n}_1\vec{n}_2}{2} \longleftarrow \text{спиновая цепочка} \\ \text{Z геодезич. поток} \end{gathered}}$$

Но $|\vec{p}| < \sqrt{s_1 s_2} = \sqrt{s(s+q)}$! $(\subset N_q)$

SU(n) случай $\qquad F_n = \dfrac{SU(n)}{S(U(1)^n)}$ $\qquad$ Система 1

Рассмотрим $N_q := (T^* F_n\,,\ \Omega_q)$ $\qquad$ n-1 магнитных зарядов

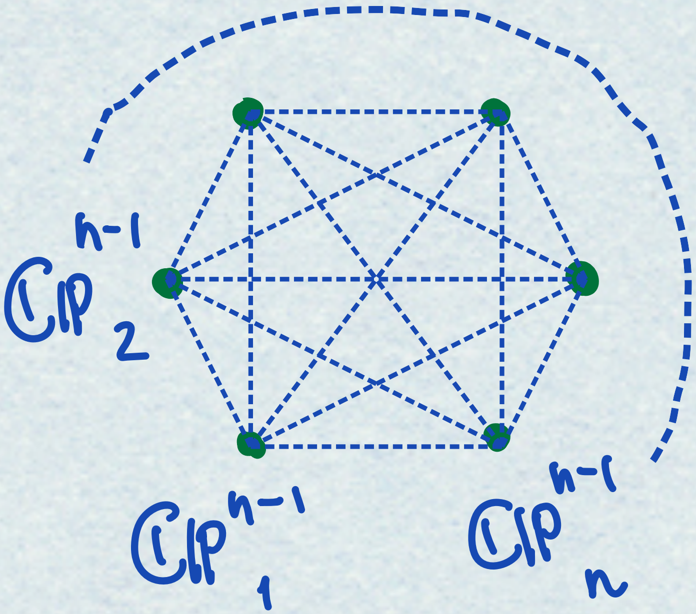
$$\pi:\ T^* F_n \mapsto F_n$$

$$\Omega_q := \sum dp_i \wedge dx_i + \pi^*(dA)$$

Гамильтониан геодез. потока

$$H_{geod} = \sum_{i,j} g_{ij}(x)\, p_i p_j$$

SU(n) спиновая цепочка



Параметризуем

$$\mathbb{CP}^{n-1}_k \rightsquigarrow z_k \in \mathbb{C}^n$$

$$z_k^+ \cdot z_k = s_k, \quad z_k \sim e^{i\alpha_k} z_k$$

Симпл. форма $\Omega = i \sum_{k=1}^{n} dz_k^+ \wedge dz_k = i\,\mathrm{Tr}(dz^+ \wedge dz)$

Дивизор $D \subset \underbrace{\mathbb{CP}^{n-1} \times \ldots \times \mathbb{CP}^{n-1}}_{n \text{ раз}}$: $\mathrm{Det}(Z) = 0$

Обозначим $\mathcal{X} = \left(\mathbb{CP}^{n-1}\right)^{\times n} \setminus D$

Пусть $Z \in \mathcal{X} \Rightarrow \operatorname{Det} Z \neq 0$

Полярное разложение: $Z = \underset{\text{унитарная}}{U}\, \underset{\substack{\text{эрмитова}\\ >0}}{K^{1/2}}$

$$\Omega\big|_{\mathcal{X}} = \underbrace{d\left(\sum_{j\geq k} K_{jk}\, \overline{u_j}\cdot du_k + \text{к.с.}\right)}_{\text{"}p\,dx\text{" на } T^*F_n} + \underbrace{i \sum_k S_k\, d\overline{u_k} \overset{\cdot}{\wedge} du_k}_{\pi^*(dA)}$$

Вероятный вид матрицы K:

$$K = \begin{pmatrix} s_1 & K_{12} & \cdots & K_{1n} \\ \overline{K_{12}} & s_2 & & \\ \vdots & & \ddots & \\ \overline{K_{1n}} & & & s_n \end{pmatrix} \quad \text{и} \quad K > 0$$

$\Rightarrow$ Условия на K_{ij} (аналог $|\vec{p}| \leq \sqrt{s_1 s_2}$)

Spec(K) – интеграл движения

Возьмём $s_i = s + q_i$ и $s \to \infty$

$\Rightarrow$ Условия исчезают!

Явные решения

Рассмотрим метрики на F_n такого вида

$$
\begin{array}{c}
\dfrac{U(n)}{U(1)^n} \\
\pi_1 \big\downarrow \\
\dfrac{U(n)}{U(2)\times U(1)^{n-2}} \\
\pi_2 \big\downarrow \\
\dfrac{U(n)}{U(3)\times U(1)^{n-3}} \\
\big\downarrow \\
\dots
\end{array}
$$

Забывающие отображения

Пусть это римановы субмерсии

$\Rightarrow$ Всего $n-1$ параметров $\alpha_2 \dots \alpha_n$

Магнитные заряды $s_1 \dots s_n$

Решение 'спиновой цепочки' $Z(t)$

$$A := Z^{+} Z(0) = K(0) = \begin{pmatrix} s_1 & K_{12}(0) & \\ \overline{K_{12}(0)} & s_2 & \\ & & \ddots \end{pmatrix}$$

$$\boxed{Z(t) = Z(0)\, Ev(t)}$$

проектор

$$Ev(t) = \overleftarrow{\prod_{k=2}^{n}} \exp\left[i \left(\alpha_{k+1} - \alpha_k\right) t \; Pr_k(A) \right]$$

Решение для логических $\boxed{U(t) = U(0)\, Ev(t)}$

$$\Uparrow$$
$$\mathcal{F}_n$$

Квантование 1

Геодезические

$$H = \sum g_{jk}(x)\, p_j p_k$$

$$\big\downarrow \qquad [x_j, p_k] = i\,\delta_{jk}\,, \qquad p_j = -i\frac{\partial}{\partial x^j}$$

$$\mathcal{H}_0 = L_2(M)$$

гильбертово
пр-во

$$\hat{H} = -\Delta$$

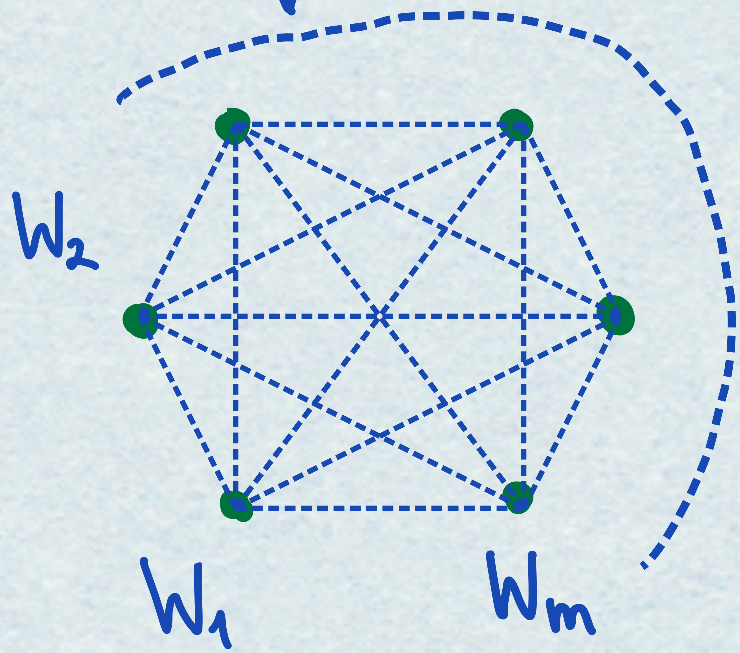
лапласиан

Квантование 2

'Спиновая цепочка'

$$H = \sum_{j<k} d_{jk}^2 \langle \mu_j, \mu_k \rangle$$

$\mathcal{H} =$



W_k

неприводимые представления $SU(n)$

$$\hat{H} = \sum_{j<k} d_{jk}^2 \, (S_j, S_k)$$

оператор Казимира в $W_j \otimes W_k$

Квантови системи снова еквивалентни

Случай $\qquad F_n = \dfrac{U(n)}{U(1)^n} \hookrightarrow \mathbb{CP}^{n-1} \times \dots \times \mathbb{CP}^{n-1}$

$$W_k = \overbrace{\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}}^{s_k} \qquad s_k = s + q_k$$

$$\lim_{s\to\infty} \overbrace{W_1 \otimes \dots \otimes W_n}^{\mathcal{H}} = L_2^{(q)}(F_n)$$

сечения обобщённых расслоений Хопфа ('магнитни гармоники')

$$\mathrm{Spec}\, \hat{H}_p \subset \mathrm{Spec}\, \hat{H}_{p+1} \subset \mathrm{Spec}(-\Delta)$$

$\mathbb{C}\mathbb{P}^1 \simeq S^2$

Вложение $\mathbb{C}\mathbb{P}^1 \hookrightarrow \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$

Спектр $E_l = l(l+1)$, $l = 0, 1, 2, \dots$

$$L_2(S^2) = \bigoplus_{l=0}^{\infty} \{Y_l^m\}_{m=-l}^{l}$$ сферические гармоники

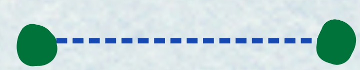
V_{2l}

$$V_s := \underbrace{\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}}_{s}$$ представление su_2

Оставим только первые 's+1' гармоник $l = 0, 1, \dots, s$

$$\boxed{\bigoplus_{l=0}^{s} V_{2l} = V_s \otimes V_s}$$

'Спиновая цепочка'



Осцилляторный формализм

q = монопольный заряд

Тамм '1931
Wu-Yang '1976

$$\mathcal{H} = \underset{V_s}{\bullet}\cdots\cdots\cdots\underset{V_{s+q}}{\bullet}$$

Гамильтониан $H = (S_1, S_2) = \sum_{\alpha=1}^{3} S_\alpha^{[V_s]} \otimes S_\alpha^{[V_{s+q}]}$

матрицы Паули

$$S_{A\alpha} = a^+_{Ai} (\sigma_\alpha)_{ij}\, a_{Aj} \qquad [a_{Ai}, a^+_{Bj}] = \delta_{AB}\,\delta_{ij}$$

Пр-во Фока: $a^+_{A_1 i_1} \dots a^+_{A_\ell i_\ell} |0\rangle$, где $a_{Ai}|0\rangle = 0$

$$a_1^+ \circ a_1 = s, \qquad a_2^+ \circ a_2 = s+q$$

$$\Rightarrow \quad \boxed{H = \left(a_1^+ \circ a_2\right)\left(a_2^+ \circ a_1\right)}$$

(аналог классического $H = |z_1^+ \circ z_2|^2$)

Пусть $q \geq 0$. Построим основн. состояние $(H=0)$:

$$\left|\begin{matrix}\text{основн.}\\ \text{состоян.}\end{matrix}\right\rangle = \sum_{i_1,\dots,i_q=1}^{2} \psi_{i_1\dots i_q} (a_2)^+_{i_1} \cdots (a_2)^+_{i_q} \times$$

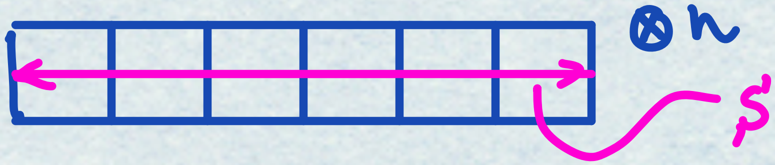
$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \times \left(\varepsilon_{lm} (a_1)^+_l (a_2)^+_m\right)^s |0\rangle$$

Представление $V_q = \underbrace{\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}}_{q}$

Спектр F_n

$$F_n \overset{\longmapsto}{\scriptstyle \pi_1} \frac{U(n)}{U(2)\times U(1)^{n-2}} \overset{\longmapsto}{\scriptstyle \pi_2} \frac{U(n)}{U(3)\times U(1)^{n-3}} \longmapsto \dots$$

π_i - забиващи / риманови субмерсии
(с напълно геодезични слоеве)

$\mathcal{H} =$ 

$$\hat{H} = \alpha_2 (S_1+S_2)^2 + \alpha_3 (S_1+S_2+S_3)^2 + \dots$$

комутиращи Казимири

Лапласиан

$$\Delta = \alpha_2 \Delta_2 + \alpha_3 \Delta_3 + \dots + \alpha_n \Delta_n$$

$[\Delta_i, \Delta_j] = 0$

Berard-Bergery, Bourguignon
'1982

Суперсимметрия (SUSY)

$$\mathcal{H} \subset \mathcal{H}_{SUSY}$$
$$\hat{H}_{SUSY}\big|_{\mathcal{H}} = \hat{H}$$

Алгебра: $\hat{H}_{SUSY} = \{Q, Q^+\}$

$$\{Q, Q\} = \{Q^+, Q^+\} = 0$$

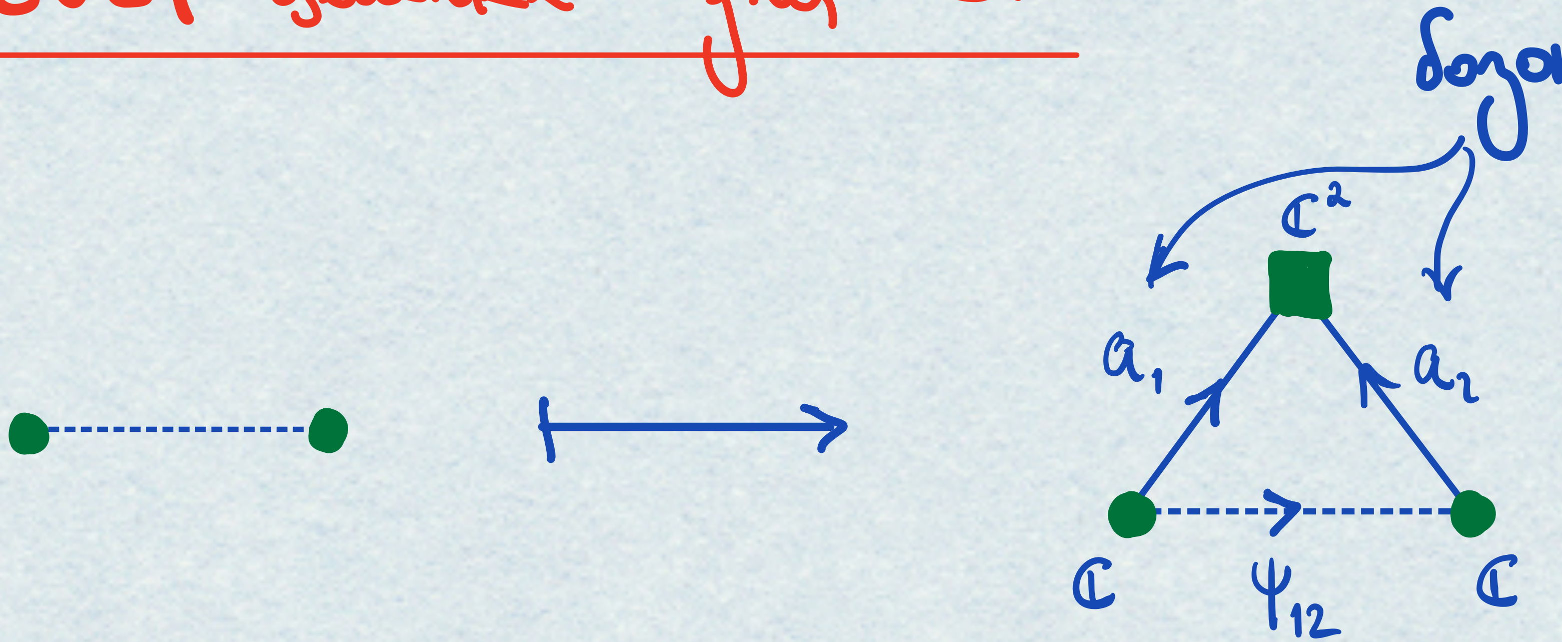
$$[\hat{H}_{SUSY}, Q] = [\hat{H}_{SUSY}, Q^+] = 0$$

Пример: гармонический осциллятор   'Nicolai 1976

$$[a, a^+] = 1\,, \quad \{\overbrace{\psi}^{\psi^2 = 0}, \psi^+\} = 1 \qquad \hat{H}_{SUSY} = \underbrace{a^+a}_{\text{обычный гарм. осциллятор}} + \psi^+\psi$$

$$Q = a^+\psi\,, \quad Q^+ = a\psi^+$$

'SUSY цепочка' для $\mathbb{CP}^1$



← фермион

DB
V. Krivorol
A. Kuzovchikov
2024

$$Q = \alpha_{12}\, \psi_{12}\, a_1^+ \circ a_2$$

↑ параметр $\in \mathbb{R}$

$$H = \{Q, Q^+\} = \alpha_{12}^2 \left(a_1^+ \circ a_2\right)\left(a_2^+ \circ a_1\right) + \ldots$$

$$\mathcal{H}_{SUSY} \supset \mathcal{H} = \{ \psi_{12} | v \rangle = 0 \}$$

Связи:

$$C_1 = a_1^+ \circ a_1 + \psi_{12}^+ \psi_{12} - s = 0$$

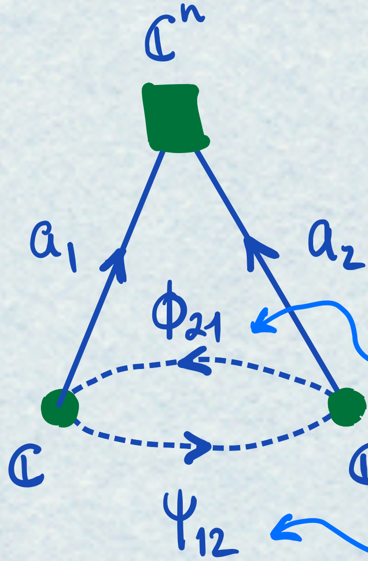
$$C_2 = a_2^+ \circ a_2 - \psi_{12}^+ \psi_{12} - (s+q) = 0$$

$$\boxed{[C_1, Q] = [C_2, Q] = 0}$$

$$H_{susy} = \{\overset{\partial}{\updownarrow}\!Q, \overset{\partial^+}{\updownarrow}\!Q^+\} \longleftrightarrow \overset{\substack{\{\partial, \partial^+\} \\ \parallel}}{\text{Лапласиан Дольбо}} \text{ на } \mathbb{CP}^1$$

$$\text{“} \quad \psi_{12}^+ \Leftrightarrow dz, \quad \psi_{12} \Leftrightarrow \iota_{\frac{\partial}{\partial z}} \quad \text{”}$$

Расширенная SUSY на $\mathbb{CP}^1$



$$Q_1 = \alpha_{12}\, \psi_{12}\, a_1^{+} \circ a_2$$

$$Q_2 = \alpha_{12}\, \phi_{21}^{+}\, a_1^{+} \circ a_2$$

Лефшец $SU(2)$ дублет

удвоение фермионов ψ_{12}, ϕ_{21}

Нет монополей

Связи:

$$C_1 = a_1^{+} \circ a_1 + \psi_{12}^{+} \psi_{12} - \phi_{21}^{+} \phi_{21} - s = 0$$

$$C_2 = a_2^{+} \circ a_2 - \psi_{12}^{+} \psi_{12} + \phi_{21}^{+} \phi_{21} - s = 0$$

$$\{Q_A, Q_B\} = 0\,, \quad \{Q_A, Q_B^{+}\} = \delta_{AB} H$$

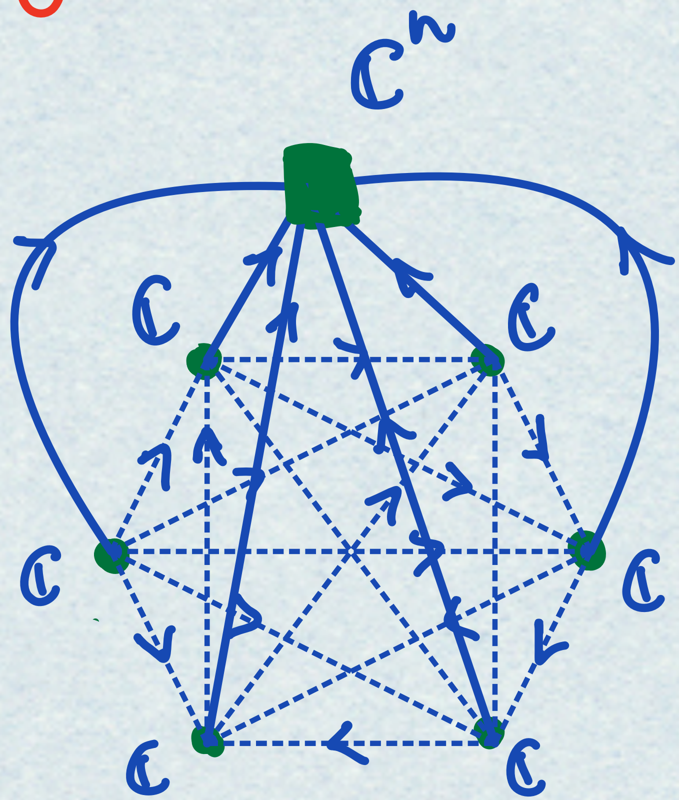
Лапласиан де Рама на $\mathbb{CP}^1$ $\{d, d^{+}\}$

SUSY цепочка для $\mathcal{F}_n$ / Дольбо

ацикличeский колчан

$\updownarrow$

комплексная структура на $\mathcal{F}_n$



кубические члены нужны, чтобы $Q^2=0$

$$Q = \sum_{A<B} \alpha_{AB}\,\psi_{AB}\, a_A^{+}\circ a_B - \sum_{A<B<C} \frac{\alpha_{AB}\alpha_{BC}}{\alpha_{AC}}\,\psi_{AB}\,\psi_{BC}\,\psi^{+}_{AC}$$

Расширенная SUSY на S_n / де Рам

$$\xi_{AB} := \begin{pmatrix} \psi_{AB} \\ \phi^{+}_{BA} \end{pmatrix}$$ SU(2) дублеты

$$Q = \sum_{A<B} \alpha_{AB}\, \xi_{AB}\, a^{+}_{A} \circ a_{B} + \sum_{A<B<C} \left(\frac{\alpha_{AB}\alpha_{AC}}{\alpha_{BC}} \xi_{AB} \left(\xi^{+}_{AC} \xi_{BC} \right) - \frac{\alpha_{AC}\alpha_{BC}}{\alpha_{AB}} \xi_{BC} \left(\xi^{+}_{AC} \xi_{AB} \right) \right)$$

SUSY алгебра $\Rightarrow$ условие когерентности

$$\frac{1}{\alpha^2_{AC}} = \frac{1}{\alpha^2_{AB}} + \frac{1}{\alpha^2_{BC}} \qquad A<B<C$$

Индекс Виттена

$\psi^+\psi \in 2\mathbb{Z}$ $\quad\psi^+\psi \in 2\mathbb{Z}+1$

$$\mathcal{H}_{SUSY} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \qquad SU(n) \circlearrowright \mathcal{H}^{(0)}, \mathcal{H}^{(1)}$$

характеры

$$W(\mathcal{H}_{SUSY}) := \chi(\mathcal{H}^{(0)}) - \chi(\mathcal{H}^{(1)})$$

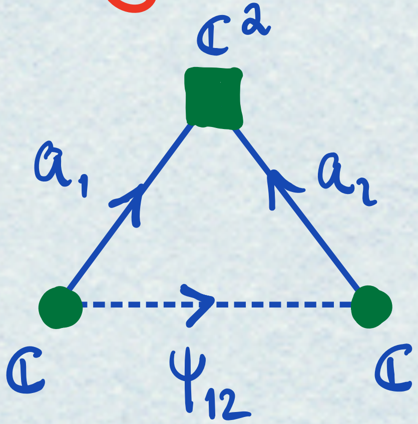
Т-ма (Виттен, 1983) $\quad W(\mathcal{H}_{SUSY}) = W(\ker(Q) \cap \ker(Q^+))$

"Q-гармонические формы"

$\mathcal{H}^{(0)}$ – чётные, $\mathcal{H}^{(1)}$ – нечётные

$\Rightarrow W(\mathcal{H}_{SUSY}) =$ (эквивариантная) эйлерова х-ка / индекс Q

Индекс Виттена для $\mathbb{CP}^1$ 'спиновой цепочки'



$$C_1 = a_1^+ \circ a_1 + \psi_{12}^+ \psi_{12} - s = 0$$

$$C_2 = a_2^+ \circ a_2 - \psi_{12}^+ \psi_{12} - (s+q) = 0$$

$$\mathcal{H}^{(0)} = V_s \otimes V_{s+q}$$
$(\psi_{12}^+ \psi_{12} = 0)$

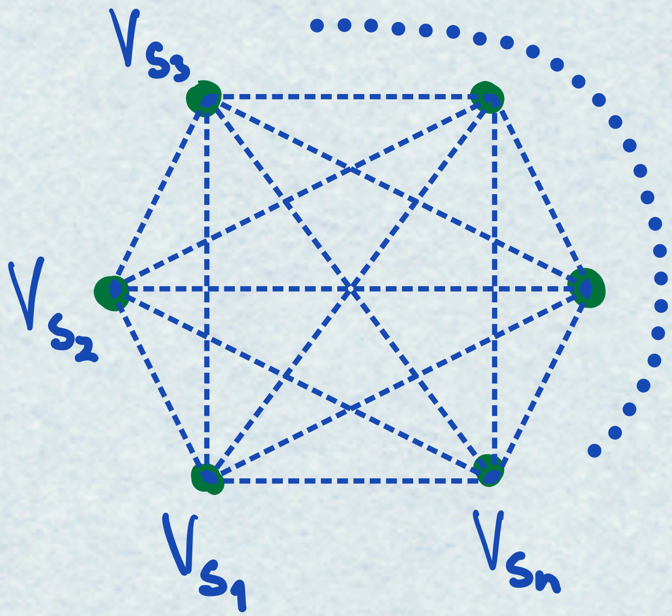
$$\mathcal{H}^{(1)} = V_{s-1} \otimes V_{s+q+1}$$
$(\psi_{12}^+ \psi_{12} = 1)$

$$W = \chi_s \chi_{s+q} - \chi_{s-1} \chi_{s+q+1} = \chi_q$$

Не зависит от s

$q+1$ состояний с $H=0$

Индекс: общий случай



$$s_A = s + q_A$$

$g \in SU(n)$

$$W = \mathrm{STr}\Big|_{c_1=c_2=0}(g) = \;?$$

1) Осцилляторная статсумма

$$Z(t|\lambda) = \mathrm{STr}\Big(\underbrace{\prod_{i=1}^{n} t_i^{J_i} \prod_{j=1}^{n} \lambda_j^{c_j}}_{=g}\Big) =$$

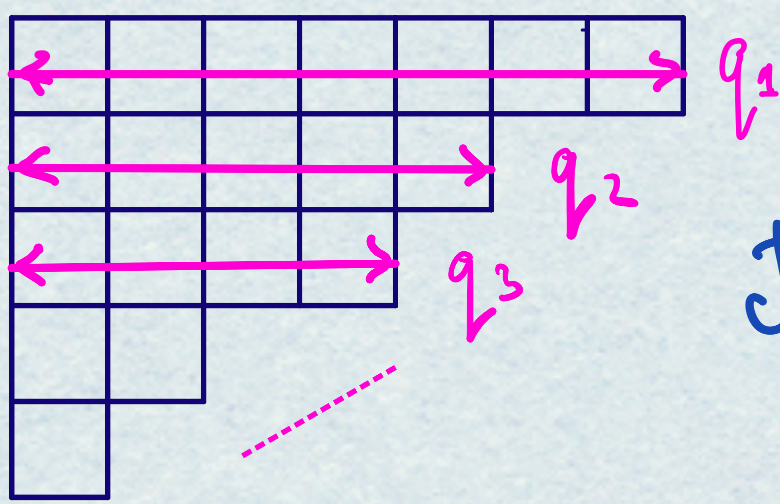
связи

$u(1)^n \subset u(n)$ генераторы

$$= \prod_{i,j=1}^{n} \frac{1}{1-t_i\lambda_j} \times \prod_{k<l} \left(1-\frac{\lambda_k}{\lambda_l}\right) \frac{1}{\lambda_1^{p_1}\lambda_2^{p_2}\lambda_3^{p_3}}$$

2) Ограничимся на $\{c_j = 0\}$ путём взятия вычетов

$$W = \oint \frac{d\lambda_1}{2\pi i \lambda_1} \cdots \oint \frac{d\lambda_n}{2\pi i \lambda_n} Z(t|\lambda) = \text{Формула Вейля для } \chi_{q_1,\ldots,q_n}$$



Теорема Бореля-Ботта-Вейля

Расшир. SUSY: W = эйлерова х-ка $F_n = n!$

Выводы / планы

* Магнитные модели на Бn | Yamaguchi '1979, Kuwabara '1988

* Вычисление спектров магнитных Δ на многообр. флагов $\mapsto$ 'спиновая цепочка'

* Теорема об индексе через осцилляторные статсуммы

* Интегрируемая задача? Явное решение? | DB, Kuzovchikov '2024

* Обобщение на SO / Sp орбиты?

* Обобщение на некомпактные орбиты?

  $L_2(\mathbb{H}_2) = \dots$

* Обобщение на ∞-мерные группы (петель...)

* Связь с 2D сигма-моделями / гармоническими отображ.

Спасибо!