Breaking of internal solitary waves in a three-layer fluid over an obstacle

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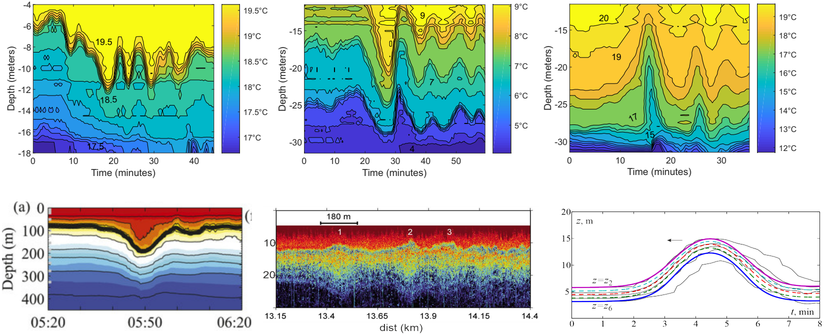
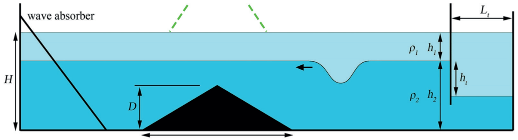
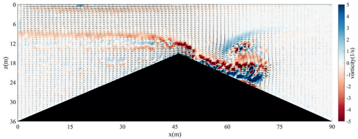
Introduction

Across the world's oceans, variations in seawater temperature and salinity stratify the water column, producing conditions where density disturbances can propagate as internal solitary waves (ISWs).



Field observations: Scotti & Pineda, 2004; Helfrich & Melville, 2006; Shroyer et al, 2010; Lien et al, 2014; Rayson et al, 2019; Liapidevskii et al.





- Field observations of ISWs (Moum et al., 2003; Lamb & Farmer, 2011; Lien et al., 2014; Zhang & Alford, 2015; Bourgault et al., 2016; Chang et al., 2021; Cyriac et al., 2023; Kozlov et al., 2023; Sun et al., 2024; Morozov, Makarenko et al; Liapidevskii et al.; ...).
- Laboratory experiments and numerical simulation (Grue et al., 1999; Chen, 2007; Chen et al, 2008; Fructus et al., 2009; Aghsaee et al., 2011; Carr et al., 2011, 2017; Zou et al. 2020; Zhao et al., 2020; Nian et al., 2023; Zheng et al., 2024; Gavrilov et al.; ...).
- Theoretical study and modelling of ISWs (Choi & Camassa, 1999; Choi, 2000; Rusås & Grue, 2002; Barros et al., 2007; Forgia & Sciortino, 2020, 2021; Derzho, 2022; Tseluiko et al., 2023; Zhao et al., 2023; Makarenko et al; Chesnokov & Liapidevskii; ...).

Our goal is to present and validate a “simple” 1D model describing the propagation and breaking of ISWs over an obstacle.

The long-wave model is based on a three-layer representation of stratified flow, taking into account dispersive effects and turbulent mixing during wave breaking.

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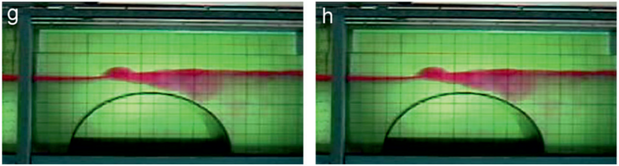
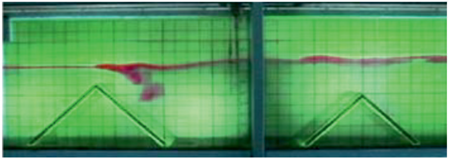




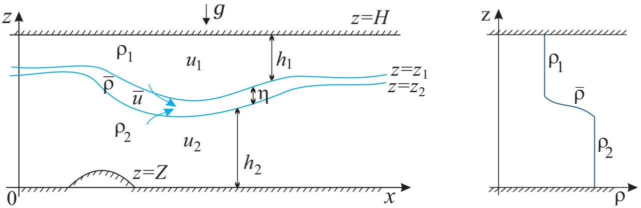
Mathematical model

We consider a 2D motion of a 3-layer fluid confined between the surfaces $z = Z(x)$ and $z = H = \mathrm{const}$. The governing equations are

$$u_x + w_z = 0, \quad \rho_t + (u\rho)_x + (w\rho)_z = 0,$$
$$(\rho u)_t + (\rho u^2)_x + (\rho uw)_z + p_x = 0,$$
$$\varepsilon^2\big((\rho w)_t + (\rho uw)_x + (\rho w^2)_z\big) + p_z = -g\rho.$$

We apply the 2-order approximation of the shallow water theory ($\varepsilon \ll 1$).
Boundary conditions: pressure p is continuous at the interfaces, and

$$uZ_x = w\big|_{z=Z}, \quad w\big|_{z=H} = 0, \quad z_{jt} + uz_{jx} - w\big|_{z=z_j} = (-1)^j M_j \quad (j = 1, 2).$$



Assumptions:

- The outer layer are homogeneous with constant densities ρ_i, the flow in them is almost potential $u_z \approx 0$;
- The intermediate layer is non-homogeneous and hydrostatic, the flow in it can be shear (or turbulent);
- The stable stratification is weak $0 < (\rho_1 - \rho_2)/\rho_2 \ll 1$, $\rho_2 < \bar{\rho} < \rho_1$, thus the Boussinesq approximation can be applied.

The conservation of energy

$$E_t + \big((E+p)u\big)_x + \big((E+p)w\big)_z = 0, \quad E = \big((u^2 + \varepsilon^2 w^2)/2 + gz\big)\rho.$$

Depth-averaged variables:

$$u_1(t,x) = \frac{1}{h_1}\int_{z_1}^{H} u\,dz, \quad u_2(t,x) = \frac{1}{h_2}\int_{Z}^{z_2} u\,dz, \quad \bar{u}(t,x) = \frac{1}{\eta}\int_{z_2}^{z_1} u\,dz,$$

$$q^2(t,x) = \frac{1}{\eta}\int_{z_2}^{z_1} (u-\bar{u})^2\,dz \quad \text{or} \quad q^2 = \frac{1}{\eta}\int_{z_2}^{z_1} (\overline{u'^2} + \overline{w'^2})\,dz.$$

Derivation of a 1D model. Main steps:

(i) We integrate the governing equations over the layer thicknesses and use the boundary conditions. All terms of order $O(\varepsilon^{2+\gamma})$ are ignored ($\gamma > 0$).

(ii) Then we apply the Boussinesq approximation and introduce buoyancy:

$$b = g(\rho_2 - \rho_1)/\rho_1 = \mathrm{const} > 0, \quad \bar{b} = g(\bar{\rho} - \rho_1)/\rho_1 > 0\,.$$

(iii) We assume symmetrical fluid entrainment $M_j = M$ and following Townsend, 1956; Pope, 2000; Liapidevskii & Teshukov, 2000, we take

$$M = \sigma q, \quad \sigma = \mathrm{const} \quad (\sigma \sim \varepsilon^{\gamma},\ \gamma > 0).$$

The depth averaging procedure gives the energy equation (its consequence)

$$\frac{\partial q}{\partial t} + \bar{u}\frac{\partial q}{\partial x} + kq\frac{\partial \bar{u}}{\partial x} = \varphi, \qquad (*)$$
$$\varphi = \frac{\sigma}{2\eta}\Big((u_1 - \bar{u})^2 + (u_2 - \bar{u})^2 - (2 + \kappa\,\mathrm{sign}(q))q^2 - b\eta\Big);$$

and the following system:

$$
\begin{aligned}
&\frac{\partial h_1}{\partial t}+\frac{\partial u_1 h_1}{\partial x}=-\sigma q, \quad \frac{\partial \eta}{\partial t}+\frac{\partial \bar{u}\eta}{\partial x}=2\sigma q,\\
&\frac{\partial h_2}{\partial t}+\frac{\partial u_2 h_2}{\partial x}=-\sigma q, \quad \frac{\partial}{\partial t}(\bar{b}\eta)+\frac{\partial \bar{u}\bar{b}\eta}{\partial x}=b\sigma q,\\
&\frac{\partial u_1}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u_1^2}{2}+\Pi\right)+\frac{\beta_1}{3h_1}\frac{\partial}{\partial x}\left(h_1^2\frac{d_1^2 h_1}{dt^2}\right)=0,\\
&\frac{\partial u_2}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u_2^2}{2}+\bar{b}\eta+bh_2+\Pi\right)+\frac{\beta_2}{3h_2}\frac{\partial}{\partial x}\left(h_2^2\frac{d_2^2 h_2}{dt^2}\right)=-\frac{\partial Z}{\partial x},\\
&\frac{\partial Q}{\partial t}+\frac{\partial}{\partial x}\Bigg(u_1^2 h_1+(\bar{u}^2+kq^2)\eta+u_2^2 h_2+\frac{\bar{b}\eta^2}{2}+\frac{bh_2}{2}+\\
&\qquad +H\Pi+\frac{\beta_1 h_1^2}{3}\frac{d_1^2 h_1}{dt^2}+\frac{\beta_2 h_2^2}{3}\frac{d_2^2 h_2}{dt^2}\Bigg)=-(\bar{b}\eta+bh_2+\Pi)\frac{\partial Z}{\partial x}.
\end{aligned}
\tag{1}
$$

Here $H(x)=h_1+\eta+h_2=H_0-Z(x)$ and $Q(t)=u_1 h_1+\bar{u}\eta+u_2 h_2$ are the total depth and flow rate; Π is the pressure at $z=H_0$; constants σ and κ are responsible for mixing and dissipation; $k=1$ or $k=0$ (depends on the definition of q^2); $\beta_i=1$ or $\beta_i=0$ (if i-th layer is hydrostatic).

The differential consequences of (1) are

$$\bar{b}_t + \bar{u}\bar{b}_x = \frac{\sigma q}{\eta}(b - 2\bar{b}), \quad \bar{u}_t + \bar{u}\bar{u}_x + 2kqq_x + \bar{b}h_{2x} +$$
$$+\Big(\bar{b} + \frac{kq^2}{\eta}\Big)\eta_x + \frac{\eta}{2}\bar{b}_x + \Pi_x = \frac{\sigma q}{\eta}(u_1 + u_2 - 2\bar{u}) - \bar{b}Z_x.$$

Obviously, if $\bar{b} = b/2$ at $t = 0$, then $\bar{b} = b/2$ for all $t > 0$.

For hydrostatic flows ($\varepsilon_i = 0$) the model with $k = 1$ and $k = 0$ gives similar results (Chesnokov et al, 2022; Liapidevskii & Chesnokov, 2022.

In what follows, we take $k = 0$, $\bar{b} = b/2$.

Equilibrium model. Let the right-hand side of equation for q is equal to zero, i.e. $\varphi = 0$. Then the fluid entrainment (mixing) is described by

$$q^2 = \max\left\{0, \frac{(u_1 - \bar{u})^2 + (u_2 - \bar{u})^2 - b\eta}{2 + \kappa}\right\}, \quad q = \sqrt{q^2}. \qquad (2)$$

Note, the presence of a sufficiently large velocity shear gives rise to the vorticity q/η in the intermediate layer.

Numerical implementation of the model

We split the governing equations into hyperbolic and elliptic parts:

$$\begin{aligned} &\frac{\partial h_i}{\partial t} + \frac{\partial u_i h_i}{\partial x} = -\sigma q, \quad (i = 1, 2) \\ &\frac{\partial K_i}{\partial t} + \frac{\partial}{\partial x}\left(u_i K_i - \frac{(u_i - \bar{u})^2}{2} + \frac{b h_i}{2} - \frac{\beta_i \alpha_i^2 h_i^2}{2}\right) = M + L_i, \end{aligned} \tag{3}$$

where

$$K_i = u_i - \bar{u} - \frac{\beta_i}{3h_i}\frac{\partial}{\partial x}\big(h_i^3 \alpha_i\big), \quad \alpha_i = \frac{\partial u_i}{\partial x} \quad (i = 1, 2) \tag{4}$$

and

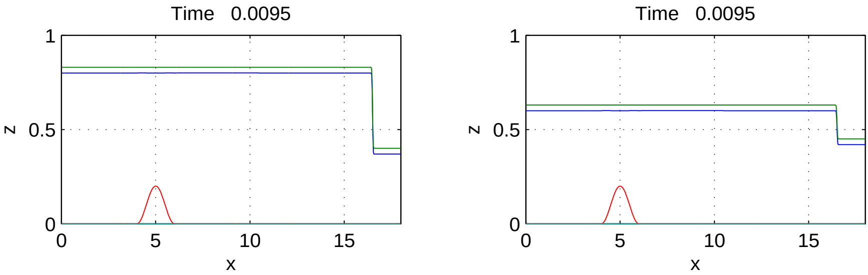
$$\begin{aligned} &M = -\frac{\sigma q}{\eta}(u_1 + u_2 - 2\bar{u}), \quad L_1 = 0, \quad L_2 = \frac{b}{2}\frac{\partial Z}{\partial x}, \\ &\eta = H_0 - Z - h_1 - h_2, \quad \bar{u} = \frac{Q - u_1 h_1 - u_2 h_2}{\eta}, \quad (\bar{b} \equiv b/2). \end{aligned}$$

Equations (3), (4) and mixing law (2) form a close system for determining the thicknesses h_i of fluid layers and the velocities u_i in them.

Numerical algorithm. Knowing the solution at time t, we

- solve of the evolutionary system (3) and find (h_i, K_i) at time $t + \Delta t$ using a Godunov-type scheme;
- determine velocities u_i as a result of solving ODEs (4) by the matrix sweep method.

Test calculation: formation of a solitary wave and its interaction with a smooth obstacle. Here and below we take $\sigma = 0.15$, $\kappa = 6$.



The left panel shows the mixing that occurs in the wave crest region. The intermediate layer increases behind the wave.

Stationary solutions

For the class of stationary solutions (or travelling waves) this model takes the following normal form ($Z = 0$):

$$h_i' = p_i, \quad p_i' = r_i, \quad u_i' = \frac{u_i p_i + \sigma q}{h_i}, \quad (i = 1, 2)$$

$$\begin{aligned} \beta_1 r_1' &= \left(2r_1 - \frac{p_1^2}{h_1}\right)\frac{\beta_1 p_1}{h_1} \\ &+ \frac{3}{u_1^2 h_1}\left(\left(\frac{u_1^2}{h_1} + \frac{\bar{u}^2}{\eta} - \frac{b}{2}\right)p_1 + \frac{\bar{u}^2}{\eta}p_2 + \sigma q S_1\right), \\ \beta_2 r_2' &= \left(2r_2 - \frac{p_2^2}{h_2}\right)\frac{\beta_2 p_2}{h_2} \\ &+ \frac{3}{u_2^2 h_2}\left(\frac{\bar{u}^2}{\eta}p_1 + \left(\frac{\bar{u}^2}{\eta} + \frac{u_2^2}{h_2} - \frac{b}{2}\right)p_2 + \sigma q S_2\right), \end{aligned} \tag{5}$$

where

$$S_i = \frac{u_i}{h_i} + \frac{4\bar{u} - u_1 - u_2}{\eta} \qquad (i = 1, 2).$$

Bifurcation from a constant flow. We are looking for solutions of (5) in the form of a solitary wave such that

$$h_i = h_{i0}, \quad \eta = \eta_0, \quad u_i = \bar{u} = U, \quad p_i = r_i = 0$$

as $x \to -\infty$. We linearize (5) on the constant solution. For determining small perturbations we have:

$$\tilde{h}_i' = \tilde{p}_i, \ \tilde{p}_i' = \tilde{r}_i, \ \tilde{u}_i' = -\frac{u_{i0}}{h_{i0}}\tilde{p}_i, \ \gamma_1 \tilde{r}_1' = a_1 \tilde{p}_1 + \tilde{p}_2, \ \gamma_2 \tilde{r}_2' = \tilde{p}_1 + a_2 \tilde{p}_2.$$

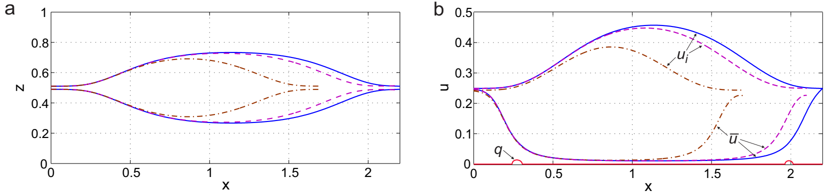
Representation of solutions: $(\tilde{h}_i, \tilde{p}_i, \tilde{r}_i, \tilde{u}_i) = (\hat{h}_i, \hat{p}_i, \hat{r}_i, \hat{u}_i)\exp(\nu x)$.
Here $\hat{h}_i$, $\hat{p}_i$, $\hat{r}_i$, and $\hat{u}_i$ are the amplitudes, and ν is a positive parameter. Substitution in the previous equations yields

$$\hat{p}_i = \nu \hat{h}_i, \quad \hat{r}_i = \nu^2 \hat{h}_i, \quad \hat{u}_i = -u_{i0}\hat{h}_i/h_{i0},$$

$$(a_1 - \gamma_1 \nu^2)\hat{h}_1 + \hat{h}_2 = 0, \quad \hat{h}_1 + (a_2 - \gamma_2 \nu^2)\hat{h}_2 = 0.$$

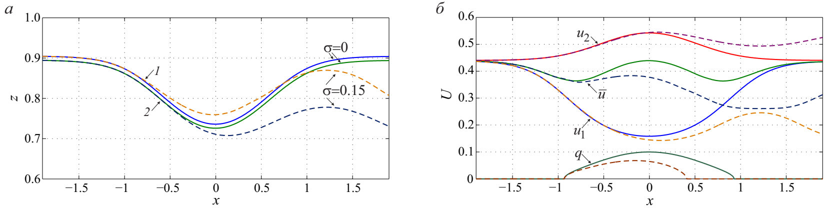
Equality of the determinant to zero allows us to find $\nu > 0$; we set h_1 and determine all other amplitudes.

Examples of soliton-like solutions. Froude number $\mathrm{Fr} = |U|/\sqrt{bH}$ defines a one-parameter family of solutions. If both outer layers are non-hydrostatic ($\beta_i = 1$), then the solution has the form of **mode-2 wave**.

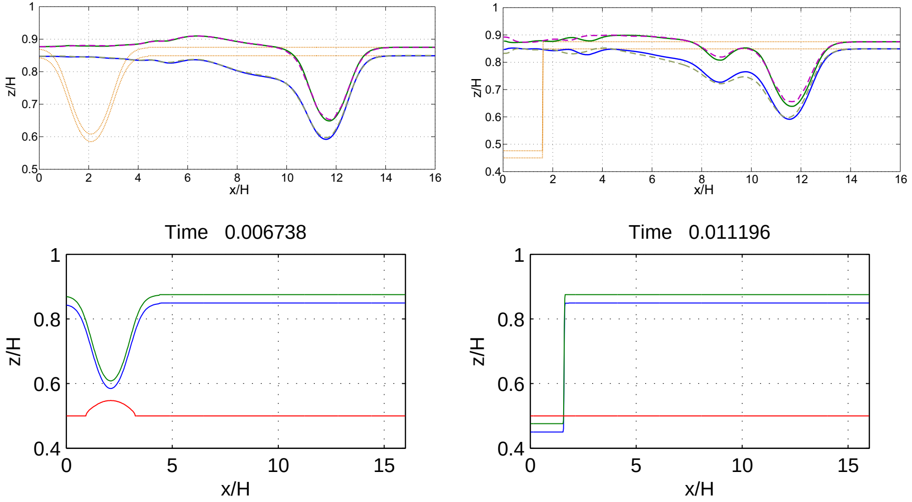


$\mathrm{Fr} = 0.249$, $\sigma = 0.15$ (solid), $\sigma = 0$ (dashed); $\mathrm{Fr} = 0.243$ – dash-dotted.

Mode-1 waves: $\beta_1 = 0$, $\beta_2 = 1$, $\eta_0 = 0.01\, H$, and $\mathrm{Fr} = 0.44$.



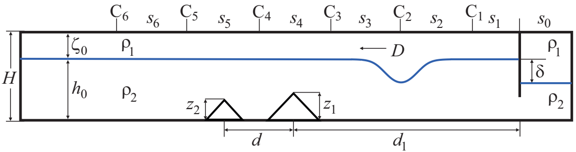
Large-amplitude waves: numerical results (dashed curves obtained by the model with only bottom non-hydrostatic layer).



Interfaces $z = H_0 - h_1$, $z = h_2$ and shear velocity $q + 0.5$.

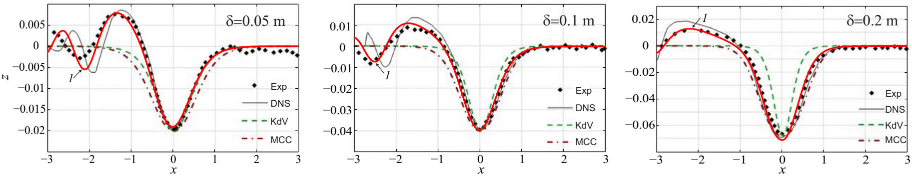
Model validation

Scheme of the experimental setup (Chen, 2007; Chen et al., 2008).



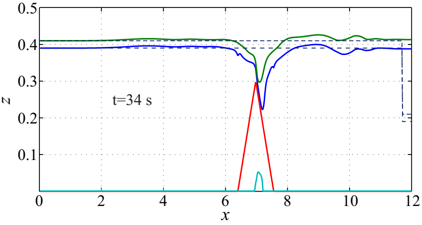
$$H = 0.5,\ L = 12,\ \zeta_0 = 0.1,\ h_0 = 0.4,\ s_0 = 0.3 \text{ m},\ b \approx 0.23 \text{ m/s}^2$$

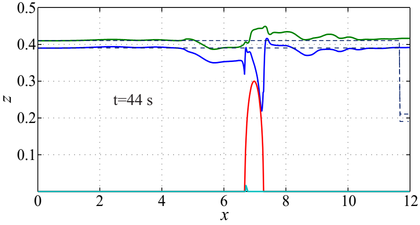
Test 1: lock-exchange problem (no obstacles).

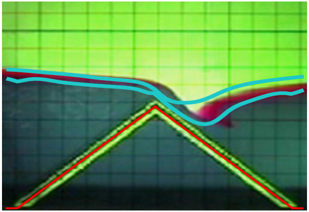


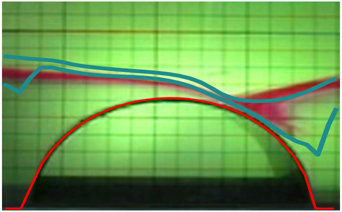
Comparison with experiment Chen, 2007 and DNS Zhu et al., 2016.

Test 2: interaction of an ISW with a single obstacle (triangle / semicylinder). Dashed curves – initial data.

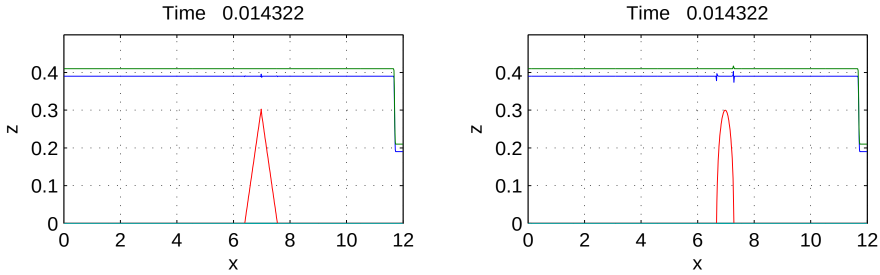




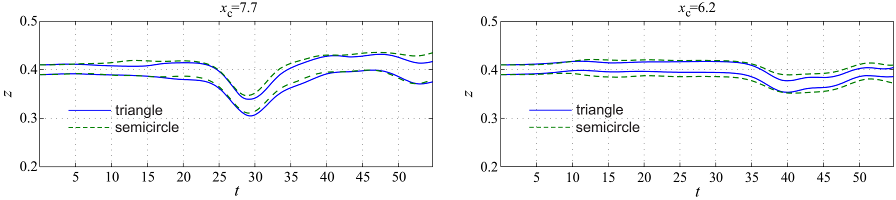




Comparison with experiment (Chen, 2007) at $t = 34$ s.

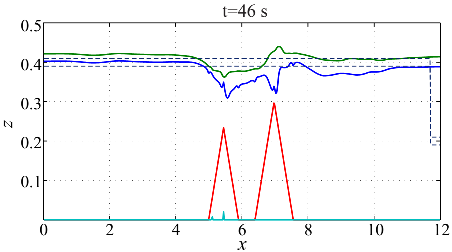


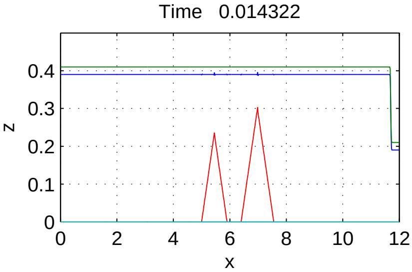
Breaking of a solitary wave over an obstacle

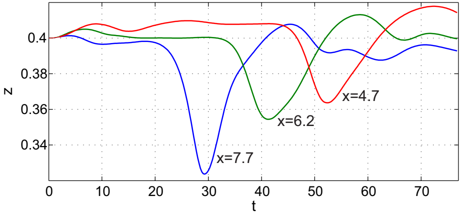


Comparison of the interfaces position at points $x = 7.7$ ($C3$) and $x = 6.2$ ($C4$) depending on time t for triangular and cylindrical obstacles.

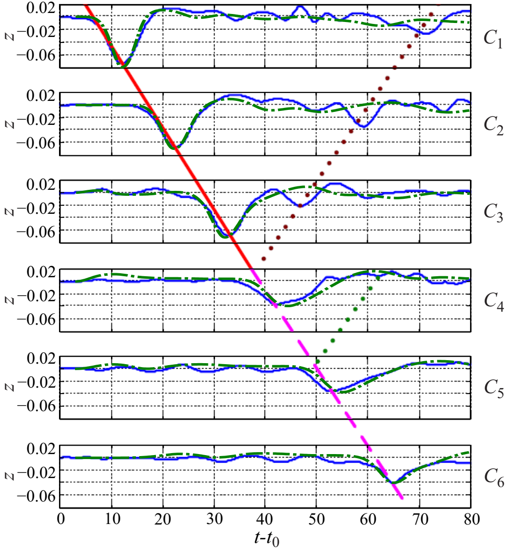
Test 3: formation of ISW and its interaction with two triangular obstacles. This calculation corresponds to the experiment (Chen at al., 2008).







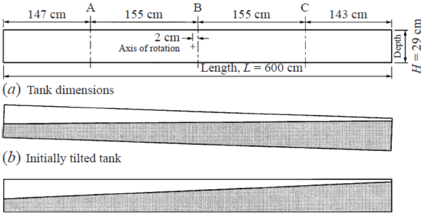
Midline $z = h_2 + \eta/2$ as a function of t at stations $C3$, $C4$ and $C5$.



Midline $z = h_2 - h_{20} + (\eta - \eta_0)/2$: blue solid curves are the experiment (Chen at al., 2008); green dashed lines — calculations by the model.

Current studies: Internal seiches

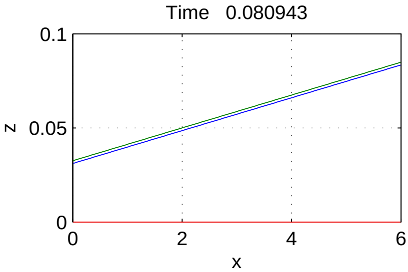
Test 1. Comparison with experiments (Horn et al., 2001).



(a) Tank dimensions

(b) Initially tilted tank

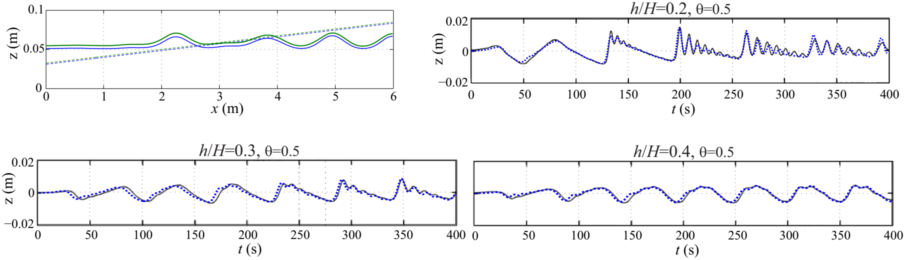
(c) Initial condition with the tank horizontal and the interface inclined



We add the terms responsible for friction to the right-hand sides of the momentum equations:

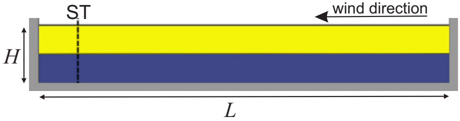
$$f_1 = -\frac{c_w u_1|u_1| + c_i(u_1 - \bar{u})|u_1 - \bar{u}|}{h_1}, \quad f_2 = -\frac{c_w u_2|u_2| + c_i(u_2 - \bar{u})|u_2 - \bar{u}|}{h_2},$$

$$\bar{f} = -\frac{c_i}{\eta}\Big((\bar{u} - u_1)|\bar{u} - u_1| + (\bar{u} - u_2)|\bar{u} - u_2|\Big), \quad c_i = (1 - \operatorname{sign} q)c_i.$$



Interface at $x = L/2$ depending on time: experiment and calculation.

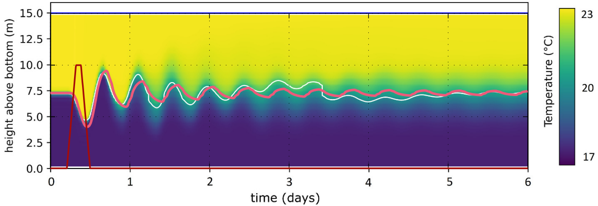
Test 2. Internal waves in a thermally stratified reservoir generated by wind (Bueno et al., 2021). Comparison with DNS.

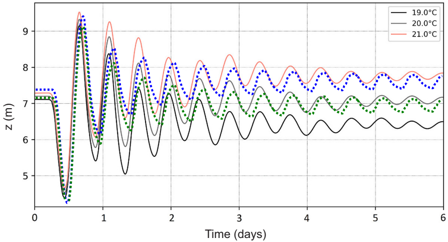


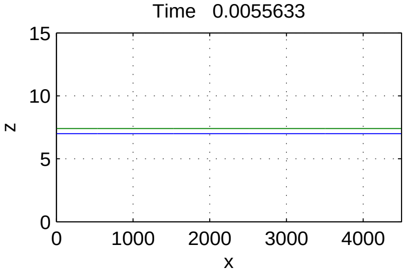
H=15 m
L=4.5 km
W_{max}=10 m/s

Modification of $f_1 = f_1 - U_*^2/h_1$ (U_* normalized wind speed).

Numerical results: DNS (Bueno et al., 2021) against the proposed 1D model (3 layers with mixing).







Conclusion

- We propose a 1D model describing the evolution of ISWs and their breaking due to interaction with a single or combined obstacle.
- The model is capable of describing the mixing processes during wave breaking. The numerical implementation is based on splitting the system into hyperbolic and elliptic parts.
- The model is validated against known experimental data and DNS.

The results are presented in the manuscript:

Liapidevskii & Chesnokov, Breaking of ISWs in a three-layer fluid over an obstacle, Comput. Math. Math. Phys. 2025. (accepted).

THANK YOU FOR ATTENTION !