Об одном подходе к построению асимптотик собственных функций оператора Лапласа в эллипсе, основанном на некомпактных лагранжевых многообразиях

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The spectral problems for the Laplace operator in the ellipse

$$\Omega = \{\frac{x_1^2}{a} + \frac{x_2^2}{b} \leq 1\} \quad a,\, b > 0 \quad \text{are constants}$$

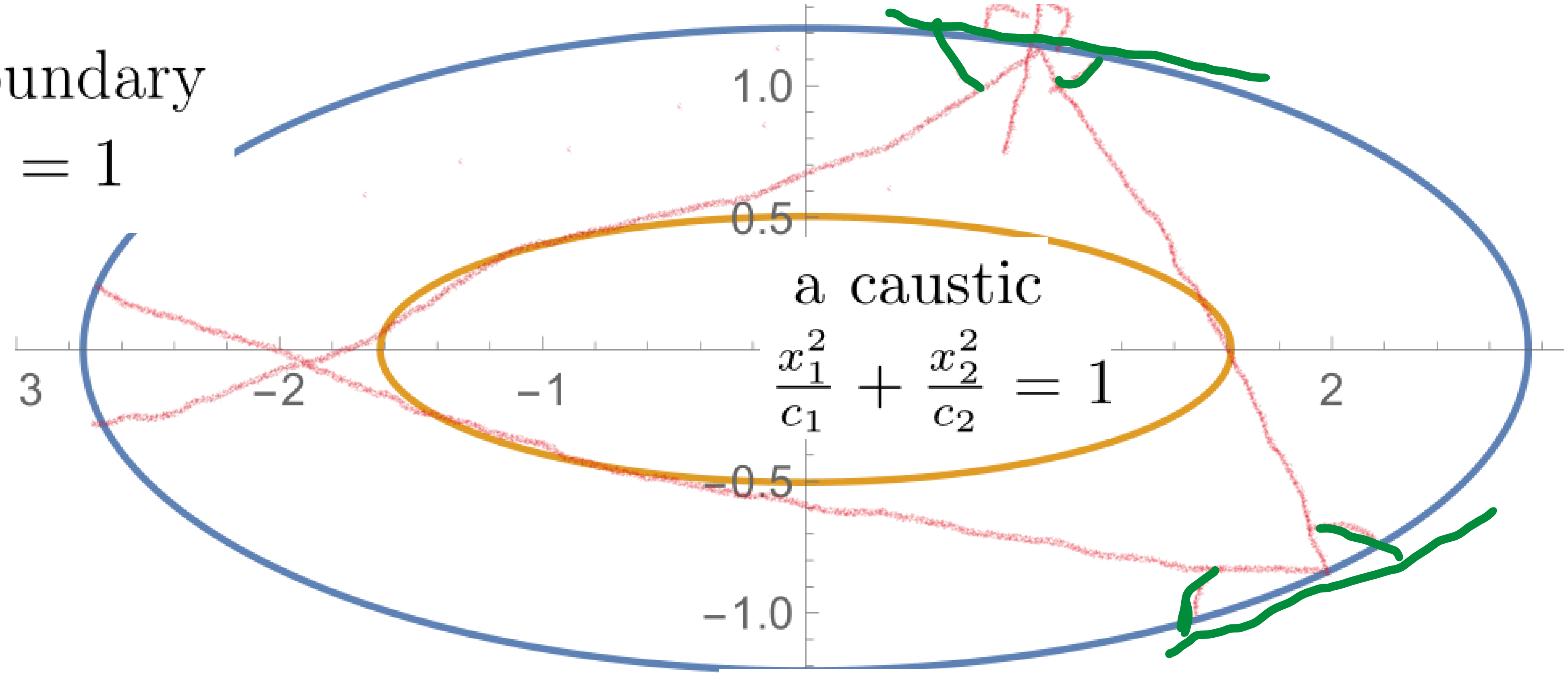
$$-\Delta w(x_1, x_2) = \lambda w(x_1, x_2), \quad (x_1, x_2) \in \Omega,$$

$$w|_{\partial\Omega} = 0 \quad \text{(Dirichlet condition) or} \quad \frac{\partial w}{\partial n}|_{\partial\Omega} = 0 \quad \text{(Neumann condition)}$$

Asymptotics for large Λ or small $h = 1/\lambda$ and their relations with Birkhoff Billiards billiards: V. F. Lazutkin, KAM Theory and Semiclassical Approximations to Eigenfunctions, 1993, Springer-Verlag

Classical Birkhoff biliards: В. В. Козлов, Д. В. Трещев, Бильярды. Генетическое введение в динамику систем с ударами, М., МГУ, 1991

The boundary
$$\frac{x_1^2}{a} + \frac{x_2^2}{b} = 1$$



Integrable system: $H = p^2$, $F = \frac{p_1^2}{a} + \frac{p_2^2}{b} - \frac{(p_1 x_2 - x_1 p_2)^2}{ab}$, $\{H, F\} = 0$

The appropriate constructed manifold Λ is a joint level surface of the Hamiltonians H and F

$$H|_\Lambda = 1, \quad F|_\Lambda = \frac{\lambda}{(c_1 + l)(c_2 + l)} = g\,(\text{const})$$

the ellipse $\frac{x_1^2}{a} + \frac{x_2^2}{b}$ is confocal for the ellipse that forms the caustic

$$a = c_1 + l,\, b = c_2 + l,$$

The answer: $w = K_\Lambda^h A$, Λ is the appropriate Lagrangian manifold constructed from trajectories, K_Λ is the Maslov canonical operator with appropriate **measure**, A is the appropriate **amplitude** $+$
$h = h_{n,k}$, which are found from the Bohr-Sommerfeld quantization condition.

Our goal is to construct a suitable Lagrangian manifold with a relatively simple natural parametrization based on the *caustics* of the problem.

The basic idea is similar to the one used in asymptotic constructions related to Bessel functions in the paper S. Yu. Dobrokhotov, D. S. Minenkov, V.E.Nazaikinskii, Representations of Bessel functions via the Maslov canonical operator, Theoret. and Math. Phys., 208:2 (2021), 1018-1037

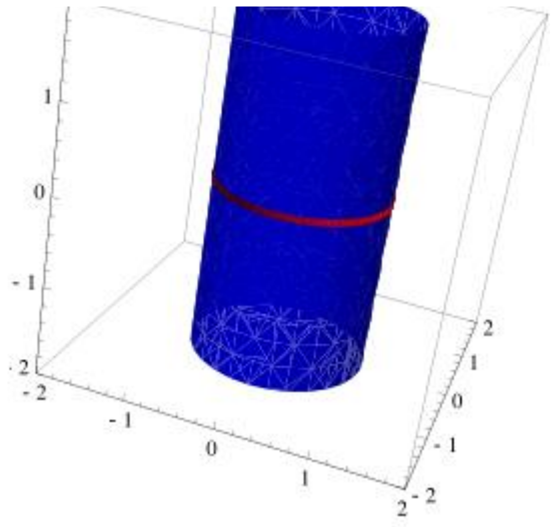
The Lagrangian manifolds and Bessel functions

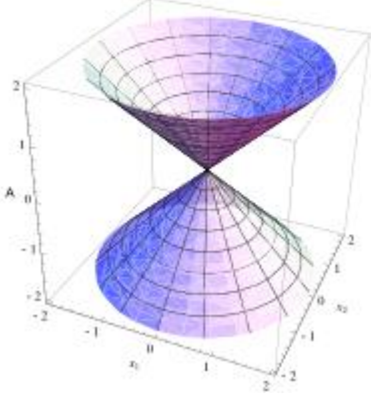
$$w = e^{in\phi}\mathbf{J}_0\left(\frac{r}{h}\right) \qquad\qquad n = \gamma h$$

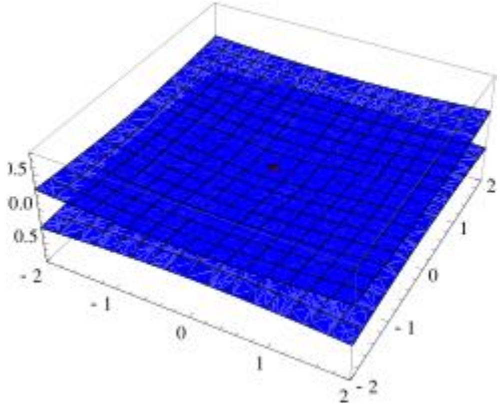
$$\hat{p}^2 w \equiv -h^2 \triangle w = w, \quad \hat{M}w \equiv (x_2\hat{p}_1 - x_1\hat{p}_2)w = \gamma w, \quad \hat{p}_j = -ih\frac{\partial}{\partial x_j},$$

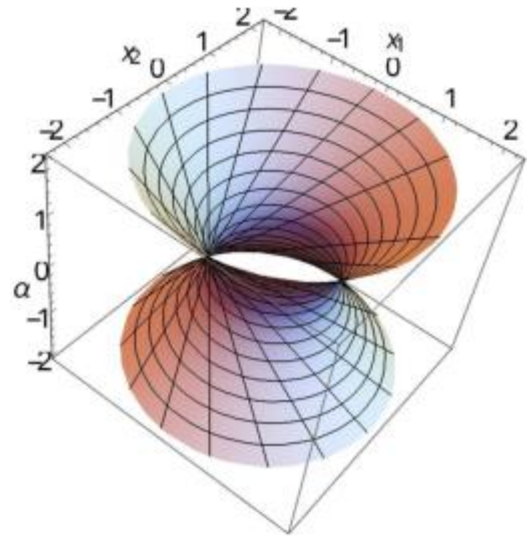
$$\Lambda^2_\gamma = \{(p,x) \in \mathbb{R}^4 : \quad p^2 = 1, \quad p_1x_2 - p_2x_1 = \gamma\} =$$

$$\{p = \mathbf{n}(\phi), \quad x = \phi\mathbf{n}(\psi) - \gamma\mathbf{n}'(\psi), \quad \psi \in \mathbb{S}^1, \phi \in \mathbb{R}\}, \quad \mathbf{n}(\phi) = \begin{pmatrix}\cos\psi \\ \sin\psi\end{pmatrix}$$









the Hamiltonian system $\dot{x} = 2p$, $\dot{p} = 0$

an ellipse $\frac{x_1^2}{c_1} + \frac{x_2^2}{c_2} = 1$ is a caustic

we put

$$x_1 = X_1(\varphi, \tau) := \sqrt{c_1}\cos\varphi - \tau \frac{2\sqrt{c_1}\sin\varphi}{\sqrt{c_1\sin^2\varphi + c_2\cos^2\varphi}},$$

$$x_2 = X_2(\varphi, \tau) := \sqrt{c_2}\sin\varphi + \tau \frac{2\sqrt{c_2}\cos\varphi}{\sqrt{c_1\sin^2\varphi + c_2\cos^2\varphi}},$$

$$p_1 = P_1(\varphi) := -\frac{\sqrt{c_1}\sin\varphi}{\sqrt{c_1\sin^2\varphi + c_2\cos^2\varphi}},$$

$$p_2 = P_2(\varphi) := \frac{\sqrt{c_2}\cos\varphi}{\sqrt{c_1\sin^2\varphi + c_2\cos^2\varphi}},$$

The surface

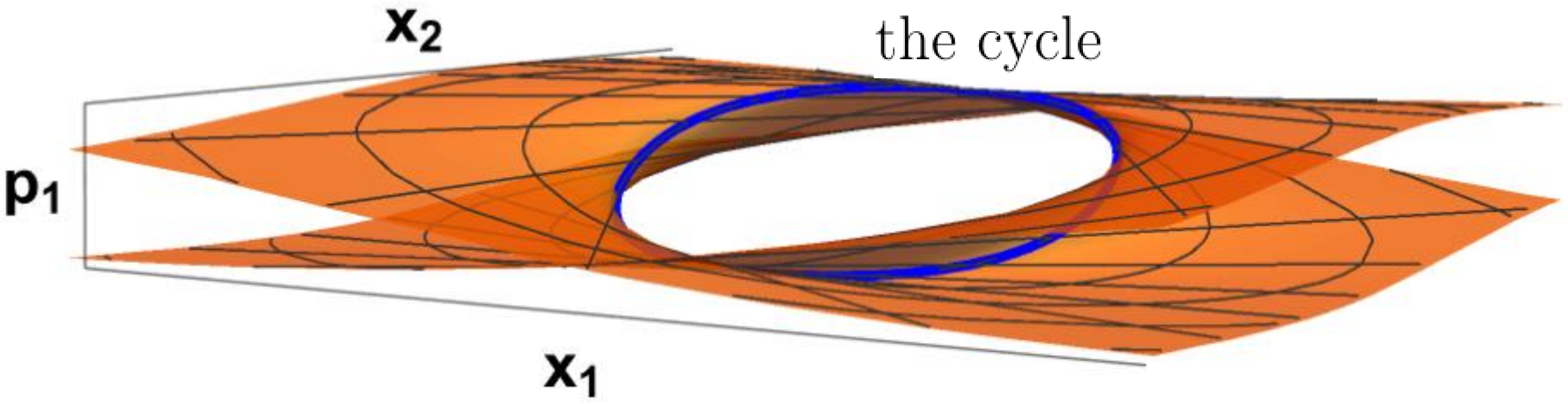
$$\Lambda = \{x_1 = X_1(\varphi, \tau),\, x_2 = X_2(\varphi, \tau),\, p_1 = P_1(\varphi),\, p_2 = P_2(\varphi) \,|\, \varphi \in S^1,\, \tau \in \mathbb{R}\}$$

is a smooth two-dimensional Lagrangian manifold in the four-dimensional phase space $\mathbb{R}^4_{(x_1,x_2,p_1,p_2)}$.

$$J = \det \frac{\partial X}{\partial(\tau, \phi)} = -\frac{4\sqrt{c_1}\sqrt{c_2}\tau}{c_1 \sin^2(\varphi) + c_2 \cos^2(\varphi)} = 0 \quad \Rightarrow \quad \tau = 0$$

Λ is **noncompact** Lagrangian manifold with a Lagrangian singularity of the fold type which is the ellipse

$$\Gamma := \left\{ \frac{x_1^2}{c_1} + \frac{x_2^2}{c_2} = 1 \right\}$$



one cycle $\Longrightarrow$ one Bohr-Sommerfeld quantization rule

$$\frac{1}{2\pi h}\int_\gamma <\mathbf{p}, d\mathbf{x}> \in \mathbb{Z} \quad \Leftrightarrow \quad = \frac{2}{\pi h}\sqrt{c_2}\mathcal{E}\left(1-\frac{c_1}{c_2}\right) = n,\ n\in\mathbb{Z},$$

where $\mathcal{E}(z) := \int_0^{\pi/2}\sqrt{1-z\sin^2\theta}d\theta$ is an elliptic function.

Remark

$$\frac{1}{\pi h}\sqrt{b(1-ag)}\mathcal{E}\left(\frac{b-a}{b(1-ag)}\right) = n,\ n\in\mathbb{Z}.$$

Action, Phases, and Indices of Charts

$$S(\phi,t)=\int_0^{\phi}\sqrt{c_1\sin^2(\theta)+c_2\cos^2(\theta)}d\theta+\int_0^t 2ds=\sqrt{c_2}E\left(\phi,1-\frac{c_1}{c_2}\right)+2t$$

where $E(\psi,z):=\int_0^{\psi}\sqrt{1-z\sin^2\theta}d\theta$

two nonsingular charts $\Omega_+=\{\tau>\delta\},\quad \Omega_-=\{\tau<-\delta\},\quad \delta>0.$

$$m_+=0,\quad m_-=1$$

$$\varphi_{\pm}(x_1,x_2)=\arctan\left(\frac{x_2\sqrt{c_1}}{x_1\sqrt{c_2}}\right)\mp\arccos\left(\frac{\sqrt{c_2}\sqrt{c_1}}{\sqrt{x_1^2c_2+x_2^2c_1}}\right),$$

$$\tau_{\pm}(x_1,x_2)=\pm\frac{1}{2}\sqrt{(x_2-\sin(\varphi_{\mp})\sqrt{c_2})^2+(x_1-\cos(\varphi_{\mp})\sqrt{c_1})^2}.$$

WKB-asymptotics with the amplitude $A(\phi)$

$$[K_\Lambda A(\varphi,\tau)](x_1,x_2) \approx e^{\frac{i\pi}{2}m_+}\frac{A_+}{\sqrt{|J_+|}}e^{\frac{iS_+}{h}} + e^{\frac{i\pi}{2}m_-}\frac{A_-}{\sqrt{|J_-|}}e^{\frac{iS_-}{h}}$$

$$= \exp\left(-\frac{i\pi}{4}\right)\exp\left(\frac{i\left(S_+ + S_-\right)}{2h}\right)\left(\left(\frac{A_+}{\sqrt{|J_+|}} + \frac{A_-}{\sqrt{|J_-|}}\right)\right.$$

$$\left.\times\cos\left(\frac{S_+ - S_-}{2h} - \frac{\pi}{4}\right) + i\left(\frac{A_+}{\sqrt{|J_+|}} - \frac{A_-}{\sqrt{|J_-|}}\right)\sin\left(\frac{S_+ - S_-}{2h} - \frac{\pi}{4}\right)\right)$$

$$S_\pm(x_1,x_2) = S(\varphi_\pm,\tau_\pm), \quad J_\pm(x_1,x_2) = J(\varphi_\pm,\tau_\pm), \quad A_\pm(x_1,x_2) = A(\varphi_\pm).$$

Uniform asymptotics via the Airy functions

$$[K_\Lambda A](x_1,x_2) \approx e^{\frac{i}{h}\Theta(x_1,x_2)}\Big(a_{ev}(x_1,x_2)\mathrm{Ai}\left(-\frac{\Psi(x_1,x_2)}{h^{2/3}}\right) + a_{odd}(x_1,x_2)\mathrm{Ai}'\left(-\frac{\Psi(x_1,x_2)}{h^{2/3}}\right)\Big)$$

Uniform asymptotics via the Airy functions

A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. V. Tsvetkova, Uniform asymptotic solution in the form of an Airy function for semiclassical bound states in one-dimensional and radially symmetric problems, Theoret. and Math. Phys., 201:3 (2019), 1742-1770

See also: Yu. A. Kordyukov, I. A. Taimanov, Quasi-Classical Approximation of Monopole Harmonics, Math. Notes, 114:6 (2023), 1285-1296

$$\mathrm{Ai}(-z) \sim \frac{\sin\left(\frac{2}{3}z^{\frac{3}{2}} + \frac{\pi}{4}\right)}{z^{\frac{1}{4}}\sqrt{\pi}}, \quad \mathrm{Ai}'(-z) \sim -\frac{\cos\left(\frac{2}{3}z^{\frac{3}{2}} + \frac{\pi}{4}\right)z^{\frac{1}{4}}}{\sqrt{\pi}}$$

Finally we have constructed generalized asymptotic eigenfunctions of the Laplace operator

$$[K_\Lambda A](x_1,x_2) \approx e^{\frac{i}{h}\Theta(x_1,x_2)}\left(a_{ev}(x_1,x_2)\mathrm{Ai}\left(-\frac{\Psi(x_1,x_2)}{h^{2/3}}\right) + a_{odd}(x_1,x_2)\mathrm{Ai}'\left(-\frac{\Psi(x_1,x_2)}{h^{2/3}}\right)\right)$$

$$\Psi(x_1,x_2) = \left(\frac{3(S_+ - S_-)}{4}\right)^{\frac{2}{3}}, \quad \Theta(x_1,x_2) = \frac{(S_+ - S_-)}{2},$$

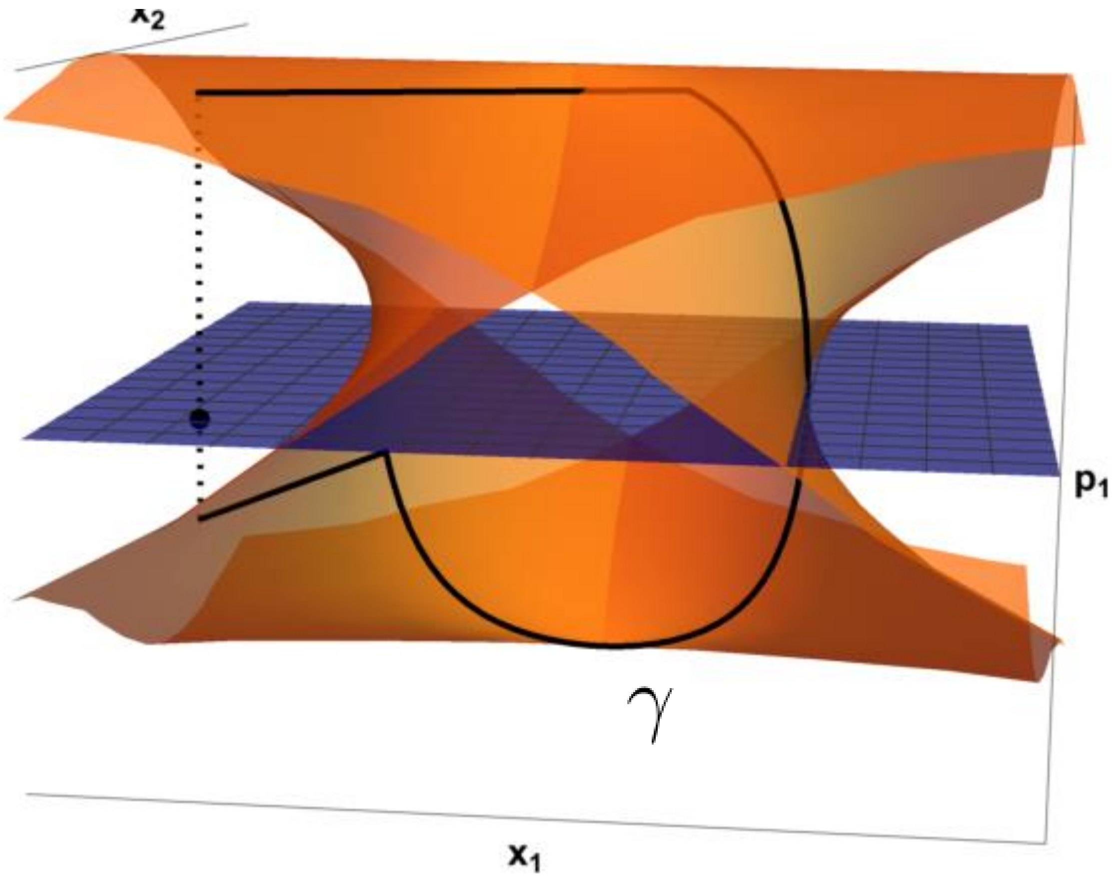
$$a_{ev}(x_1,x_2) = e^{-\frac{i\pi}{4}}\sqrt{\pi}\left(\frac{A_+}{\sqrt{|J_+|}} + \frac{A_-}{\sqrt{|J_-|}}\right)\left(\frac{3}{2}\left(\frac{S_+ - S_-}{2h}\right)\right)^{\frac{1}{6}},$$

$$a_{odd}(x_1,x_2) = ie^{-\frac{i\pi}{4}}\sqrt{\pi}\left(\frac{A_+}{\sqrt{|J_+|}} - \frac{A_-}{\sqrt{|J_-|}}\right)\left(\frac{3}{2}\left(\frac{S_+ + S_-}{2h}\right)\right)^{-\frac{1}{6}}$$

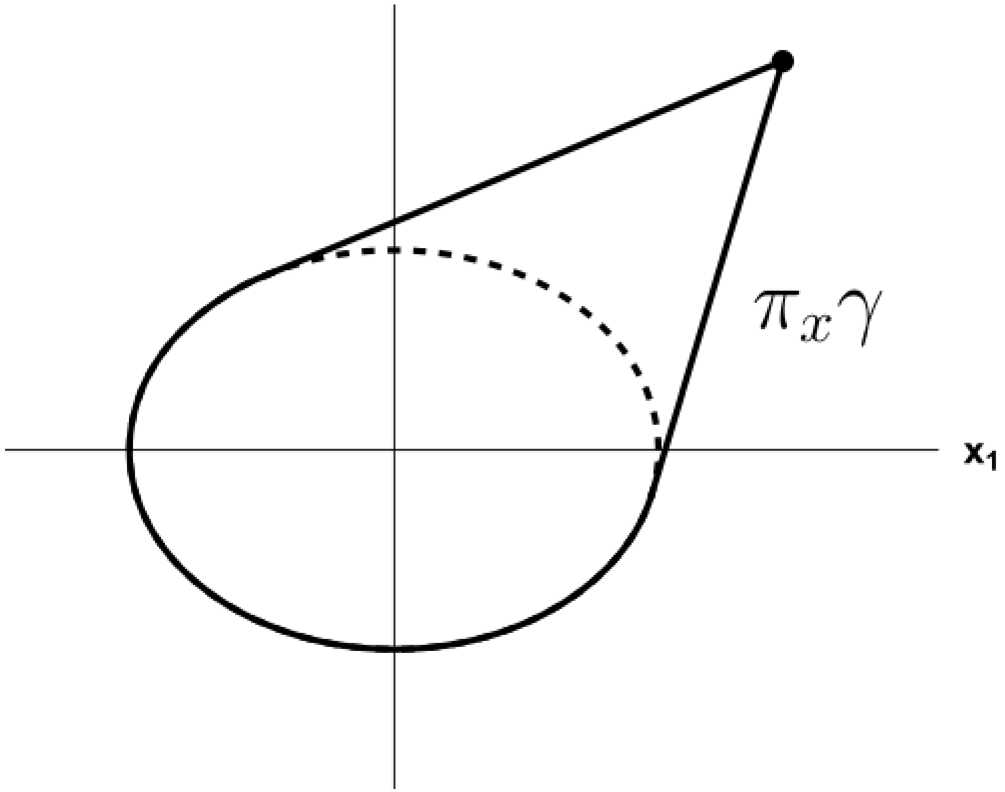
Restriction of Lagrangian manifolds and manifolds with boundary

First we try to choose the boundary in such a way that
$S(\varphi, \tau) \Longrightarrow S^{+}(x) - S^{-}(x) = 2\pi h(k - 1/4) \Longrightarrow$

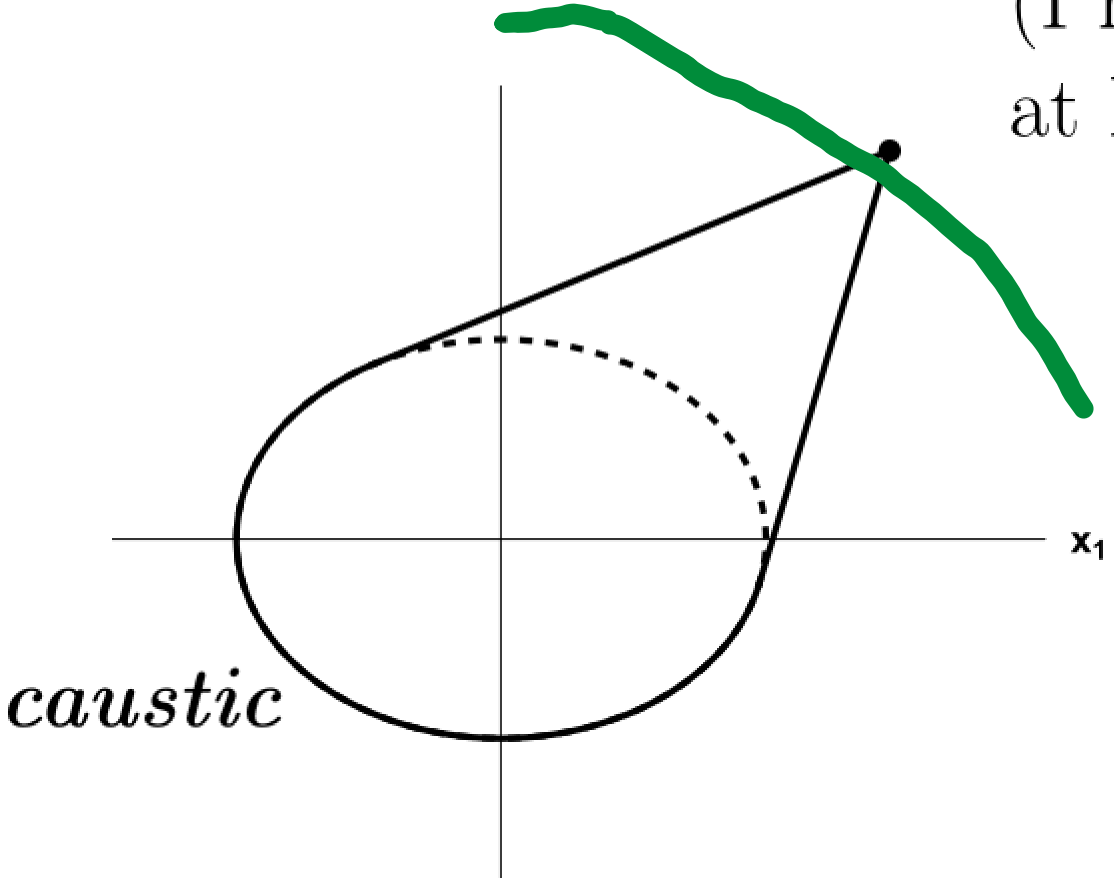
$$\frac{1}{2\pi h}\int_{\gamma} PdX = -\frac{1}{4} + k$$



(a) paths on the Lagrangian manifold



The Bohr-Sommerfeld rule provides a candidate for the boundary on which the Dirichlet condition can be satisfied



the sofocus ellipse
(I heard about it from A.T.Fomenko at his talk in June 2024)

The Bohr-Sommerfeld rule does not require the integrability,
but it is enough for construction of solutions to the boundary-value problem.
One needs to choose the correct amplitude.

$$A(\varphi) = \frac{1}{\sqrt{K(\varphi)}}, \quad K(\varphi) := \sqrt{c_1 \sin^2\varphi + c_2 \cos^2\varphi},$$

$$a_{ev} = \frac{\sqrt{\pi}}{2\sqrt[4]{c_1 c_2}} \frac{(\Psi)^{1/4}}{\sqrt[4]{\frac{x_1^2}{c_1} + \frac{x_2^2}{c_2} - 1}}, \quad a_{odd} \equiv 0. \qquad \Longleftarrow \quad \text{the integrability}$$

$$[K_\Lambda A](x_1, x_2) \approx e^{\frac{i}{h}\Theta(x_1,x_2)} a_{ev}(x_1, x_2) \mathrm{Ai}\left(-\frac{\Psi(x_1, x_2)}{h^{2/3}}\right)$$

$$\Psi(x_1, x_2) = \left(\frac{3(S_+ - S_-)}{4}\right)^{\frac{2}{3}}, \quad \Theta(x_1, x_2) = \frac{(S_+ - S_-)}{2}$$

ASYMPTOTICS IN ELLIPTIC COORDINATES

Canonical Change of Coordinates

$$x_1 = \sqrt{c_1 - c_2}\cosh u \cos v, \quad p_1 = \frac{\sinh u \cos v p_u - \cosh u \sin v p_v}{\frac{\sqrt{c_1 - c_2}}{2}(\cosh 2u - \cos 2v)},$$

$$x_2 = \sqrt{c_1 - c_2}\sinh u \sin v, \quad p_2 = \frac{\sinh u \cos v p_v + \cosh u \sin v p_u}{\frac{\sqrt{c_1 - c_2}}{2}(\cosh 2u - \cos 2v)}.$$

$$H = \frac{2}{(c_1 - c_2)(\cosh 2u - \cos 2v)}(p_u^2 + p_v^2),$$

$$F = \frac{1}{ab(b-a)(\cosh 2u - \cos 2v)}\left(p_u^2((a-b)\cos 2v - a - b) + p_v^2((a-b)\cosh 2u - a - b)\right)$$

Classical separation of variables

$$L_1(u, p_u) := p_u^2 - \frac{(c_1 - c_2)}{2} \cosh 2u + \frac{c_1 + c_2}{2} = 0$$

$$L_2(v, p_v) := p_v^2 + \frac{(c_1 - c_2)}{2} \cos 2v - \frac{c_1 + c_2}{2} = 0$$

Quantum separation of variables $\quad w = R(u)\Phi(v)$

$$\frac{1}{R(u)} \left(-h^2 R''(u) - \frac{c_1 - c_2}{2} \cosh(2u) R(u) \right) = \frac{1}{\Phi(v)} \left(h^2 \Phi''(v) - \frac{c_1 - c_2}{2} \cos(2v) \Phi(v) \right) = -s,$$

$$L_1(u, \widehat{p_u}) := -h^2 R''(u) + \left(-\frac{(c_1 - c_2)}{2} \cosh(2u) + s \right) R(u) = 0 \quad \text{– modified Mathieu equation,}$$

$$L_2(v, \widehat{p_v}) := -h^2 \Phi''(v) + \left(\frac{(c_1 - c_2)}{2} \cos(2v) - s \right) \Phi(v) = 0 \quad \text{– Mathieu equation,}$$

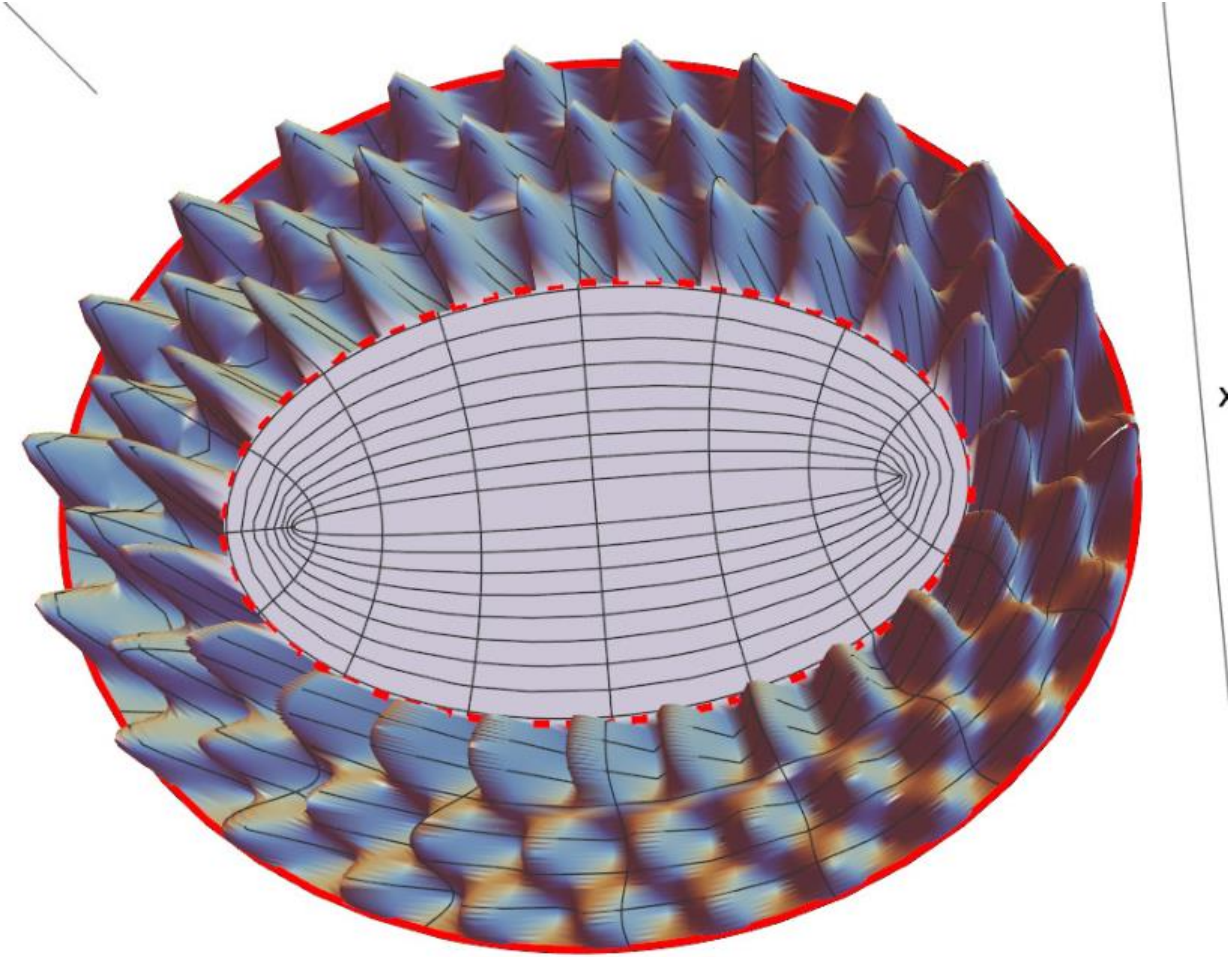
where $\widehat{p_u} = -ih\frac{\partial}{\partial u}$, $\widehat{p_v} = -ih\frac{\partial}{\partial v}$, and $s = \frac{c_1+c_2}{2} = \frac{a+b-2abg}{2}$.

$$\widetilde{\Lambda} = \{L_1 = 0,\ L_2 = 0\}, \qquad p_u = \pm\sqrt{(c_1 - c_2)\cosh^2 u - c_1}, \quad p_v = \sqrt{c_1 - (c_1 - c_2)\cos^2 v}.$$

$$K_{\widetilde{\Lambda}}[1](u,v) \approx 2\sqrt{\pi}e^{i\pi/4} \cdot \frac{e^{\frac{i}{h}\sqrt{c_2}E\left(v,1-\frac{c_1}{c_2}\right)}}{\sqrt[4]{|c_1 - (c_1 - c_2)\cos^2 v|}} \cdot \frac{|T(u)|^{1/4}}{\sqrt[4]{|(c_1 - c_2)\cosh^2 u - c_1|}} \cdot \mathrm{Ai}\left(-\frac{T(u)}{h^{2/3}}\right)$$

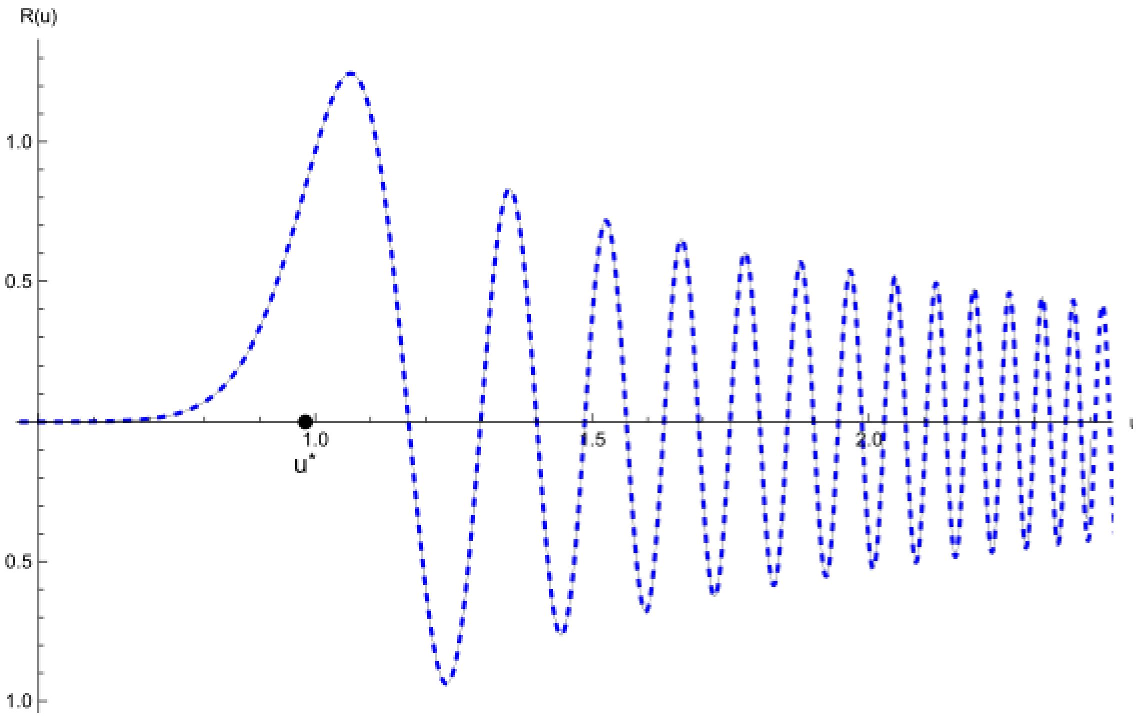
$$T(u) = \begin{cases} \left(\frac{3}{2}\sqrt{c_2}\left(E\left(iu, 1 - \frac{c_1}{c_2}\right) - E\left(iu^*, 1 - \frac{c_1}{c_2}\right)\right)\right)^{2/3}, & u > u^* \\ -\left(\frac{3}{2}i\sqrt{c_2}\left(E\left(iu, 1 - \frac{c_1}{c_2}\right) - E\left(iu^*, 1 - \frac{c_1}{c_2}\right)\right)\right)^{2/3}, & 0 \leqslant u < u^*. \end{cases}$$

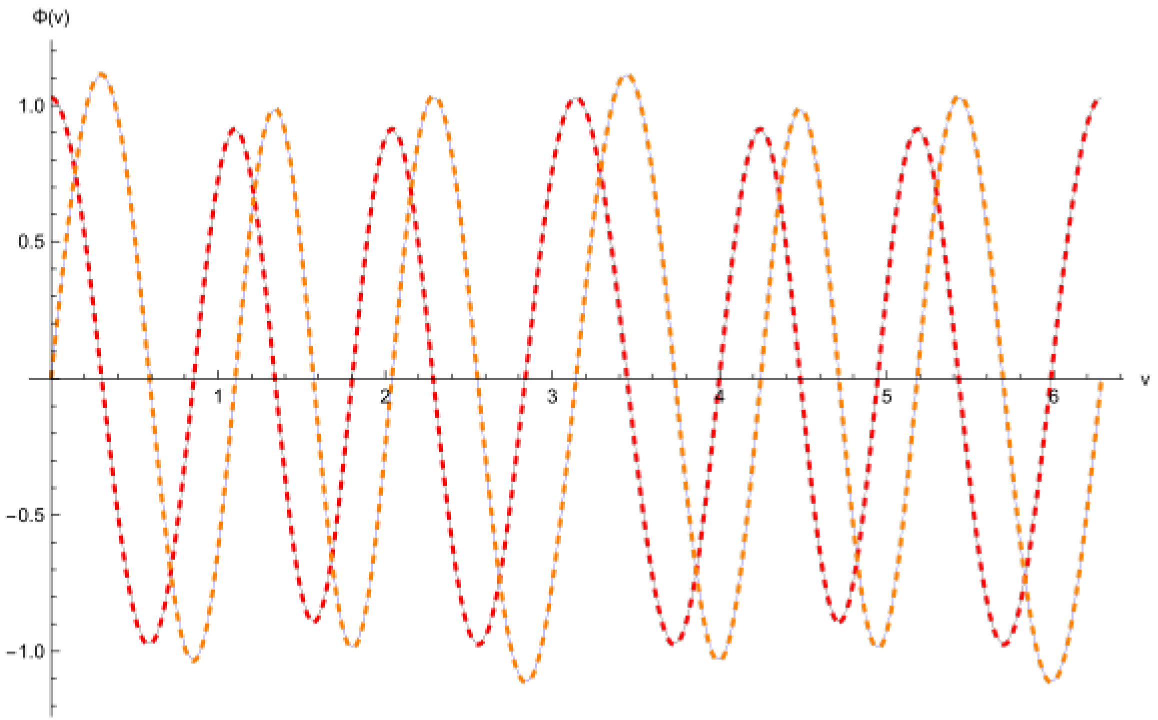
$$E\left(iu, z\right) = i\int_0^u \sqrt{1 + z\sinh^2 s}\, ds. \qquad\qquad u^* = \mathrm{arch}\left(\sqrt{\frac{c_1}{c_1 - c_2}}\right)$$



Comparison of Mathieu functions with their asymptotics:

they are practically the same





S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. V. Tsvetkova, and A. V. Turin, On an Approach to Constructing Asymptotics of Eigenfunctions of the Laplace Operator in an Ellipse Associated with Noncompact Lagrangian Manifolds, Russian Journal of Mathematical Physics, 2025, Vol. 32, No. 1 (to appear)

Open questions. 1) is it possible to construct a small spectral series for a separate convex caustic (not elliptical) without the requirement of integrability? 2) Is it possible to transfer these considerations to billiards in dimensions greater than 2?

Thank you for your attention

Спасибо за внимание!

Дорогой Володя, поздравляю с юбилеем!