Sound propagation in a slowly changing ocean.

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Dynamics in Siberia, 2025

Models for sound propagation in the ocean

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\omega^2 n^2(y,x)\,U=0 \quad \text{and} \quad i\frac{\partial U}{\partial x}+\frac{\partial^2 U}{\partial y^2}+\omega^2 n^2(y,x)\,U=0.$$

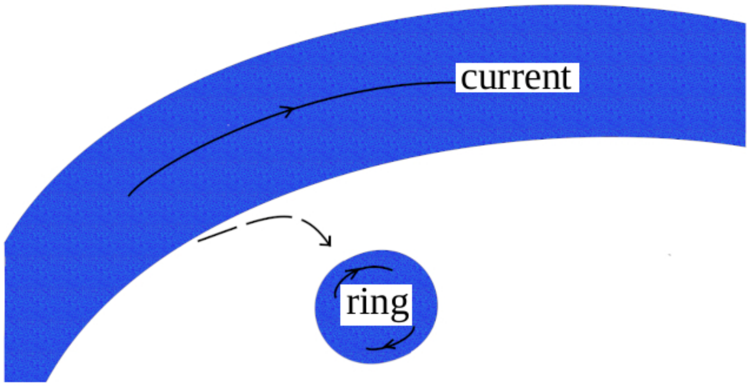
U is the acoustic field pressure,
ω is a sound frequency,
n is a refraction index,
x and y are is the horizontal and vertical coordinates.

1. Ocean as a layered medium: $n=n(y)$,

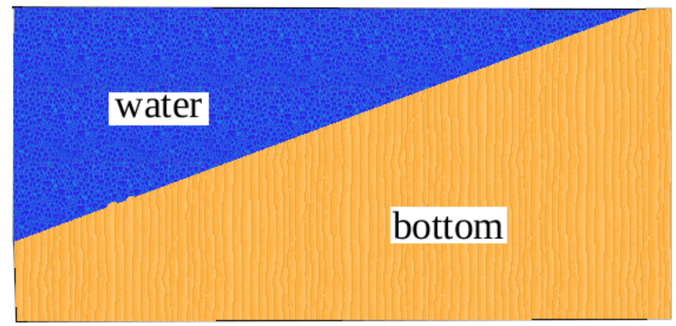
Water: $0<y<H$, its depth: H; bottom: $y>H$.

Free ocean surface: $U(x,0)=0$, bottom: U, U_x are continuous at $y=H$, conditions at ∞ depend on the physical problem.

Ocean adiabatic inhomogeneities



synoptic ring



upslope

Two-dimensional models:

$$n = \begin{cases} n_-(y), & x < x_- \\ n(\varepsilon x, y), & x_- < x < x_+ \\ n_+(y), & x > x_+ \end{cases} \qquad n = \begin{cases} n_0 > 1, & 0 < y < -\varepsilon x, \\ & x < 0 \\ 1 & \text{otherwise} \end{cases}$$

$$n = n(y, \varepsilon x), \qquad 0 < \varepsilon << 1$$

Normal waves for equation $\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\omega^2 n^2(y,\varepsilon x)\,U=0.$

When there exist $\psi_k(\cdot,\xi)\in L^2(\mathbb{R}_+)$ and $\mu_k(\xi)\in\mathbb{R}$ satisfying

$$\psi_{yy}+\omega^2 n^2(y,\xi)\psi=\omega^2\mu^2\psi,\quad \psi(0,\xi)=0,$$

one constructs *adiabatic normal wave*, a formal asymptotic solution,

$$U_k(x,y)=e^{\frac{i\omega}{\varepsilon}\int\limits_{\xi_0}^{\varepsilon x}\mu_k(\xi)\,d\xi}\sum_{l\geq 0}\varepsilon^l u_{k,l}(y,\varepsilon x),\qquad u_{k,0}=\psi_k. \tag{1}$$

If $n=n(y)$, by separation of variables, one constructs a *normal wave*, an exact solution,

$$U_k(x,y)=e^{i\omega\mu_k x}\psi_k(y).$$

Solutions (1) are constructed by "asymptotic separation of variables".

Model problem for synoptic rings

Recall that $n = \begin{cases} n_-(y), & x < x_- \\ n_+(y), & x > x_+ \end{cases}$. One looks for solutions

$$\mathcal{U}_k = U_k^- + \text{reflected waves}, \quad \varepsilon x < \xi_-,$$

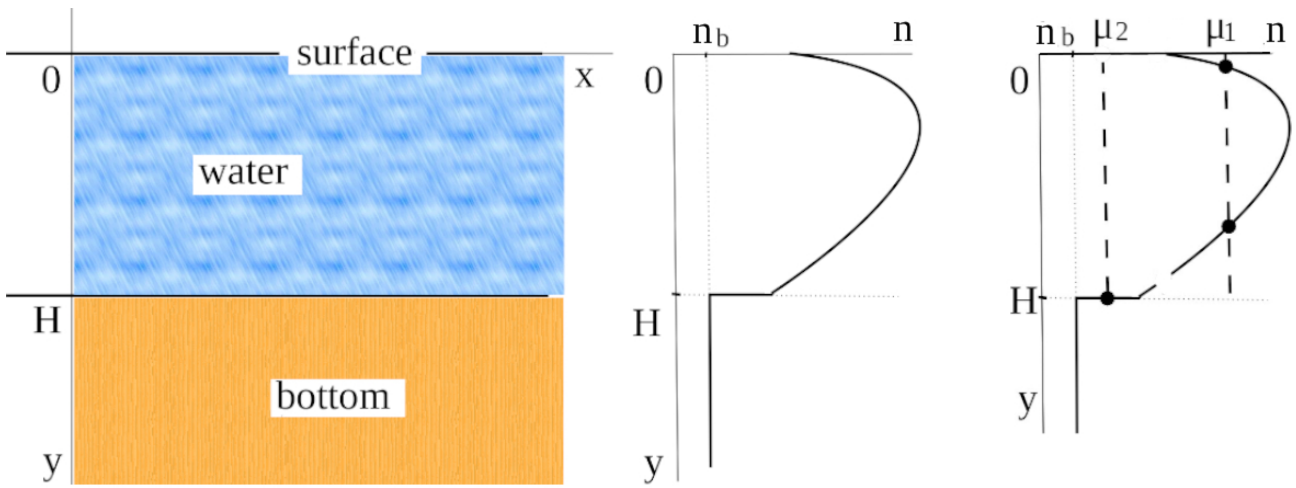
$$\mathcal{U}_k = \sum_l t_{kl} U_l^+, \qquad \varepsilon x > \xi_+,$$

$U_k^\pm$ correspond to $n_\pm$.

Problem: describe asymptotics of t_{kl} as $\varepsilon \to 0$.

V.S. Buldyrev, 1981: *if* $\omega\varepsilon \to 0$, *then* $\|t\|_{kl} \approx \delta_{kl}\, t_{kk}$
(formal asymptotic expansions)

Difficulty: for synoptic ring problems $\omega\varepsilon$ can be **large**!



Typical index of refraction

Branch points

Branch points: points $y(\xi)$ such that $n(y,\xi) = \mu_k(\xi)$.

V.A. Borovikov, A.V. Popov (early 80-ies): *Assume that,* $\forall \xi$*,*
- $n(\cdot,\xi)$ *has one non-degenerate maximum,*
- *the number of branch points* $n(y,\xi) = \mu_k(\xi)$ *is constant,*
- *for a given* $m \in \mathbb{N}$*,* $\omega\varepsilon^m \to 0$*.*

Then $t_{k,l} \approx \delta_{kl}\, t_{kk}$
(two-scale Cherry-type formal asymptotic expansions).

For a given k, assume that

- $n(0,\xi) < \mu_k(\xi) \quad \xi < \xi_-$, • $n(0,\xi) > \mu_k(\xi) \quad \xi > \xi_+$,
- between ξ_- and ξ_+, there is one point where $n(0,\xi) = \mu_k(\xi)$,
- at this point $\frac{\partial n}{\partial \xi}(0,\xi) > \frac{\partial \mu_k}{\partial \xi}(\xi)$.

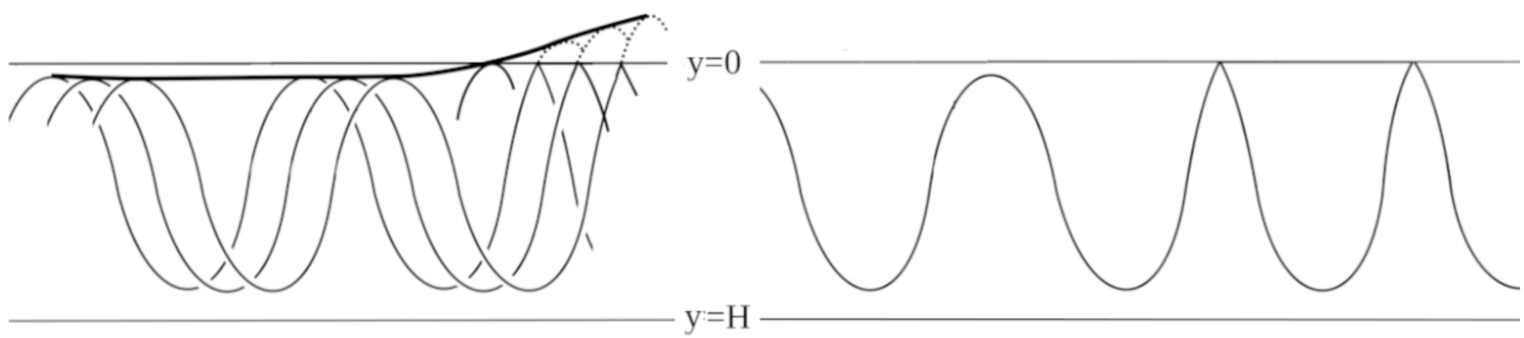
As $\omega\varepsilon^2 \to 0$, one has (V.S.Buslaev, A.A.Fedotov, 1987)

$$t_{kl} \approx e^{i\gamma_{kl}} \int_0^1 \Phi(a, at)\; e^{2i\left(\pi t(k-l) - a^{3/2}\int\limits_0^t F(s)\,ds\right) + \frac{i\pi t}{2}} dt$$

$$F(s) = \sqrt{s} + \zeta(-1/2, s/2), \quad a_k = \omega^{2/3}\varepsilon\alpha_k,$$

Φ is expressed in terms of Airy functions, γ_{kl} and α_k are expressed in terms of some geometric objects.

To describe U_k, one solves an initial-boundary value problem for the ray method. A family of rays is associated to U_k. For $\varepsilon x < \xi_-$ these rays differ by horizontal translations, and each ray is periodic.

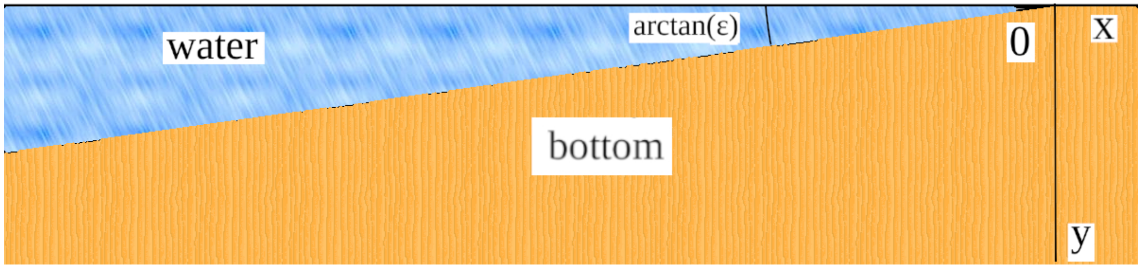


One has to solve a problem similar to the problem of diffraction on a smooth convex curve. This leads to Φ.

F enters into the formula describing the adiabatic invariant increment along the rays that begin to reflect from the sea surface.

Underwater upslope sound propagation

A.D. Pierce (1982), J. M. Arnold and L. B. Felsen (1983), ... , V.M. Babich



$$i\frac{\partial U}{\partial x} = -\frac{\partial^2 U}{\partial y^2} + v(y, \varepsilon x)U, \quad 0 < y < \infty, \quad U|_{x=0} = 0.$$

$$v(y,\xi) := -\omega^2 n^2(y,\xi) = \begin{cases} -1, & 0 \le y \le -\xi, \\ 0, & \text{otherwise.} \end{cases}$$

$$H(\xi)\psi = -\psi'' + v(y,\xi)\psi, \quad H(\xi) \text{ acts in } L^2(0,\infty), \quad \psi(0,\xi) = 0.$$

Exact solutions U and U_n

$$U(y,x,p) = e^{ix} \sum_{k=p+\varepsilon l,\, l\in\mathbb{Z}} e^{-ik^2x} sin(ky)\, R(k), \quad 0 \le y < -\varepsilon x,$$

(plane wave $e^{-i(p^2-1)x+ipy}$ and all the reflected waves)

$$U(y,x,p) = \sum_{k=p+\varepsilon l,\, l\in\mathbb{Z}} e^{-ip_1^2(k)x+ip_1(k)y}\, T(k)\, R(k), \quad y < -\varepsilon x.$$

(all the “refracted” waves)

- $U(\cdot,x,p) \in C^1 \Rightarrow$ difference equation for R, formulas for T, p_1.
- R is multivalued; convergence $\Rightarrow$ an univalued branch of R.

U is periodic in $p \Rightarrow$ Fourier series:

$$U(y,x,p) = \sqrt{\frac{\varepsilon}{\pi}} \sum_{m\in\mathbb{Z}} e^{\frac{2\pi imp}{\varepsilon}} U_m(y,x).$$

Integral representation for U_m. For $0 < y < -\varepsilon x$,

$$U_m(y,x) = \frac{e^{ix}}{\sqrt{\varepsilon\pi}} \int_{-\infty}^{\infty} e^{-i(p^2\xi + 2\pi m p)/\varepsilon} sin(py)\, R(p)\, dp, \qquad \xi = \varepsilon x.$$

$$R\left(p+\frac{\varepsilon}{2}\right) = \rho(p) R\left(p-\frac{\varepsilon}{2}\right), \qquad \rho(p) = \frac{Q(p)-p}{Q(p)+p}. \qquad Q(p) = \sqrt{p^2-1},$$

Branch points: $p = \pm\Big(1 + j\varepsilon/2\Big)$, $j \in \mathbb{N}$. One has $R(-p) = 1/R(p)$,

$$-i\varepsilon \frac{d}{dp} \ln R(p) = \pi + \sqrt{2\varepsilon}\, \zeta\left(\frac{p-1}{\varepsilon}\right) + O\Big(\varepsilon^{\frac{3}{2}} + |p-1|^{\frac{3}{2}}\Big), \qquad p \sim 1,$$

$$\zeta(t) = \lim_{L\to+\infty} \left(\sum_{l=0}^{L-1} \Big(l + 1/2 - t\Big)^{-\frac{1}{2}} - 2L^{\frac{1}{2}} \right).$$

Adiabatic asymptotics of Ψ_n : $\varepsilon t < \tau_n$

Consider $H(\xi) = -\frac{d^2}{dx^2} + v(\cdot, \xi)$ with the Dirichlet b.c. at zero.

$$\sigma(H) = \sigma_{ac}(H) + \sigma_d(H),\ \sigma_{ac}(H) = [0, +\infty),\ \sigma_d(H) \subset (-\infty, 0).$$

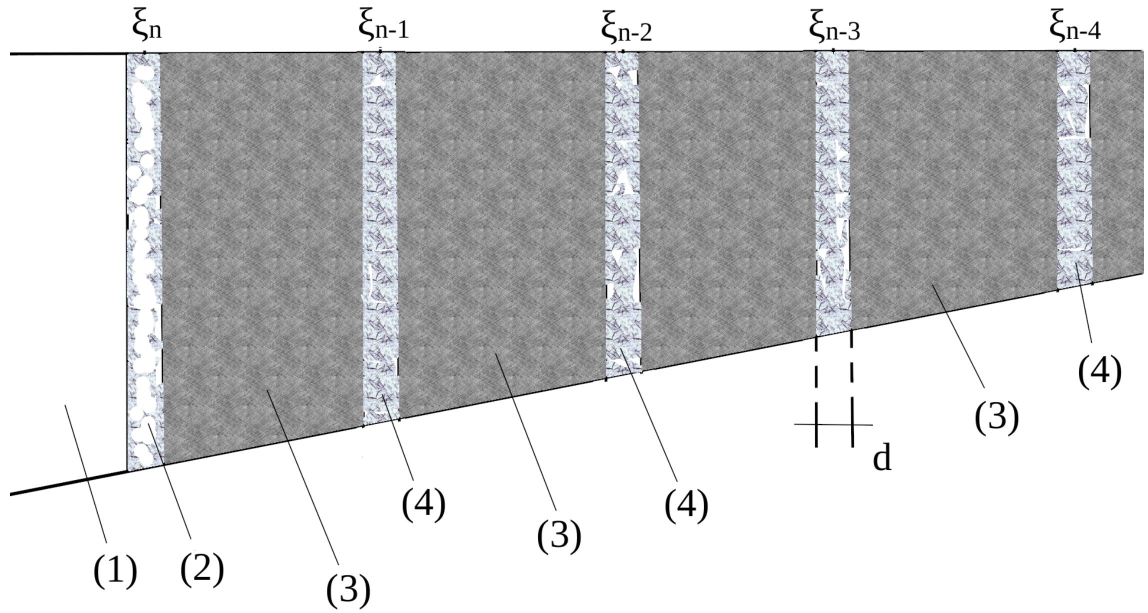
If $\xi_{k-1} < \xi < \xi_k$, $\xi_k = 1 - \pi k + \frac{\pi}{2}$, $k \in \mathbb{N}$, H has k eigenvalues. As ξ increases $E_k(\xi)$ moves to 0, and, at $\xi = \xi_k$, it disappears.

Fix $C_1 < C_2 < \xi_k$. Let ε be sufficiently small. If $C_1 \leq \varepsilon x \leq C_2 < \xi_k$ and $0 \leq y \leq 1 - \varepsilon x$, then

$$U_k(y, x) \sim e^{-\frac{i}{\varepsilon} \int\limits_{\xi_k}^{\varepsilon x} E_k(\xi)\, d\xi} \sum_{m=0}^{\infty} \varepsilon^m \psi_{k,m}(y, \varepsilon x).$$

U_k behaves like an adiabatic normal wave!

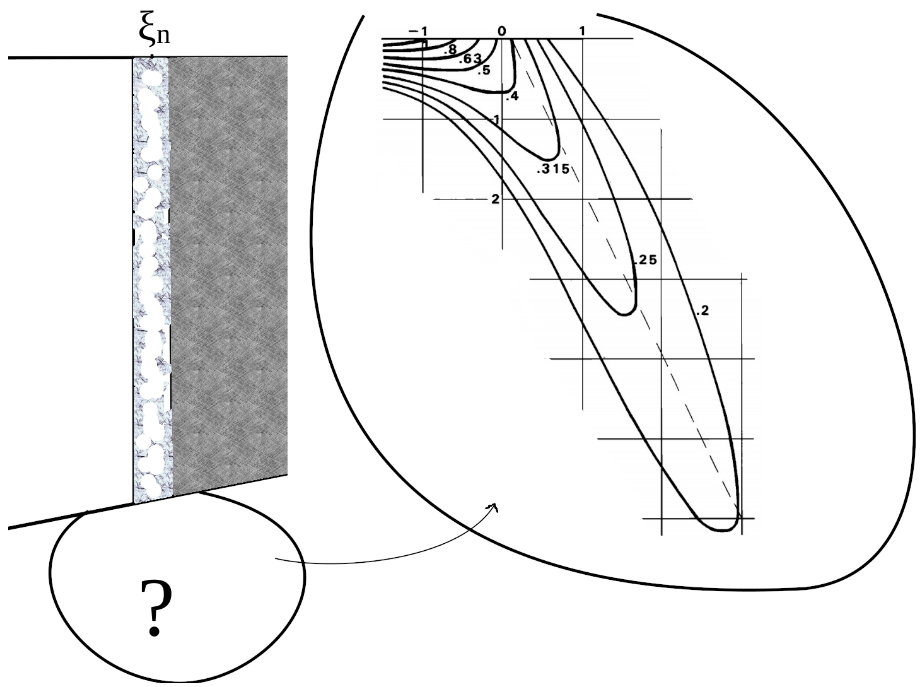
Aftermath. What happens in the water? $\xi = \epsilon x$



(1) $U_n \sim 1$ (2) $U_n \sim \epsilon^{\frac{1}{6}}$ (3) $U_n \sim \epsilon$ (4) $U_n \sim \epsilon^{\frac{2}{3}}$ $d \sim \varepsilon^{\frac{1}{3}}$

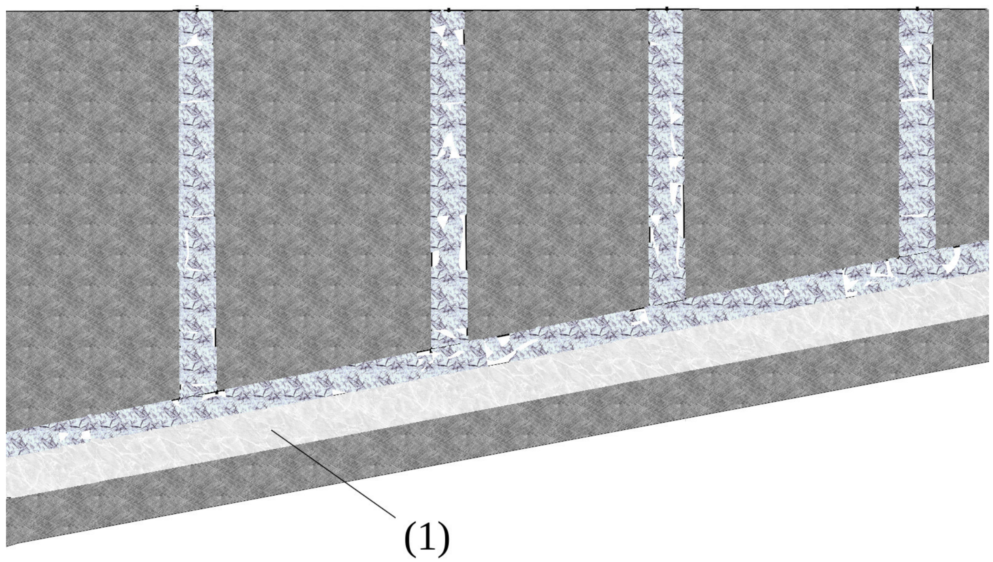
(2) $U_n \sim$ Airy functions of complex arguments,
(4) $U_n \sim$ new special functions
(A.Fedotov, 2020)

What happens in the bottom? Answer 1.



$U_n \sim$ is a combination of Airy functions with non-trivial complex functions as their arguments
(Pierce, Felson and others, 1980-ies, Fedotov, Sergeev 2022)

What happens in the bottom? Answer 2.



$$(1) \quad U_n \sim \frac{\xi}{(\xi-\xi_n)^{3/2}} \exp\left(-\frac{1}{\varepsilon}\frac{\eta^3}{12(\xi-\xi_n)^3}\right), \quad \xi = \epsilon x, \;\; \eta = \epsilon(y+\epsilon x).$$

(A.Fedotov, W. Sergeev, 2024)