Jacobians of rational curves corresponding to principal cells.

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Dynamics in Siberia
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Periodic lattice formed by solitary waves:

How to construct such solutions?

They correspond to Riemann surfaces, **close to rational degenerations.**

We call them **almost degenerate**.

KP-II equation and shallow water waves

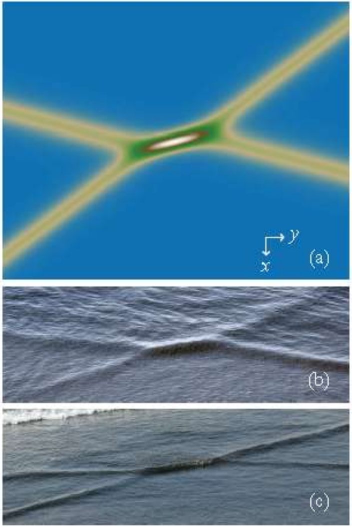
The Kadomtsev-Petviashvili-II (KP-II) solutions is one of the models used in the theory of shallow water waves.

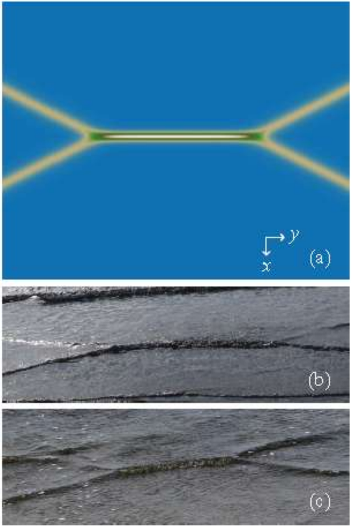
$$(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0,$$

Of course, the question of applicability of KP-II to the water vawes is rather delicate.

Nevertheless, some KP-II multiline solitons resembles real photos of the ocean surface.

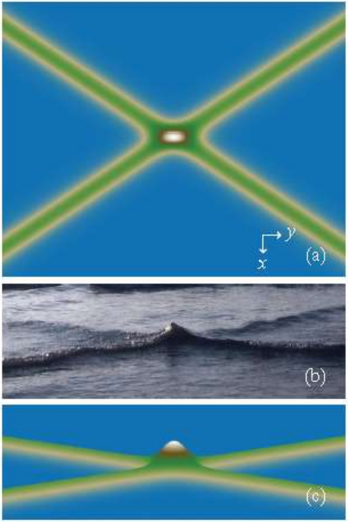
KP-II multisoliton solutions and water waves

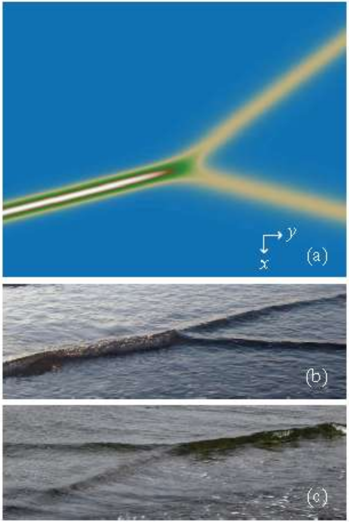




M.A. Ablowitz and D.E. Baldwin Phys. Rev E , v. 86 (2012)

KP-II multisoliton solutions and water waves





Special KP-II solutions: periodic structures formed by solitons

We are interested in **real regular** KP-II solutions.
The Kadomtsev-Petviashvili-II equation

$$(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0,$$

can be written as compatibility condition for the following pair of linear operators:

$$(\partial_y - B_2)\Psi(P,x,y,t) = 0, \qquad (\partial_t - B_3)\Psi(P,x,y,t) = 0,$$

where

$$B_2 = \partial_x^2 + u, \quad B_3 = \partial_x^3 + \frac{3}{4}\Big(\partial_x \circ u + u \circ \partial_x\Big) + w,$$

$$u = u(x,y,t), \quad w = w(x,y,t), \quad \partial_x w(x,y,t) = \frac{3}{4}\partial_y u(x,y,t).$$

How to construct periodic structures formed by solitons?

One has to use Riemann surfaces close to rational degenerations.

The first step is the following:

If we have a multiline soliton solution, how to reconstruct it from the finite-gap approach?

Then we can make the Riemann surface regular.

Idea of S.P. Novikov: **real regular** soliton solution should be obtained by degenerating **real regular** finite-gap solutions.

Example of a multisoliton solution

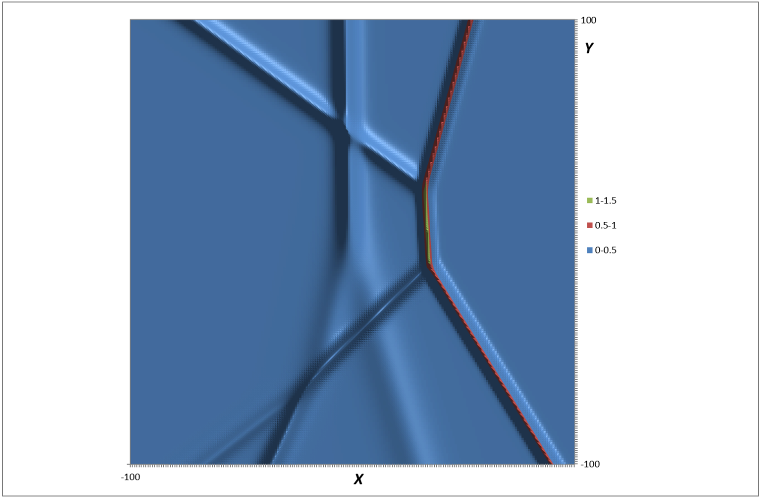


Figure: Level plot for a solution solution corresponding to a point of $Gr(3,7)$ at a fixed time $t = t_0$. (Thanks to Tanya Filonetz.)

The spectral data:

1) M phases (complex numbers) k_j.

2) A $N \times M$, $N < M$ matrix $A = (a_{i\,j})$, $j = 1, \ldots, M$.

Denote: $\vec{t} = (x, y, t)$, $\theta_j = k_j x + k_j^2 y + k_j^3 t$

Let

$$f^{(i)}(\vec{t}) = \sum_{j=1}^{M} a_{i\,j} e^{\theta_j}, \quad i = 1, \ldots, N.$$

Define their Wronskian

$$\tau(\vec{t}) = Wr(f^{(1)}, \ldots, f^{(N)}) = \det \begin{vmatrix} f^{(1)} & f^{(2)} & \ldots & f^{(N)} \\ \partial_x f^{(1)} & \partial_x f^{(2)} & \ldots & \partial_x f^{(N)} \\ \ldots & \ldots & \ldots & \ldots \\ \partial_x^{N-1} f^{(1)} & \partial_x^{N-1} f^{(2)} & \ldots & \partial_x^{N-1} f^{(N)} \end{vmatrix}$$

Then the function

$$u(\vec{t}) = 2\partial_x^2 \log(\tau(\vec{t}))$$

satisfies KP-2 equation.

Multisoliton KP-II solutions

We have

$$\tau(\vec{t}) = Wr(f^{(1)}, \ldots, f^{(N)}) \equiv \sum_I \Delta_I(A) \prod_{i_1<i_2,\, i_1,i_2\in I} (k_{i_2} - k_{i_1}) e^{\sum\limits_{i\in I} \theta_i} \quad (1)$$

where the sum is other all N–element ordered subsets I in $\{1, 2, \ldots, M\}$, *i.e.* $I = \{1 \leq i_1 < i_2 < \cdots < i_N < M\}$ and $\Delta_I(A)$ are the maximal minors of the matrix A.

Properties of multisoliton solutions:

M. Boiti, F. Pempinelli, A.K. Pogrebkov, B. Prinari (2001)
Yu. Kodama and collaborators (2003–...)

Real regular multiline soliton KP-II solutions

KP-II equation vs KP-II hierarchy:

3 times: $(x, y, t) \quad \Leftrightarrow \quad \vec{t} = (t_1 = x, t_2 = y, t_3 = t, t_4, t_5 \ldots),$

$$\theta_j = k_j x + k_j^2 y + k_j^3 t \quad \Leftrightarrow \quad \theta_j = \sum_{l=1}^{\infty} k_j^l t_l.$$

We assume that only finite number of times is non-zero.

Theorem (Malanyuk, 1991) If $u(\vec{t})$ is a **real regular** solution of the **full KP hierarchy** if and only if

1. All k_j are real.
2. If we assume that $k_1 < \cdots < k_M$, then all $\Delta_I(A) \geq 0$.

Theorem (Kodama, Williams, 2013) If $u(x, y, t)$ is a **real regular** of the **KP equation**, then

1. All k_j are real.
2. If we assume that $k_1 < \cdots < k_M$, then all $\Delta_I(A) \geq 0$.

Totally non-negative Grassmannians

Let C be a $N \times N$ non-degenerate matrix. Then the matrices A and $\tilde{A} = CA$ generate the same KP solution.

We know that $N \times M$ matrices A, $\tilde{A}$ of rank N defines the same point of Grassmannian $Gr(N, M)$ iff

$$\tilde{A} = CA, \quad \det C \neq 0.$$

Therefore real regular multiline soliton solutions of KP-II are defined by the following pair:

1. Collection of M real phases k_j. We assume that they are ordered in ascending order: $k_1 < \cdots < k_M$.
2. A point of Grassmannian $Gr(N, M)$ such that all Plücker coordinates are non-negative: $\Delta_I(A) \geq 0$ (we assume that at least one of them is positive).

Finite-gap KP-II solutions

The spectral data (Krichever, 1976):

1. A finite genus g compact Riemann surface Γ with a marked point P_0;
2. A local parameter $1/\zeta$ near P_0;
3. A non-special divisor $\mathcal{D} = \gamma_1 + \ldots + \gamma_g$ of degree g in Γ.

The Its-Matveev formula:

$$u(\vec{t}) = 2\partial_x^2 \log \theta(x\vec{W}_1 + y\vec{W}_2 + t\vec{W}_3 + \vec{C}) + 2\hat{\omega}_{11}.$$

$$\theta(z) = \theta(\vec{z}|B) = \sum_{\substack{n_j\in\mathbb{Z}\\ j=1,\ldots,g}} \exp\left[\pi i \sum_{k,l=1}^{g} B_{kl} n_k n_l + 2\pi i \sum_{k=1}^{g} z_k n_k\right].$$

Real regular finite-gap KP-II solutions

The necessary and sufficient conditions on spectral data to generate real regular KP-II hierarchy solutions (Dubrovin, Natanzon, 1988), from **regular** curves.

1. Γ possesses an antiholomorphic involution $\sigma : \Gamma \to \Gamma$, $\sigma^2 = \mathrm{id}$, which has the maximal possible number of fixed components (real ovals). This number is equal to $g+1$, therefore (Γ, σ) is an $\mathtt{M}$-curve.
2. P_0 lies in one of the ovals, and each other oval contains exactly one divisor points. The oval containing P_0 is called "infinite" and all other ovals are called "finite".

Remark. For double periodic real regular $u(x, y)$ the spectral curve is always $\mathtt{M}$-curve (may be of infinite genus) with divisor at correct positions (Krichever 1989).
Remark. We assume that $\sigma\zeta = \bar{\zeta}$.

Remark. For degenerate curves these conditions are sufficient but not necessary.

M-curves

Let Γ be a compact complex curve Γ of genus g.

A **real structure** on Γ is an antiholomorpjic involution σ, i.e. an antiholomorphic map $\sigma : \Gamma \to \Gamma$ such that $\sigma^2 = id$. σ is also called complex conjugation, and we say that Γ is a real curve.

The fixed points of σ: $\sigma\gamma = \gamma$ are called **real points** of Γ. They form a set of k smooth curves (real ovals), where $0 \leq k \leq g+1$.

A curve Γ with an an antiholomorpjic involution σ is called `M`-**curve** if $k = g+1$, i.e. it has the **maximal possible** number of real ovals.

M-curves

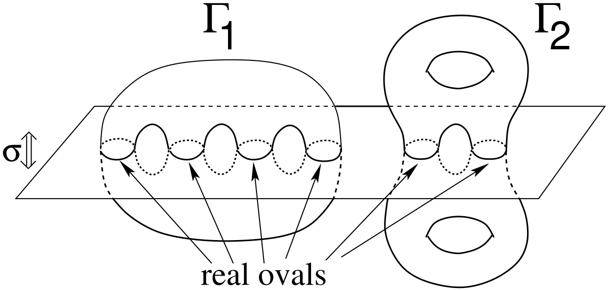


Figure: Two curves of genus 3 with antiholomorphic involution σ. Here σ is the reflection with respect to the horizontal plane, Γ_1 has 4 real ovals, i.e. Γ_1 is an $\mathtt{M}$-curve. Γ_2 has only 2 real ovals, i.e. Γ_2 is not an $\mathtt{M}$-curve.

Solitons as degenerations of finite-gap solutions

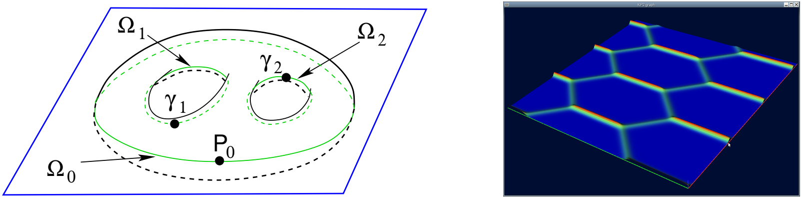


Figure: Real regular quasi–periodic solutions to KP-II are associated to Krichever data on regular M-curve.

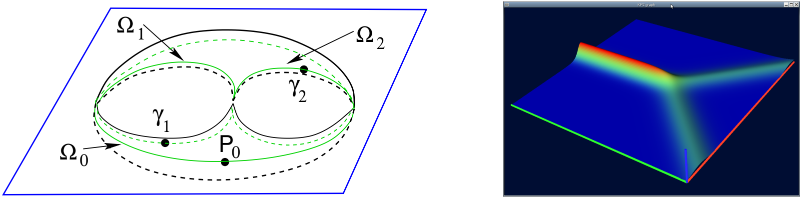


Figure: Real regular bounded multi-soliton solutions may be associated to Krichever data on rational degenerations of M–curves.

How to associate a degenerate M and a proper divisor to a point of a totally non-negative Grassmannian?

We use Alexander Postnikov paper: Total positivity, Grassmannians, and networks, arXiv:math/0609764 [math.CO].

Let us recall positroid cell decomposition of $Gr^{\geq 0}(N, M)$ playing an essential role. Positroid cells may be described as intersections of Gelfand-Serganova strata with the totally non-negative part of real Grassamnnians. They form ad subdivision of intersections of Schubert cells with totally non-negative part .

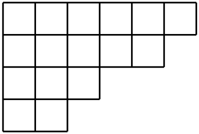
Let us remark, that the positroid decomposition, described below, is a CW-decomposition (Postnikov), moreover, this complex is homeomorphic to a ball (Galashin, Karp, Lam, 2018).

Totally non-negative Grassmannians – Postnikov

Schubert cells:

Each point of $Gr(N, M)$ can be uniquely represented by a matrix in reduced row echelon form

$$\begin{bmatrix} 1 & a_{1\,2} & 0 & a_{1\,4} & a_{1\,5} & 0 & a_{1\,7} & 0 & a_{1\,9} & a_{1\,10} \\ 0 & 0 & 1 & a_{2\,4} & a_{25} & 0 & a_{2\,7} & 0 & a_{2\,9} & a_{2\,10} \\ 0 & 0 & 0 & 0 & 0 & 1 & a_{3\,7} & 0 & a_{3\,9} & a_{3\,10} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a_{4\,9} & a_{4\,10} \end{bmatrix}$$



Here 1,3,6,8 are pivot column and 2,4,5,7,9,10 are non-pivot columns.

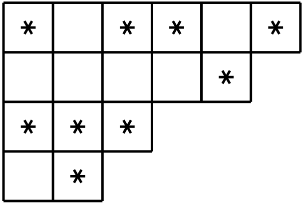
The graph on the left is called Young tableau (English notation), associated with this matrix.

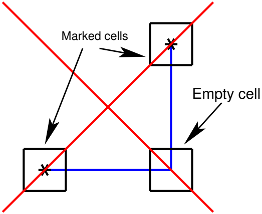
The numbers $a_{i\,j}$ are the coordinates on the set of all matrices with the fixed Young tableaux (Schubert cells).

The positroid cells are enumerated by the following pairs:

1. Young tableau;
2. A subset S of the cells of this tableau.

We mark the cels from this subset by the sign $*$.





Le-rule: such configuration is forbidden.

The subset S is an arbitrary subset satisfying the following **Le-rule:** configurations in which an empty cell is simultaneously "blocked" by marked cells from above and from the left are forbidden.

To each Le-diagram a Le-graph is associated.

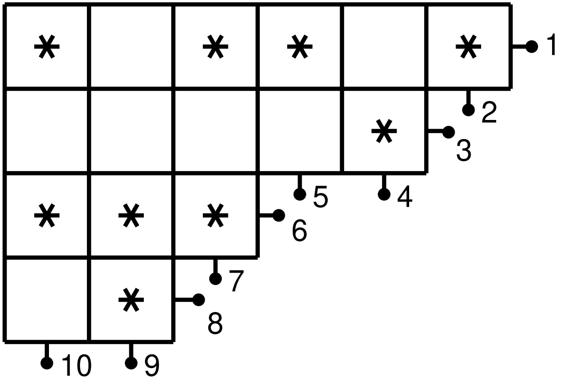


Figure: *Step 1: we write the numbers of the pivot columns at the right border of the tableau and the number of non-pivot columns at the bottom. Here 1,3,6,8 are pivot column and 2,4,5,7,9,10 are non-pivot columns. Denote the set of pivot columns by I. Here* $I = \{1, 3, 6, 8\}$, $i_1 = 1$, $i_2 = 3$, $i_3 = 6$, $i_4 = 6$,

To each Le-diagram a Le-graph is associated.

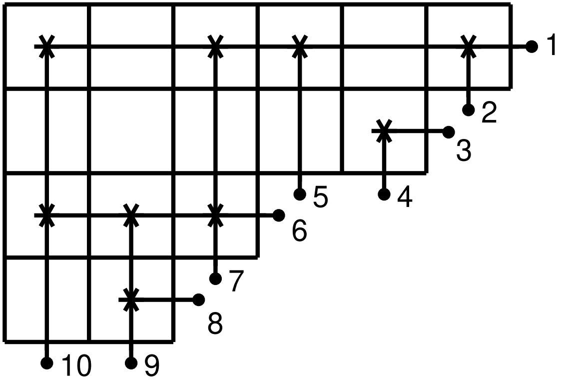


Figure: *Step 2: From each marked cell we draw a line to the left and down up to the boundary. The Le-rule implies that resulting graph is planar.*

To each Le-diagram a Le-graph is associated.

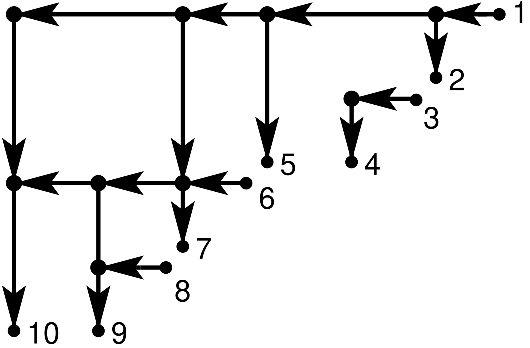


Figure: *Step 3: The following orientation is introduced: all horizontal edges are oriented to the left, all vertical edges are oriented down.*

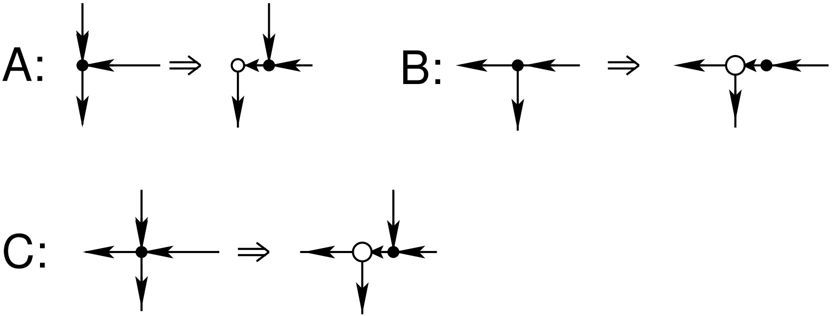


Figure: *Step 4: We transform the diagram into a bi-colored diagram with edges of valency 2 and 3 using the following rules.*

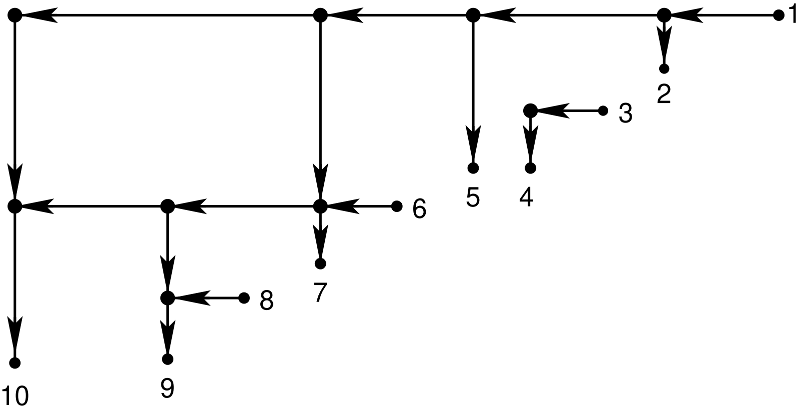


Figure: *Step 4: The diagram from the previous slide before transformation*

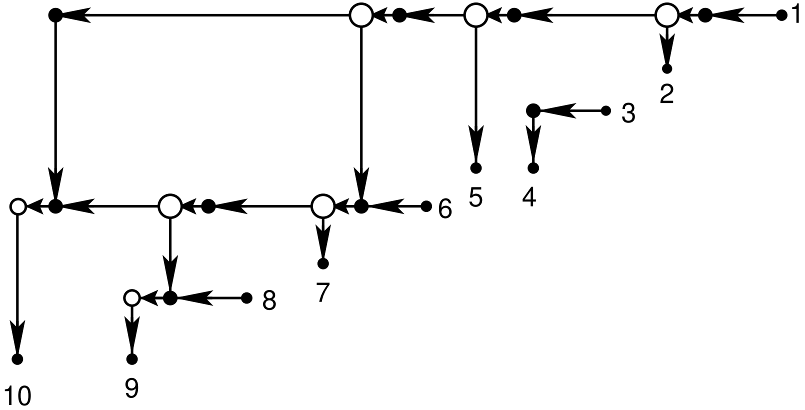


Figure: *Step 4: The diagram from the previous slide after transformation*

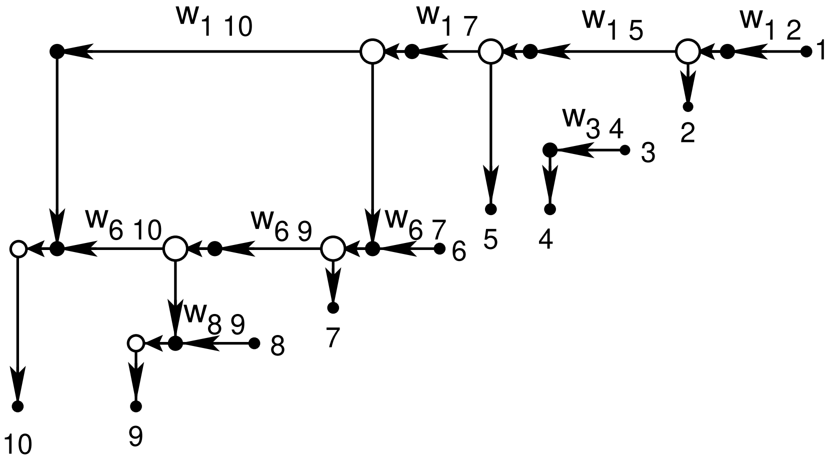


Figure: *Step 5: We assign positive weights to horizontal edges ending at black vertices.*

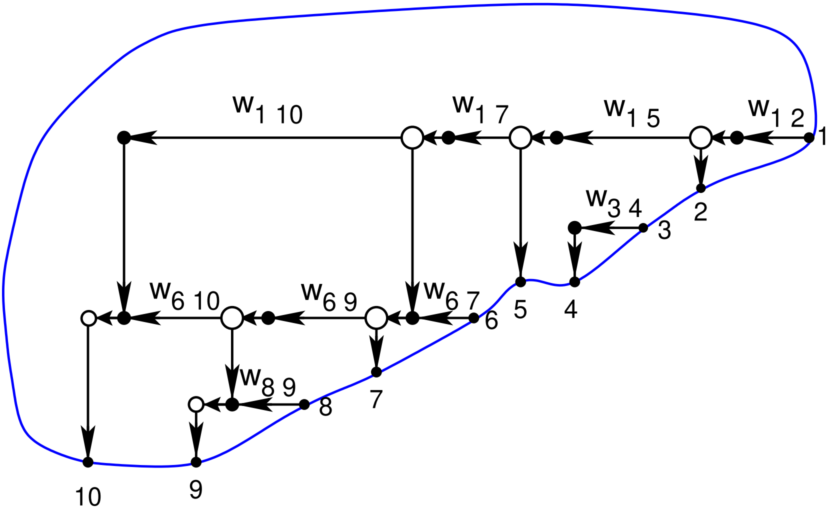


Figure: *Step 6: We assume that the graph lies in a disk and the boundary vertices of the graph belong to the boundary of this disk.*

Totally non-negative Grassmannians – Postnikov

The number of markded cells D in the diagram coincides with the number of w-s, i.e. D is the dimension of the cell.

Theorem (Postnikov) The parameters w_{ij} are coordinates on the corresponding cell.

The birational equivalence between the positive octant in $\mathbb{R}^D$ and the corresponding cell is the following:

Denote the pivot columns by $i_1, \ldots, i_N$ and non-pivot columns by $j_1, \ldots, j_{M-N}$. The element $a_{r\,j}$ of the matrix A in reduced row echelon form (RREF) from non-pivot column is defined by:

$$a_{r\,j} = (-1)^{\sigma_{i_r j}} \sum_{P: i_r \mapsto j} \left(\prod_{e \in P} w_e \right), j > i_r,$$

where the sum is over all paths P from the boundary source b_{i_r} to the boundary sink b_j, and $\sigma_{i_r j}$ is the number of pivot elements $i_s \in I$ such that $i_r < i_s < j$.

Totally non-negative Grassmannians

Definition. Boundary edges corresponding to pivot columns are called **sources**. Boundary edges corresponding to non-pivot columns are called **sinks**.

Theorem (Postnikov) Let $\tilde{I} = \{i_{r_1}, \ldots, i_{r_{N-k}}\}$ be a collection of pivot columns and $\tilde{J} = \{j_{s_1}, \ldots, j_{s_k}\}$ be a collection of non-pivot columns. Then for the minor Δ_J formed by this collection of columns one has the following formula:

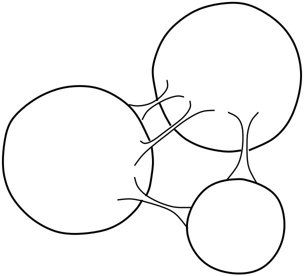
$$\Delta_J = \sum_{P \in \mathcal{P}} W(P),$$

where $\mathcal{P}$ denotes the set of all collections of k oriented paths $P = P_1 \cup P_2 \cup \ldots \cup P_k$ starting at $I \backslash \tilde{I}$ and ending at $\tilde{J}$, such that $P_k \cap P_l = \emptyset$ if $k \neq l$, $W(P)$ denotes the product of all weights at the edges belonging to P.

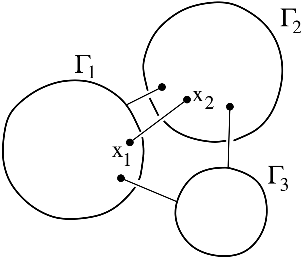
Degenerated algebraic curves

To construct soliton solutions we use maximallly degenerated spectral curves: a union of Riemann spheres with connected pairs of **real** points.

Smooth Riemann surface

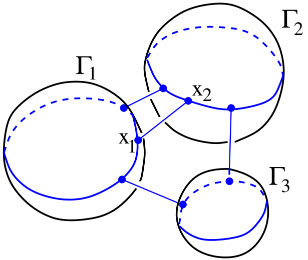


Rational degeneration

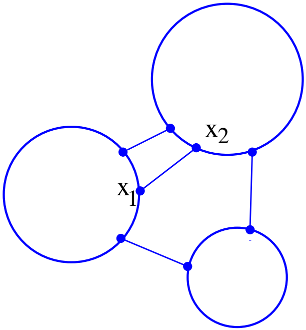


Algebraical model: a function ψ on Γ is a collection of functions $\psi_k(\gamma_k)$, $\gamma_k \in \Gamma_k$. A pair of points $x_i \in \Gamma_i$, $x_j \in \Gamma_j$ is connected if $\psi_i(x_i) \equiv \psi_j(x_j)$. At the picture $\psi_1(x_1) \equiv \psi_2(\gamma_2)$.

It is very convenient to draw only the real part of the curve.



A rational M-curve.



Real part of this M-curve.

The circles on the right picture denote the real points of the Riemann spheres – the sets of points with real coordinates including the infinite ones.

For `M`-curve we have a **planar** diagram.

Such rational curves and their Jacobians were studied by many authors, including:

Andrei Tyurin, "Quantization, Classical and Quantum Field Theory and Theta Functions", *CRM Monograph Series,* v. 21 (2003), 136 pp.

I.V. Artamkin, "Canonical maps of punctured curves with simplest singularities", *Sb.Math.*, **195**:5 (2004), 615–642.

I.V. Artamkin, "Topologically trivial bundles on curves with the simplest singularities", *Proc. Steklov Inst. Math.,* **246** (2004).

I.V. Artamkin, "Discrete Torelli theorem", *Sb.Math.*, 2006, **197**:8 (2006), 1109–1120.

Oda T., Seshadri C.S., "Compactifications of the generalized Jacobian variety" *Trans. Amer. Math. Soc.,* **253** (1979), pp. 1–90.

In the Tyurin's book such curves were called $\mathtt{m}$ − **curves.**

We study the curves which are both $\mathtt{m}$ and $\mathtt{M}$.

We are concentrated on a special real component of the Jacobian, which we call **DN** (Dubrovin-Natanzon).

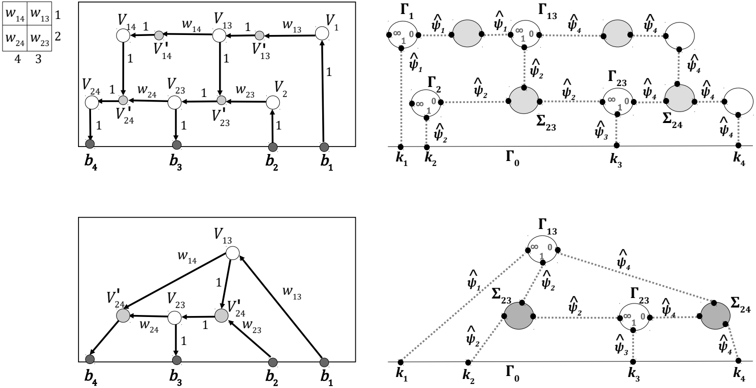
We do not discuss compactifications of the Jacobian now. We will stay **inside** the open cell.

The curve $\Gamma(\mathcal{G})$. Let $\mathcal{K} = \{\kappa_1 < \cdots < \kappa_M\}$ and let $\mathcal{G}$ be a graph representing a posiroid cell with $g+1$ faces. Then the curve $\Gamma = \Gamma(\mathcal{G})$ is associated to $\mathcal{G}$ using the following correspondence after reflecting the graph w.r.t. a line orthogonal to the one containing the boundary vertices.

Table: The graph $\mathcal{G}$ vs the reducible rational curve Γ

$\mathcal{G}$	Γ
Boundary of disk	Copy of $\mathbb{CP}^1$ denoted Γ_0
Boundary vertex b_l	Marked point κ_l on Γ_0
Black vertex V'_s	Copy of $\mathbb{CP}^1$ denoted Σ_s
White vertex V_l	Copy of $\mathbb{CP}^1$ denoted Γ_l
Internal Edge	Double point
Face	Oval

Example: main cell in $Gr^{TP}(2,4)$.



Top: The Le-network $\mathcal{N}_T$ [left] and the topological model of the spectral curve $\Gamma(\mathcal{N}_T)$ [right] for the point $[A] \in Gr^{\mathrm{TP}}(2,4)$ Bottom: the reduced Le-network $\mathcal{N}_{T,\mathrm{red}}$ [left] and the topological model of the spectral curve $\Gamma(\mathcal{N}_{T,\mathrm{red}})$ [right].

Theorem. A.-G. Let $[A]$ be a point of totally non-negative Grassmannian, x_0 be a **generic** normalization point. Then the corresponding divisor satisfies the Dubrovin-Natanzon condition:

- All divisor points lie at finite real ovals;
- each finite real oval contains exactly one divisor point.

At some times the divisor points may collide at double points. A resolution of singularities is required.

Let us discuss the main cell case.

For simplicity we do not do reflection with respect to the vertical line!

The main cell (Totally positive Grassmannians).

The main cell correspond to the totally **positive** Grassmannianas.

*	*	*	*	*	*
*	*	*	*	*	*
*	*	*	*	*	*
*	*	*	*	*	*

Figure: The main cell

- The Young diagram is rectangular;
- All cells of the Young diagram are filled.

The main cell (Totally positive Grassmannians).

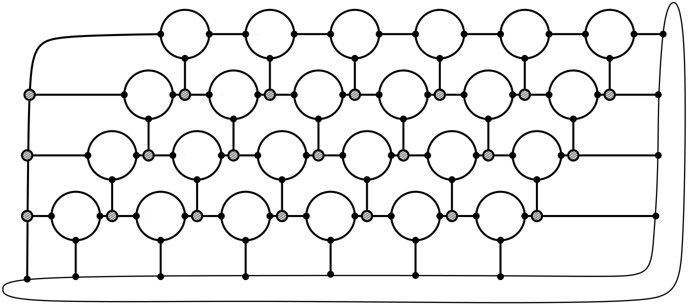


Figure: The curve corresponding to the main cell.

The *b*-cycles correspond to **finite** faces of the graph. They are oriented clockwise.

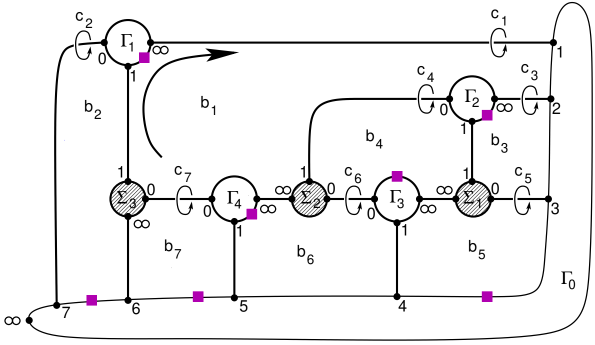


Figure: Canonical basis of cycles. Here:

$$a_1 = c_1, \quad a_2 = c_2, \quad a_3 = c_3 + c_1 \quad a_4 = c_4 + c_1$$

$$a_5 = c_3 + c_3 + c_1, \quad a_6 = c_6 + c_4 + c_1, \quad a_7 = c_7 + c_1,$$

The basic differentials

The canonical differential ω_k is defined in the following way (Tyurin, Artamkin):

1. If the cycle b_k does not pass through a vertex (black or white), then the restriction of ω_k to the corresponding copy of $\mathbb{C}P^1$ is equal to zero.
2. If the cycle b_k passes through an internal vertex (black or white), the differential ω_k has exactly two first-order poles, with residue 1 at the outgoing point and with residue -1 at the incoming point.
3. If the cycle passes through a boundary vertex, it has first-order poles on Γ_0 with residues 1 at outgoing vertexes and residues -1 at incoming vertexes.

We assume:

$$\oint_{a_j} \omega_k = 2\pi i \delta_{jk}.$$

We use the following choice of integration constants for the Abel transform. Let $\mathcal{D} = \gamma_1 + \ldots + \gamma_g$, where g is the arithmetic genus of Γ. We have few divisor points at Γ_0, and exactly one at each internal white Γ_j, $j > 0$. Let the cycle b_l pass through the components $\Gamma_{j_1}, \ldots, \Gamma_{j_{r_l}}$. Then

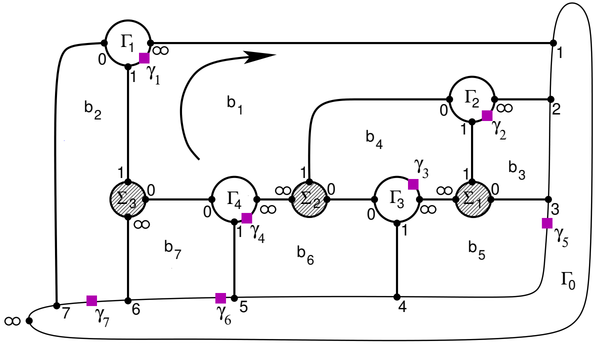
$$A_l(\mathcal{D}) = \sum_{\gamma_s \in \Gamma_0} \int_{\infty}^{\gamma_s} \omega_l \Big|_{\Gamma_0} + \sum_{\substack{j_s \neq 0 \\ j_s \in \{j_1, \ldots, j_{r_l}\}}} \int_{P_{0,j_s}}^{\gamma_{j_s}} \omega_l \Big|_{\Gamma_{j_s}},$$

where P_{0,j_s} denotes the unique marked point on Γ_{j_s} through which the cycle b_l does not pass.

A divisor $\mathcal{D}$ satisfies the Dubrovin-Natanzon condition iff

$$\operatorname{Im} A_l(\mathcal{D}) = \pi \mod (2\pi) \text{ for all } l = 1, \ldots, g. \qquad (2)$$

Abel transform



For the Abel transform for γ_1 we integrate ω_1 from 0 at Γ_1, and ω_2 from ∞ at Γ_1. Al integrals at Γ_0 are calculated from ∞.

Basic singularities on the space of divisors

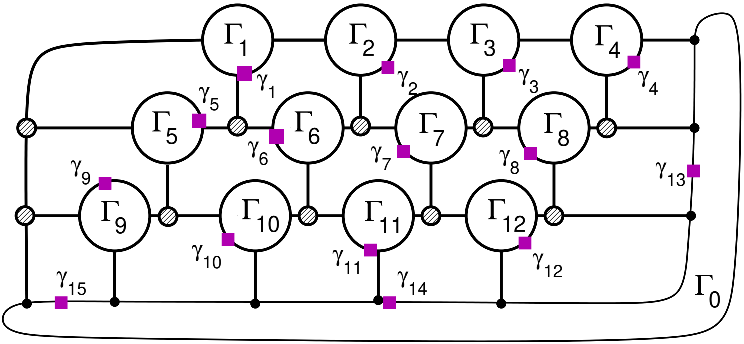


Figure: Two basic singular configurations.

- Divisor points γ_{11} and γ_{14} pass through a double point;
- Divisor points γ_1, γ_5 and γ_6 pass through a triple point;

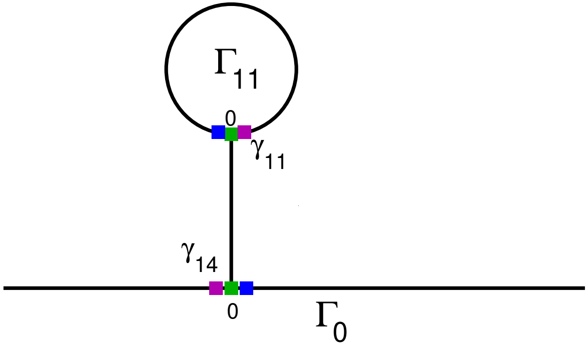


Figure: Two basic singular configurations.

Magenta, green and blue denote 3 subsequent positions of divisor. We have to do standard blow-up at the point $z_{11} = z_{14} = 0$.

$$\mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}P^1, \quad (z_{11}, z_{14}) \to (z_{11}, z_{14}, (z_{11} : z_{14})).$$

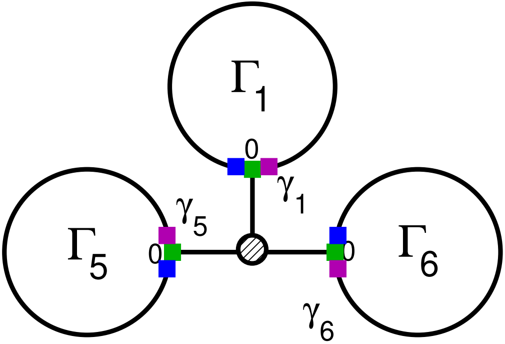


Figure: Two basic singular configurations.

Magenta, green and blue denote 3 subsequent positions of divisor. We have to do standard blow-up at the point $z_1 = z_5 = z_6 = 0$.

$$\mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}P^1, \quad (z_1, z_5, z_6) \to (z_1, z_5, z_6, (z_1 : z_5 : z_6)).$$

Theorem. A.-G. Inside the main cell only two types of singular configurations described above may take place.

The proof is based on the following lemma:

Lemma The for a given time $\vec{t}$ the wave function of KP can not vanish identically on components of Γ containing divisor points.
The sketch of proof. For a given time $\vec{t}$ the wave function:

- does not vanish identically on Γ_0, therefore it is equal to zero at the divisor points only;
- has no zeroes at the infinite oval;
- is constant on "black" components;
- is linear rational on "white" components.

The main cell case

Let us mark all components at which the wave function vanishes identically.

Lemma

- If a black component is connected to a marked white, it is also marked.
- The upper row contains no marked components.
- The left column contains no marked components.
- If a white component is connected with two marked components, it is also marked.

Lemma

Each connected component of the marked area has the form of triangle; next row of the triangle has exactly one component less then the previous one.

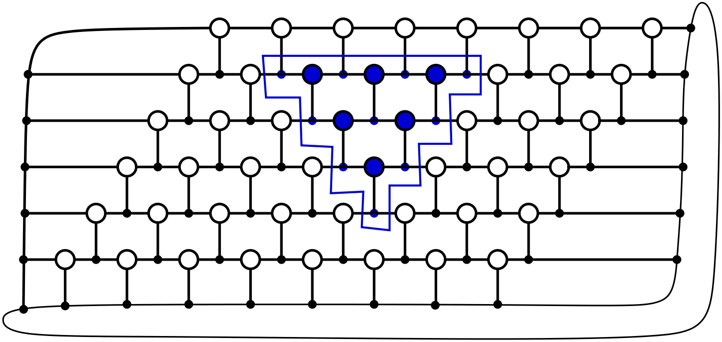


Figure: The shape of one marked area.

Final argument:
The number of internal faces in the triangular marked area is smaller than the number of white components in it. Therefore is not sufficient place for Dubrovin-Natanzon divisor.