Maslov's Canonical Operator: New Formulas and Efficient Asymptotics

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Conference "Dynamics in Siberia"

Sobolev Institute of Mathematics and Novosibirsk State University,
Novosibirsk, February 24–March 1, 2025

Viktor Pavlovich Maslov (1930-2023)

This year is the 60th anniversary of V.P. Maslov's book

"Perturbation Theory and Asymptotic Methods"
(Moscow, Moscow State University Publ., 1965),

which was approved for printing on June 26, 1965 and which can be viewed as the first detailed exposition of the

Canonical Operator

Viktor Pavlovich was my Teacher.   I dedicate this talk to his benevolent memory.

What Is the talk about?

- Since 1965, there have been many publications on the canonical operator, notably,

"Semiclassical Approximation for Equations of Quantum Mechanics"
(Moscow, Nauka, 1976) by V.P. Maslov and M.V. Fedoryuk
so far, arguably, one of the most accessible monograph expositions of the theory

- These publications were by the effort of quite a few people working on this and related topics

V.M. Babich, V.A. Borovikov, V.S. Buldyrev, S.Yu. Dobrokhotov, V.G. Danilov, M.V. Karasev, M.Ya.Kelbert, Yu.A. Kravtsov, A.S.Kryukovskii, V.V. Kucherenko, V.F. Lazutkin, D.S. Lukin, Yu.I.Orlov, E.A.Palkin, A.Yu. Shafarevich, V.E. Shatalov, S.Yu. Slavyanov, B.Yu. Sternin, B.R. Vainberg, V.M. Vorob'ev, P.N. Zevandrov …

M.V. Berry, J.J. Duistermaat, L. Hörmander, A. Melin, R.B. Melrose, F.W.J. Olver, J. Sjöstrand, G.A. Uhlmann …

(The list is, of course, **very** incomplete, and I have not even tried to mention any of the more recent developments…)

- On the other hand, in the last 60+ years, the computational tools available to researchers have dramatically changed. Technical computing systems (*Wolfram Mathematica*; *MatLab*) have appeared that provide fundamentally new possibilities for the implementation and visualization of mathematical objects but at the same time necessitate reshaping the construction of the canonical operator so as to ensure efficient application of such systems.
- Further, there arise new problems (such as equations with singularities) to which the "classical" canonical operator cannot be applied and hence needs appropriate generalizations and modifications.

In the talk, I discuss some of the results obtained in these two directions in the recent 10+ years at the Laboratory of Mechanics of Natural Hazards, Ishlinsky Institute for Problems in Mechanics, RAS, by a team including S.Yu. Dobrokhotov, A.I. Shafarevich, and myself (general theoretical questions) and A.Yu. Anikin, D.S. Minenkov, A.A. Tolchennikov, A.V. Tsvetkova, as well as younger scientists and postgraduate students (specific problems and applications), sometimes in collaboration with scientists from other institutions and countries (J. Brüning, G. Makrakis, M. Rouleux, B. Tirozzi and others)

Equations with small parameter

- Consider equations of the form $\hat{H}u = 0,$ where $\hat{H} = H(x, \hat{p}),$ $\hat{p} = -i\mu \frac{\partial}{\partial x},$
  is a differential operator with a small parameter $\mu > 0$ multiplying the derivatives
- Examples include

Schrödinger equation $H(x, p) = \frac{p^2}{2m} + V(x) - E$ or $H(x, p) = \frac{p^2}{2m} + V(x) + p_0,$

$\hat{p}_0 = -i\mu \frac{\partial}{\partial t},$ $\mu = h$ is Planck's constant

Wave equation $H(x, p) = p_0^2 - c^2(x) p^2$

Helmholtz equation $H(x, p) = p^2 - n^2(x)$

(for both:) $\mu = \omega^{-1}, \quad \omega \to \infty$ is frequency

Semiclassical asymptotics and Maslov's canonical operator

Semiclassical asymptotics = rapidly oscillating asymptotic solutions of $\hat{H}u = 0$ as $\mu \to 0$

Simplest example: WKB solutions $u(x,\mu) = \exp\left(\frac{iS(x)}{\mu}\right)a(x),$

where $S(x)$ satisfies the Hamilton-Jacobi equation

$$H\left(x, \frac{\partial S}{\partial x}\right) = 0$$

The equation $p = \frac{\partial S}{\partial x}$ specifies the Lagrangian manifold Λ associated with u.

Lagrangian: $\dim \Lambda = n, \quad i^*\omega^2 = 0, \qquad \omega^2 = dp_1 \wedge dx_1 + \cdots + dp_n \wedge dx_n$

This manifold is diffeomorphically projected onto the coordinate x-plane.
In general, this is not the case; we just have a Lagrangian manifold Λ such that $H(x,p) = 0$ on Λ. $\Rightarrow$ WKB is not enough; use canonical operator instead.

Canonical operator as a black box

Main geometric objects:

- Lagrangian manifold $\Lambda \subset \mathbb{R}^{2n}$, $x = X(\alpha)$, $p = P(\alpha)$, $\sum_j dP_j(\alpha) \wedge dX_j(\alpha) = 0$
- volume form $d\sigma$ на Λ

Quantization condition

$$\frac{2}{\pi\mu}[\omega^1] + \text{ind} \equiv 0 \bmod 4 \quad \text{in} \quad H^1(\Lambda, \mathbb{R}); \quad \omega^1 = p\,dq;$$

$$\text{ind}\,\gamma = \frac{1}{2\pi} \operatorname*{var\,arg}_{\gamma} \frac{d(X_1 - iP_1) \wedge \cdots \wedge d(X_n - iP_n)}{d(X_1 + iP_1) \wedge \cdots \wedge d(X_n + iP_n)}$$

Under these conditions there is a well-defined canonical operator

$K_{(\Lambda, d\sigma)}\colon C_0^\infty(\Lambda) \longrightarrow C_\mu^\infty(\mathbb{R}^n)$ (rapidly oscillating functions on $\mathbb{R}^n$)

$$u \in C_\mu^\infty(\mathbb{R}^n) \iff \sup_{\mu \in (0,1]} \|(1 - \mu^2\Delta + x^2)^{k/2} u\| < \infty \quad \forall k$$

Commutation formulas with a μ-ΨDO

$$\widehat{P} = P\left(\overset{2}{x}, -\mu\frac{\overset{1}{\partial}}{\partial x}, \mu\right), \quad P(x,p,\mu) = P_0(x,\mu) + \mu P_1(x,\mu) + \dots$$

Then we have two commutation formulas

$$\widehat{P}K_{(\Lambda,d\sigma)}A = K_{(\Lambda,d_\sigma)}(P_0\big|_\Lambda A) + O(\mu)$$

If $P_0 = 0$ on Λ, then $\hat{P}K_{(\Lambda,d\sigma)}A = -i\mu K_{(\Lambda,d\sigma)}\Pi A + O\left(\mu^2\right)$

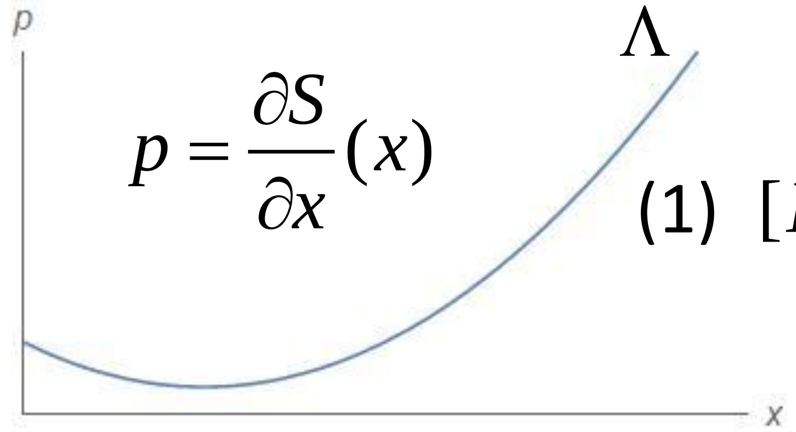
$$\Pi = (V(P_0) + iP_{sub})\big|_\Lambda, \qquad P_{sub} = P_1 + \frac{i}{2}\sum_{j=1}^{n}\frac{\partial^2 P_0}{\partial x_j \partial p_j}$$

Algorithm for solving $\widehat{P}u = 0$ (e.g. $\widehat{P} = \widehat{H} - E$)

- Find Λ such that quantization conditions hold and $P_0\big|_\Lambda = 0$
- Solve the transport equation $\Pi A = 0$ and set $u = K_{(\Lambda,d\sigma)}A$

Canonical operator K : functions on $\Lambda \mapsto$ functions on $\mathbf{R}_x$ (classical construction, 1D case)

(1)

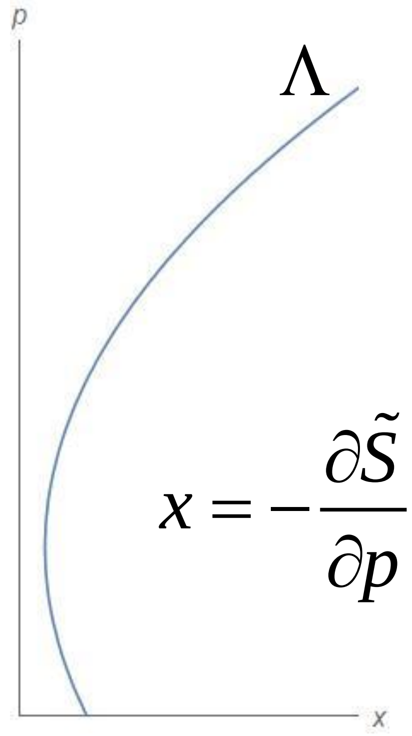


$$p = \frac{\partial S}{\partial x}(x)$$

ϕ is a function on Λ ; $d\sigma$ is a measure on Λ

(1) $$[K\phi](x) = \text{ WKB element} = \exp\left(\frac{iS(x)}{\mu}\right)\phi(x)\left(\frac{d\sigma}{dx}\right)^{1/2}$$

(2)



(2) WKB element $$= \exp\left(\frac{i\tilde{S}(p)}{\mu}\right)\phi(p)\left(\frac{d\sigma}{dp}\right)^{1/2}$$

wrong variable!

To obtain a function of x, rotate the picture by $\pi / 2$

Fock quantization of the rotation gives the Fourier transform

$$x = -\frac{\partial \tilde{S}}{\partial p}(p)$$

$$[K\phi](x) = \left(\frac{i}{2\pi\mu}\right)^{1/2} \int \exp\left(\frac{i(\tilde{S}(p) + px)}{\mu}\right)\phi(p)\left(\frac{d\sigma}{dp}\right)^{1/2} dp$$

The main disadvantage of the “classical” canonical operator is the use of coordinate systems of the form $(x_{j_1},\ldots,x_{j_s},p_{j_{s+1}},\ldots,p_{j_n})$, very often unnatural in the problem being solved. Hence plenty of charts, partitions of unity, etc. This necessitates looking for more computationally efficient formulas for the canonical operator.

Main New Developments

1. Reworking and improving known asymptotic formulas so that they could be implemented efficiently on such systems

   - New formulas in arbitrary coordinates on the Lagrangian manifold
   - Representations via special functions in neighborhoods of caustics

2. Developing modifications of the canonical operator based on a wider class of phase spaces and Lagrangian manifolds so as to extend the class of problems where the canonical operator applies

New Formulas

General Oscillating Integrals

- Nondegenerate phase function $\Phi(x,\theta)$ defined on $V \subset \mathbb{R}^n_x \times \mathbb{R}^m_\theta$: the differentials $d(\Phi_{\theta_1}),\dots,d(\Phi_{\theta_m})$ are linearly independent on

$$C_\Phi = \{(x,\theta) \in V : \Phi_\theta(x,\theta) = 0\}$$

- Lagrangian manifold $\Lambda_\Phi = \jmath_\Phi(C_\Phi)$,

$$\jmath_\Phi : C_\Phi \ \to \ \mathbb{R}^{2n}_{(x,p)}, \ (x,\theta) \ \mapsto \ (x, \Phi_x(x,\theta))$$

- Oscillating integral

$$[K^\mu_\Phi A](x,\mu) = \frac{e^{i\pi\frac{m}{4}}}{(2\pi\mu)^{m/2}} \int_{\mathbb{R}^m} e^{\frac{i}{\mu}\Phi(x,\theta)} B(x,\theta,\mu)\, d\theta_1 \cdots d\theta_m,$$

$$B(x,\theta,\mu) = |F_{(\Phi,d\sigma)}(x,\theta)|^{1/2} A(\jmath(x,\theta),\mu)$$

$$(x,\theta) \in C_\Phi$$

$$F_{(\Phi,d\sigma)} = \frac{\jmath_\Phi(d\sigma) \wedge d(\Phi_{\theta_1}) \wedge \cdots \wedge d(\Phi_{\theta_m})}{dx_1 \wedge \cdots \wedge dx_m \wedge d\theta_1 \wedge \dots d\theta_m}$$

L. Hörmander (1971)
Fourier integral operators

Universal Phase Function and the Main Formula

- Lagrangian manifold $\Lambda \subset \mathbb{R}^{2n}_{(x,p)}$, $x = X(\alpha), p = P(\alpha), \alpha = (\alpha_1, \ldots, \alpha_n)$ loc. coord.
- Action $\tau(\alpha)$: $d\tau = \langle P, dX \rangle$, measure (volume form) $d\sigma = \sigma(\alpha) d\alpha_1 \wedge \ldots \wedge d\alpha_n$

Universal phase function

$$S(x,\alpha) = \tau(\alpha) + \langle P(\alpha), x - X(\alpha) \rangle$$

$$\langle a,b \rangle = \sum_{j=1}^{n} a_j b_j$$

- Local canonical operator:

$$\alpha_0 \in \Lambda, \operatorname{rank} X_\alpha(\alpha_0) = k \quad \Rightarrow \quad \alpha = (\phi, \theta) \equiv (\phi_1, \ldots, \phi_k, \theta_1, \ldots, \theta_{n-k}), \ \operatorname{rank} X_\phi = k$$

$$\det \Pi(\phi,\theta) X_\phi(\phi,\theta) \neq 0; \qquad \Pi(x - X) = 0 \ \Rightarrow \phi = \phi(x,\theta)$$

$$\Phi(x,\theta) = S(x, \phi(x,\theta), \theta)$$

$$[K^\mu_\Lambda A](x) = \frac{e^{i\pi m/4}}{(2\pi\mu)^{m/2}} \int e^{\frac{i}{\mu}\Phi(x,\theta)} ((\sigma \det M)^{1/2} A)(\phi(x,\theta),\theta)\, d\theta$$

$$M(\phi,\theta) = \left(\Pi^*(\phi,\theta) \quad P_\phi(\phi,\theta) \Pi(\phi,\theta) X_\theta(\phi,\theta) \right)$$

Simplest Case: Eikonal Coordinates

Assume that locally on Λ one has $d\tau = \langle P, dX \rangle \neq 0.$

Then there exist functions $\psi = (\psi_1, \ldots, \psi_{n-1})$ on Λ such that (τ, ψ) are local coordinates on Λ

Eikonal coordinates

$$x = X(\tau,\psi),\ p = P(\tau,\psi)$$

Define $\tau = \tau(x,\psi)$:

$$\langle P(\tau,\psi), x - X(\tau,\psi) \rangle = 0$$

Assume that $\det\begin{pmatrix} P & P_\psi \end{pmatrix} \neq 0.$

Let $\sigma = \sigma(\tau,\psi)$ be the density of the measure, $d\sigma = \sigma\, d\tau \wedge d\psi_1 \wedge \cdots \wedge d\psi_{n-1}$

$$[K_{(\Lambda,d\sigma)}A](x) = \left(\frac{i}{2\pi\mu}\right)^{(n-1)/2} \int e^{i\frac{\tau}{\mu}} A(\tau,\psi)\sqrt{\sigma \det(P\ P_\psi)}\,\Big|_{\tau=\tau(x,\psi)} d\psi_1 \cdots d\psi_{n-1}$$

Example: Bessel functions of integer index

Reminder:
Equation

$$r^2 \frac{d^2 v(r)}{dr^2} + r \frac{dv(r)}{dr} + (r^2 - n^2) v(r) = 0$$

Integral representation

$$\mathbf{J}_n(r) = \frac{1}{2\pi} \oint e^{i(r \sin\theta - n\theta)} \, d\theta$$

Let us pretend we do not know the integral representation and try to obtain the asymptotics as $n \to \infty$ and/or $r \to \infty$

$$x = (x_1, x_2) \in \mathbb{R}^2, \qquad (r, \varphi) \quad \text{polar coordinates}$$

Consider the function $u(x,\mu) = e^{in\varphi} \mathbf{J}_n\left(\frac{r}{\mu}\right), \qquad n = \frac{\gamma}{\mu}$

$$\hat{H}_1 u \equiv -\mu^2 \Delta u = u \qquad\qquad H_1(x,p) = p^2 \equiv p_1^2 + p_2^2$$

$$\hat{H}_2 u = x_1\left(-i\mu\frac{\partial}{\partial x_2}\right)u - x_2\left(-i\mu\frac{\partial}{\partial x_1}\right)u = \gamma u \qquad\qquad H_2(x,p) = x_1 p_2 - x_2 p_1$$

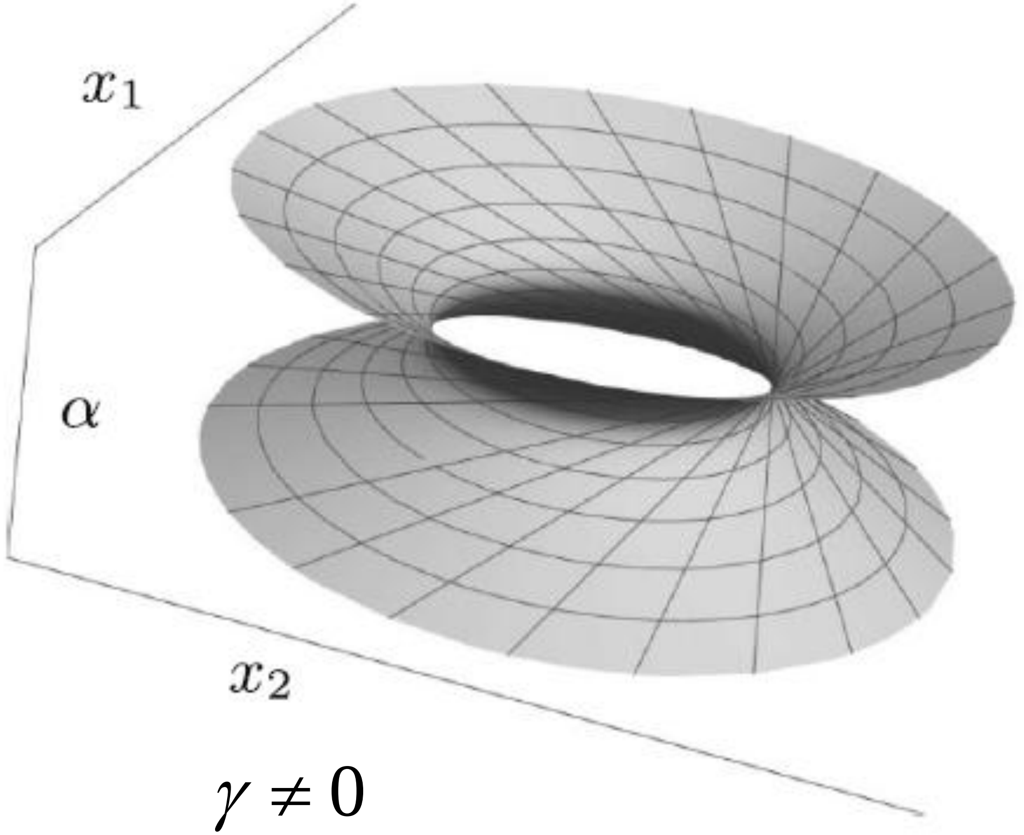
Hamiltonians in involution; “Liouville” Lagrangian manifold (their common level)

Lagrangian manifold

$$\Lambda_\gamma = \{(x,p) \in \mathbb{R}^4 : p^2 = 1,\ x_1 p_2 - x_2 p_1 = \gamma\}$$

Parametric description:

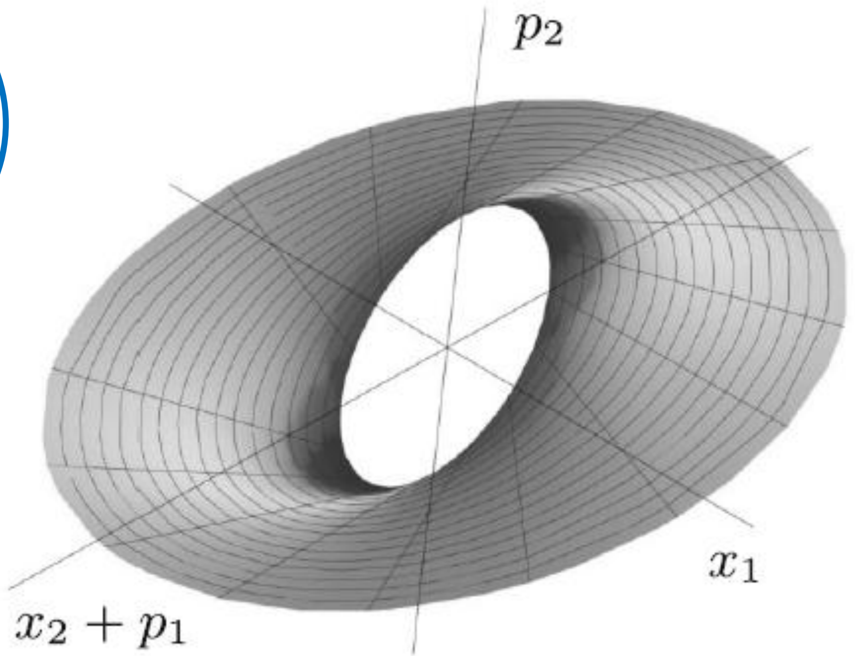
$$x = X(\alpha,\psi) \equiv \alpha \mathbf{n}(\psi) - \gamma \mathbf{n}'(\psi),$$
$$p = P(\alpha,\psi) \equiv \mathbf{n}(\psi),$$
$$\alpha \in \mathbb{R}, \quad \psi \in \mathbb{S}^1$$



$\gamma \neq 0$

$\alpha = 0$

focal points



$\gamma = 0,\ \alpha > 0$

$$x = X(\alpha, \psi) \equiv \alpha \mathbf{n}(\psi) - \gamma \mathbf{n}'(\psi),$$

$$p = P(\alpha, \psi) \equiv \mathbf{n}(\psi),$$

$$\alpha \in \mathbb{R}, \quad \psi \in \mathbb{S}^1$$

$$\mathbf{n}(\psi) = \begin{pmatrix} \cos\psi \\ \sin\psi \end{pmatrix}$$

Action (eikonal) $\tau(\alpha,\psi) = \alpha + \gamma\psi$

(τ,ψ) eikonal coordinates

$$\det(P\ P_\psi) = 1$$

$d\sigma = d\alpha \wedge d\psi = d\tau \wedge d\psi$ invariant measure

$$\langle P, x - X \rangle = 0 \quad \Rightarrow \quad \tau(x,\psi) = \langle \mathbf{n}(\psi), x \rangle + \gamma\psi$$

$A = 1$ solution of the transport equation

$$[K_{(\Lambda,d\sigma)}A](x) = \left(\frac{i}{2\pi\mu}\right)^{(n-1)/2} \int e^{i\frac{\tau}{\mu}} A(\tau,\psi)\sqrt{\sigma \det(P\ P_\psi)}\,\Big|_{\tau=\tau(x,\psi)} d\psi_1 \cdots d\psi_{n-1}$$

$$[K_{(\Lambda_\gamma,d\sigma)}1](x) = \left(\frac{i}{2\pi\mu}\right)^{1/2} \int e^{\frac{i}{\mu}\langle \mathbf{n}(\psi),x\rangle + \gamma\psi}\, d\psi \Rightarrow \quad \mathbf{J}_n(r) = \frac{1}{2\pi} \oint e^{i(r\sin\theta - n\theta)}\, d\theta$$

REPRESENTATION
VIA SPECIAL FUNCTIONS
IN A NEIGHBORHOOD OF CAUSTICS

General Principles

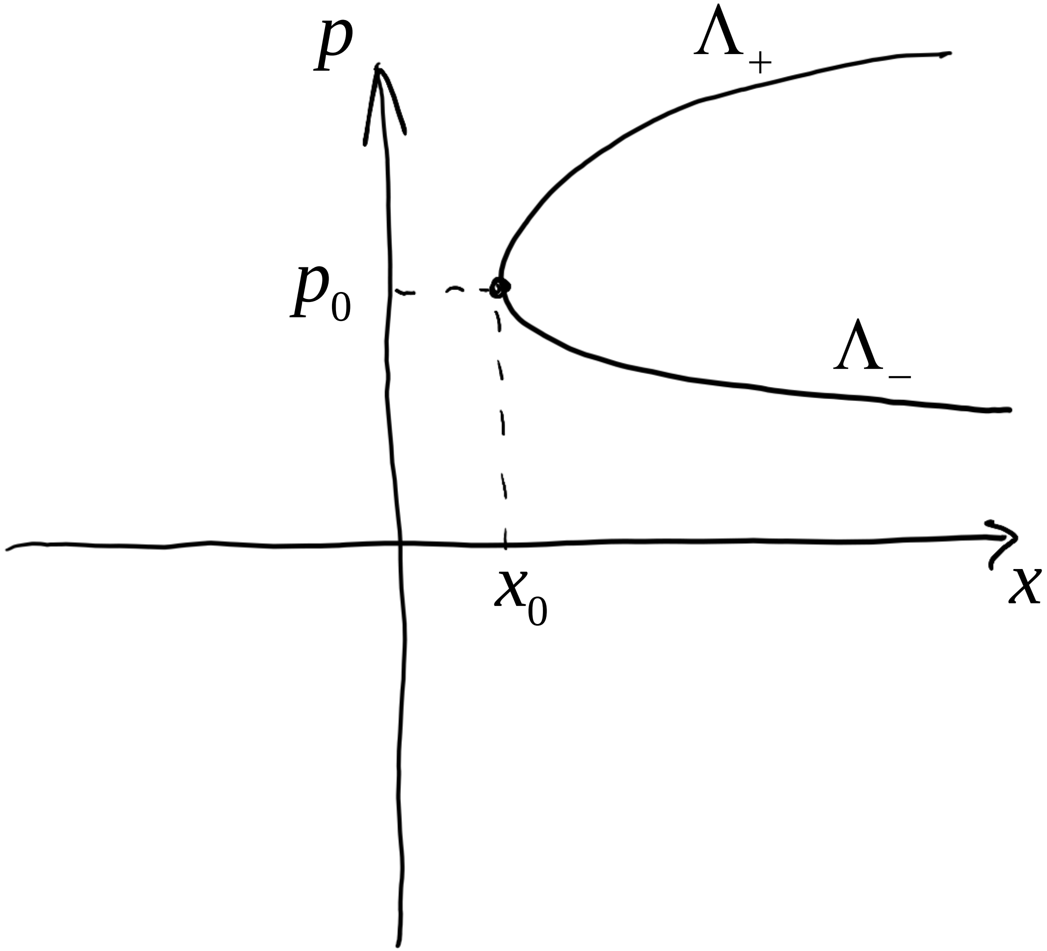
- In semiclassical problems (and ray expansions) in the vicinity of caustics (that is, the projections of Lagrangian singularities onto the configuration space), the asymptotic behavior of the solution determined by special functions, with rare exceptions, is given in parametric form.
- Suitable parameters in this representation are given by the coordinates on the corresponding Lagrangian manifold.
- Although WKB solutions (or ray expansions) do not work in the vicinity of caustics, the (multivalued) phases, Jacobians and amplitudes are determined, and it is these objects that participate in the efficient representation of asymptotic solutions in the vicinity of caustics with the use of special functions.

Studies of rapidly oscillating integrals with degenerating and merging stationary points

- V.P. Maslov, M.V. Fedoryuk, V.M. Babich, V.S. Buldyrev, I.A. Molotkov, V.A. Borovikov, D.S. Lukin, E.A. Palkin, A.S. Kryukovsky, S.Yu. Slavyanov, Yu.A. Kravtsov, Yu.I. Orlov …
- D. Ludwig, J.B. Keller, F.W.J. Olver , M.V. Berry, C.J. Howls …

and many other people…

Asymptotics in a Neighborhood of a Generic Fold



$$\Lambda = \{x = X(\alpha), p = P(\alpha)\}$$

$$\tau(\alpha): \quad d\tau = P(\alpha)\,dX(\alpha)$$

Canonical operator on a Lagrangian manifold with a turning point

$\alpha = \alpha_\pm(x)$ specify $\Lambda_\pm$

$\alpha = \alpha(p)$

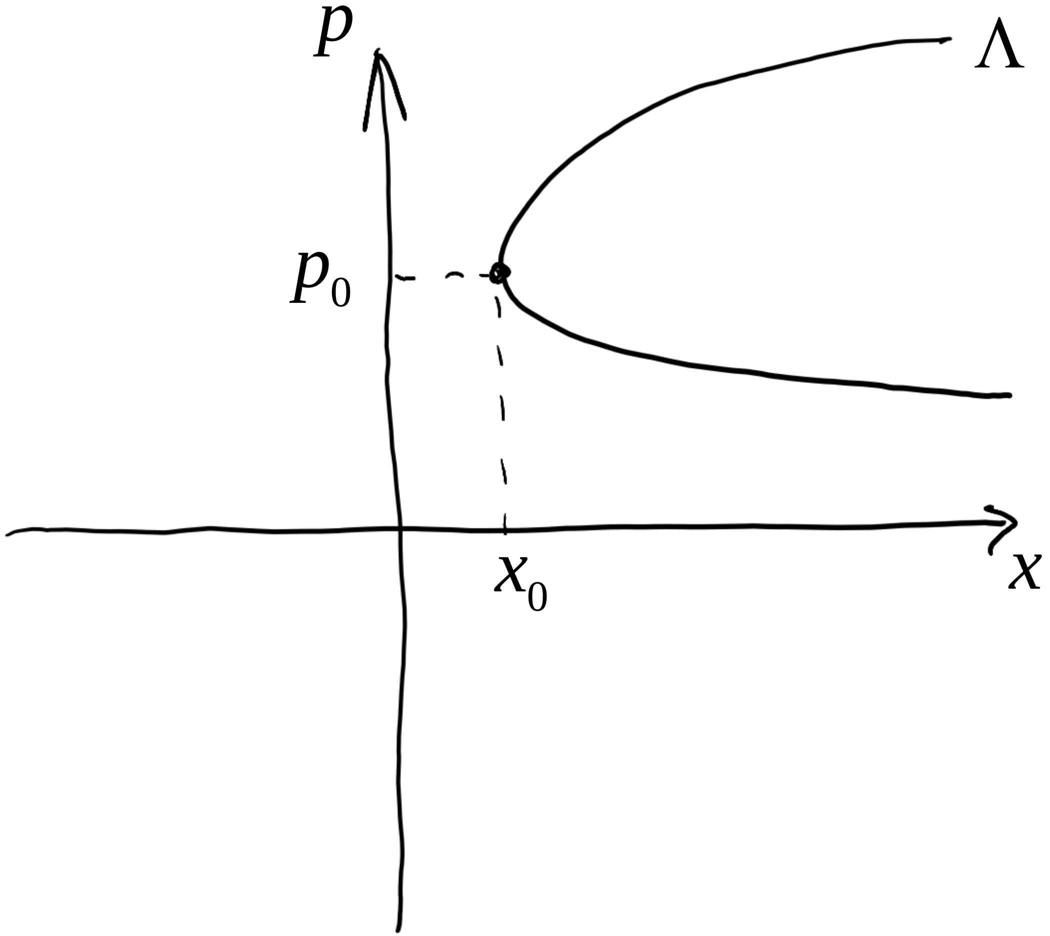
To define the canonical operator, we need:

$$S_\pm(x) = \tau(\alpha_\pm(x))$$

$$S(p) = (\tau(\alpha) - P(\alpha)X(\alpha))\,|_{\alpha=\alpha(p)}$$

$$J_\pm(x) = \frac{\partial X}{\partial \alpha}(\alpha_\pm(x))$$

$$J(p) = \frac{\partial P}{\partial \alpha}(\alpha(p))$$



Let $\varphi(\alpha)$ be a function on Λ

$$[K_\Lambda \varphi](x) = ?$$

(1) $x > x_0 + \delta, \quad \delta > 0$

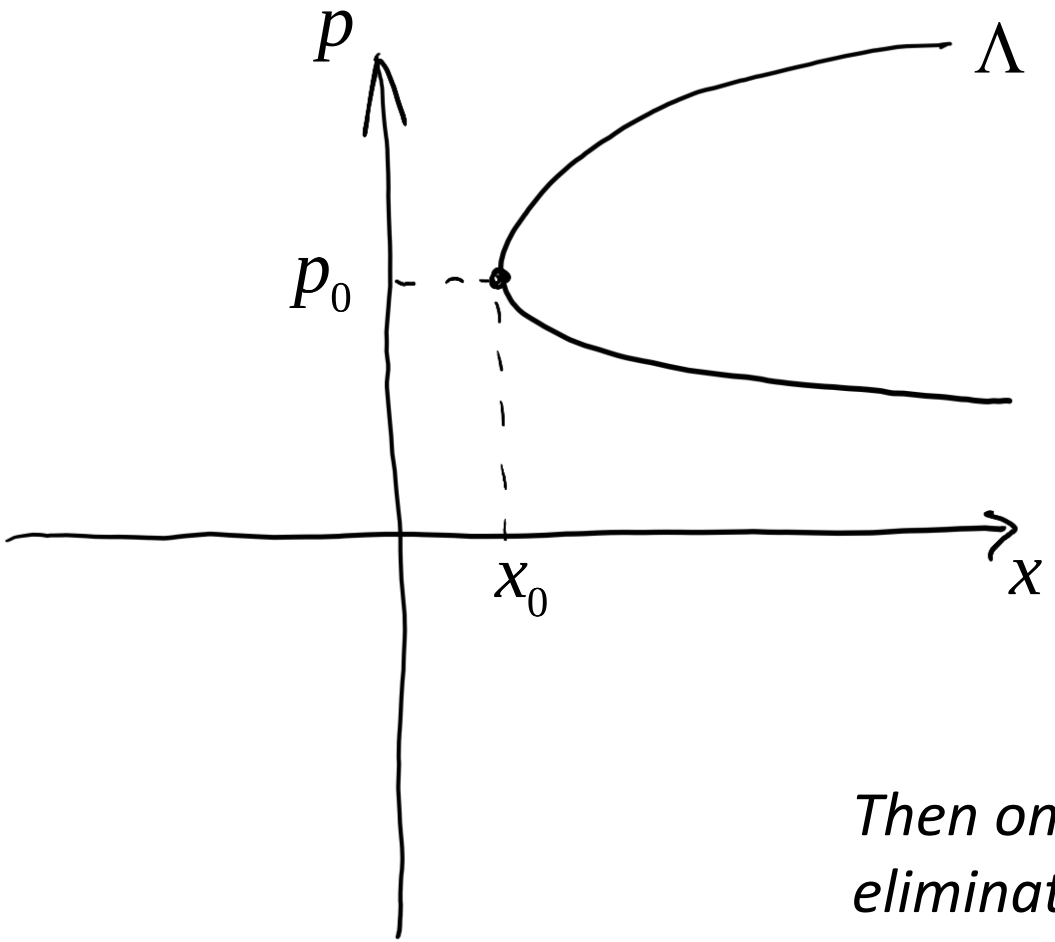
$$[K_\Lambda \varphi](x) = e^{\frac{i}{\mu} S_+(x)} \frac{\varphi(\alpha_+(x))}{\sqrt{J_+(x)}} + e^{\frac{i}{\mu} S_-(x)} \frac{\varphi(\alpha_-(x))}{\sqrt{J_-(x)}}$$

(2) $|x - x_0| < \delta$

$$[K_\Lambda \varphi](x) = \left(\frac{i}{2\pi\mu} \right)^{1/2} \int e^{\frac{i}{\mu}(px + S(p))} \frac{\varphi(\alpha(p))}{\sqrt{J(p)}} dp$$

(3) $x < x_0 - \delta$

$$[K_\Lambda \varphi](x) = O(\mu^\infty)$$



Fold (Lagrangian singularity of the type A_2)

Arnold, Varchenko, Gussein-Zade

$$x = f(p), \qquad f'(p_0) = 0, \quad f''(p_0) \neq 0$$

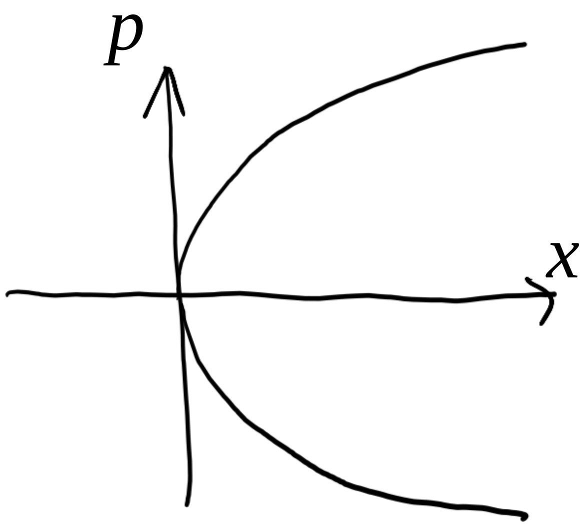
Then one can simplify the canonical operator, namely, eliminate the integral by expressing it via the Airy function

Airy function $$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(tz+t^3/3)}dt$$

$$y'' - zy = 0$$

How to do that?

(a) Consider the standard fold $x = p^2$ $\alpha = p$



$$[K_\Lambda \varphi](x) = \left(\frac{i}{2\pi\mu}\right)^{1/2} \int_{-\infty}^{\infty} e^{\frac{i}{\mu}\left(px - \frac{p^3}{3}\right)} a(p)\, dp$$

$a = \varphi / \sqrt{J}$

$$a(p) = 1 \quad \Rightarrow \quad [K_\Lambda \varphi](x) = \left(\frac{2\pi i}{\mu}\right)^{1/2} \mathrm{Ai}\left(-\frac{x}{\mu}\right)$$

$$a(p) \not\equiv 1 \quad \Rightarrow \quad a(p) = F_0(p^2) + pF_1(p^2)$$

$$[K_\Lambda \varphi](x) = \left(\frac{i}{2\pi\mu}\right)^{1/2} \int_{-\infty}^{\infty} e^{\frac{i}{\mu}\left(px - \frac{p^3}{3}\right)} (F_0(p^2) + pF_1(p^2))\, dp$$

$$= \left(\frac{i}{2\pi\mu}\right)^{1/2} \int_{-\infty}^{\infty} e^{\frac{i}{\mu}\left(px - \frac{p^3}{3}\right)} (F_0(x) + pF_1(x))\, dp + \text{lower-order terms}$$

$$F(x) - F(p^2) = (x - p^2)G(x,p) = G(x,p)\frac{\partial}{\partial p}\left(px - \frac{p^3}{3}\right)$$

integration by parts

$$= \left(\frac{2\pi i}{\mu}\right)^{1/2} \left(\mathrm{Ai}\left(-\frac{x}{\mu}\right) F_0(x) + i\,\mathrm{Ai}'\left(-\frac{x}{\mu}\right) F_0(x) \right)$$

(b) General A_2 fold: change of variables reduces the case to the standard fold,
and we obtain Airy function (and its derivative) of the composite argument.

How to compute these expressions more easily? There is a simple trick.

Asymptotics as $z \to +\infty$:

$$\mathrm{Ai}(-z) = \frac{1}{z^{1/4}\sqrt{\pi}}\left(\cos\left(\frac{2}{3}z^{3/2} - \frac{\pi}{4}\right) + O(z^{-3/2})\right)$$

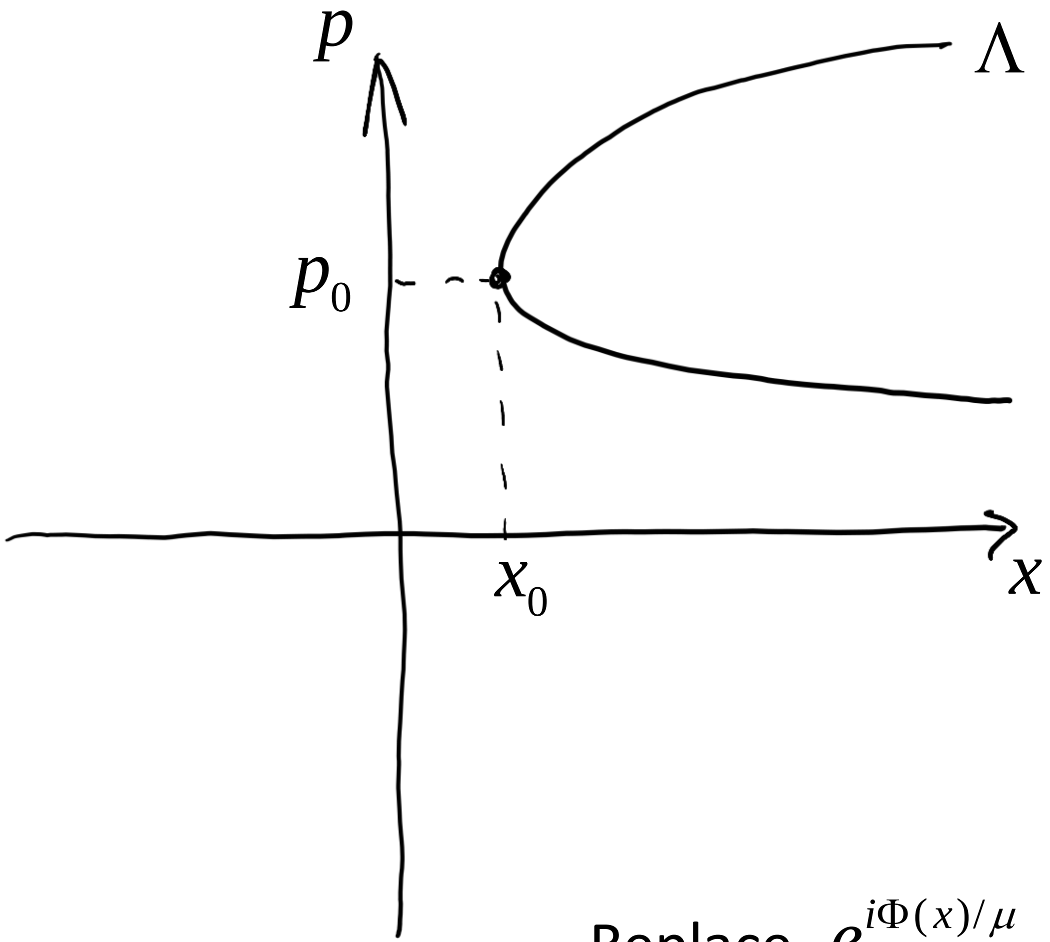
$$\mathrm{Ai}'(-z) = -\frac{z^{1/4}}{\sqrt{\pi}}\left(\sin\left(\frac{2}{3}z^{3/2} - \frac{\pi}{4}\right) + O(z^{-3/2})\right)$$

$$\mathcal{E}(w) = e^{iw}\left(1 + O\left(\frac{1}{w}\right)\right)$$

$$w \to +\infty$$

Let us make the “fake” exponential

$$\mathcal{E}(w) = \sqrt{\pi}e^{i\pi/4}\left[\left(\frac{3}{2}w\right)^{1/6}\mathrm{Ai}\left(-\left(\frac{3}{2}w\right)^{2/3}\right) - i\left(\frac{3}{2}w\right)^{-1/6}\mathrm{Ai}'\left(-\left(\frac{3}{2}w\right)^{2/3}\right)\right], \quad w > 0$$



In the nonsingular charts:

$$[K_\Lambda \varphi](x) = e^{\frac{i}{\mu}S_+(x)}\frac{\varphi(\alpha_+(x))}{\sqrt{J_+(x)}} + e^{\frac{i}{\mu}S_-(x)}\frac{\varphi(\alpha_-(x))}{\sqrt{J_-(x)}}$$

Set

$$T(x) = \frac{1}{2}(S_+(x) + S_-(x)),$$

$$\Phi(x) = \frac{1}{2}(S_+(x) - S_-(x))$$

Replace $e^{i\Phi(x)/\mu}$ by the fake exponential $\mathcal{E}(\Phi(x)/\mu)$:

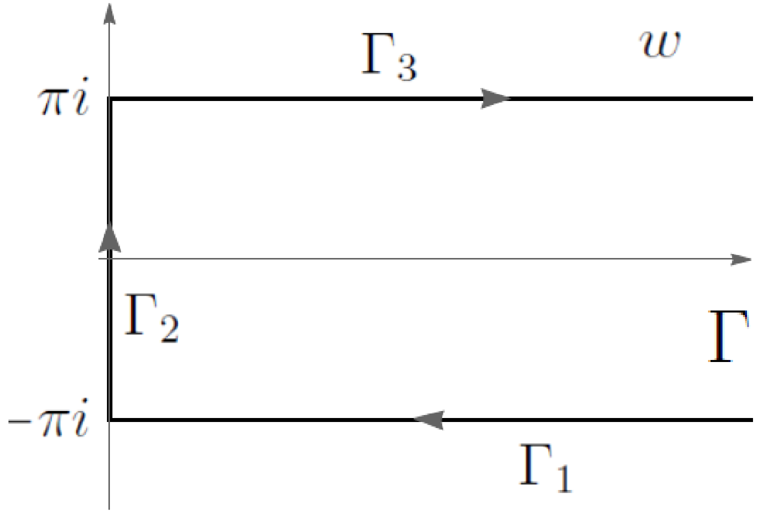
Good everywhere!

$$[K_\Lambda \varphi](x) = e^{i\frac{T(x)}{\mu}}\left[\mathcal{E}\left(\frac{\Phi(x)}{\mu}\right)\frac{\varphi(\alpha_+(x))}{\sqrt{J_+(x)}} + \mathcal{E}\left(-\frac{\Phi(x)}{\mu}\right)\frac{\varphi(\alpha_-(x))}{\sqrt{J_-(x)}}\right]$$

Asymptotics via Airy functions. Bessel functions of arbitrary index

- Schläfli formula

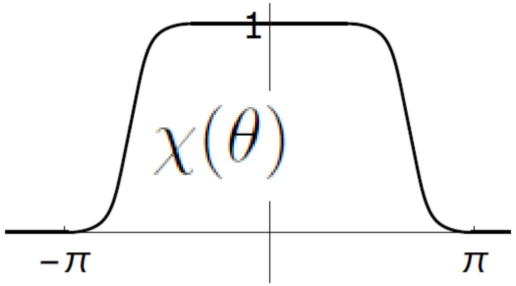
$$\mathbf{J}_\nu(z) = \frac{1}{2\pi i}\int_\Gamma e^{z\sinh w - \nu w}\,dw$$



$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

- Problem: asymptotics of $\mathbf{J}_\nu(z)$ as $\sqrt{z^2+\nu^2} \to \infty$. Then

$$\mathbf{J}_\nu(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(z\sin\theta - \nu\theta)}\chi(\theta)\,d\theta + O((z^2+\nu^2)^{-\infty})$$

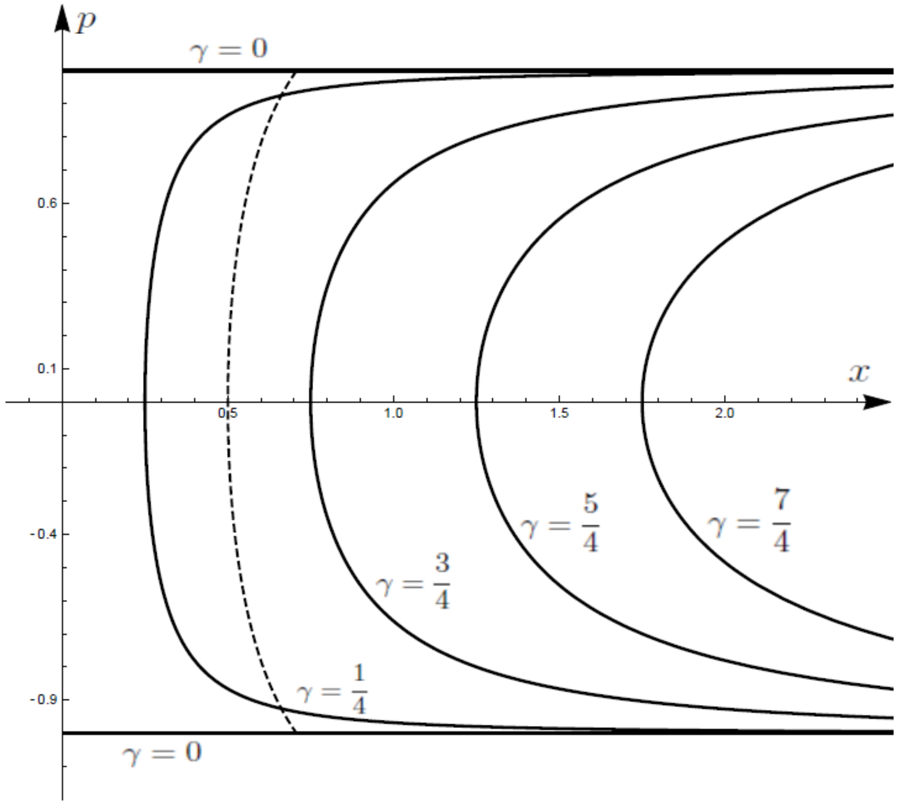


- This is oscillatory integral with small parameter $h = (z^2+\nu^2)^{-1/2}$ and variables $x = zh$, $\gamma = \nu h,\ x > 0,\ \gamma \geq 0,\ x^2 + \gamma^2 = 1$. Canonical operator

Lagrangian manifold for the oscillatory integral

- Lagrangian manifold L_γ is given by the parametric equations

$$x = \sqrt{\alpha^2 + \gamma^2}, \qquad p = \frac{\alpha}{\sqrt{\alpha^2 + \gamma^2}}, \qquad \alpha \in \mathbb{R}$$



($x = |\alpha|,\ p = \operatorname{sign}\alpha$ for $\gamma = 0$)

dotted line: $x^2 + \gamma^2 = \frac{1}{2}$

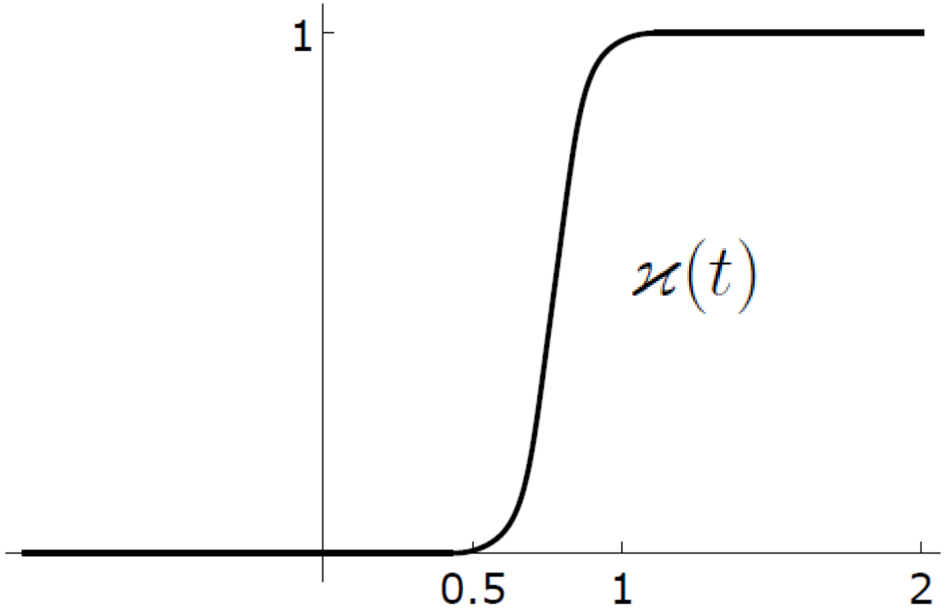
Nonparametric equation: $p^2 + \dfrac{\gamma^2}{x^2} = 1$

Canonical operator on L_γ gives the desired asymptotics. Focal points ➔ Airy functions

Bessel function via the canonical operator

- **Theorem.** One has the asymptotics

$$\varkappa(x^2+\gamma^2)\mathbf{J}_{\gamma/h}\left(\frac{x}{h}\right) = \frac{h^{1/2}e^{-\frac{i\pi}{4}}}{(2\pi)^{1/2}}[K^h_{L_\gamma,d\sigma}\varkappa(\alpha^2+2\gamma^2)](x) + O_{L_\gamma}(h^{3/2})$$



$$d\sigma = \frac{d\alpha}{\sqrt{\alpha^2+\gamma^2}}$$

Denote the amplitude by

$$A(\alpha) = \varkappa(\alpha^2+2\gamma^2)$$

Global expression via the Airy function

- Unique simple singular point ➔ canonical operator expressed via Airy function

$$[K^h_{L_\gamma,d\sigma}A](x) = 2\sqrt{\pi}e^{i\frac{\pi}{4}}\left(\frac{3\tau(\alpha(x))}{2h\alpha^3(x)}\right)^{1/6}\mathrm{Ai}\left(-\left(\frac{3\tau(\alpha(x))}{2h}\right)^{2/3}\right)\varkappa(x^2+\gamma^2)$$

$$\tau(\alpha) = \alpha - \gamma\arctan\frac{\alpha}{\gamma} \qquad \alpha(x) = \sqrt{x^2-\gamma^2}$$

- Holds not only for $x > \gamma$ but also for all $x > 0$ by continuation to $x < \gamma$:

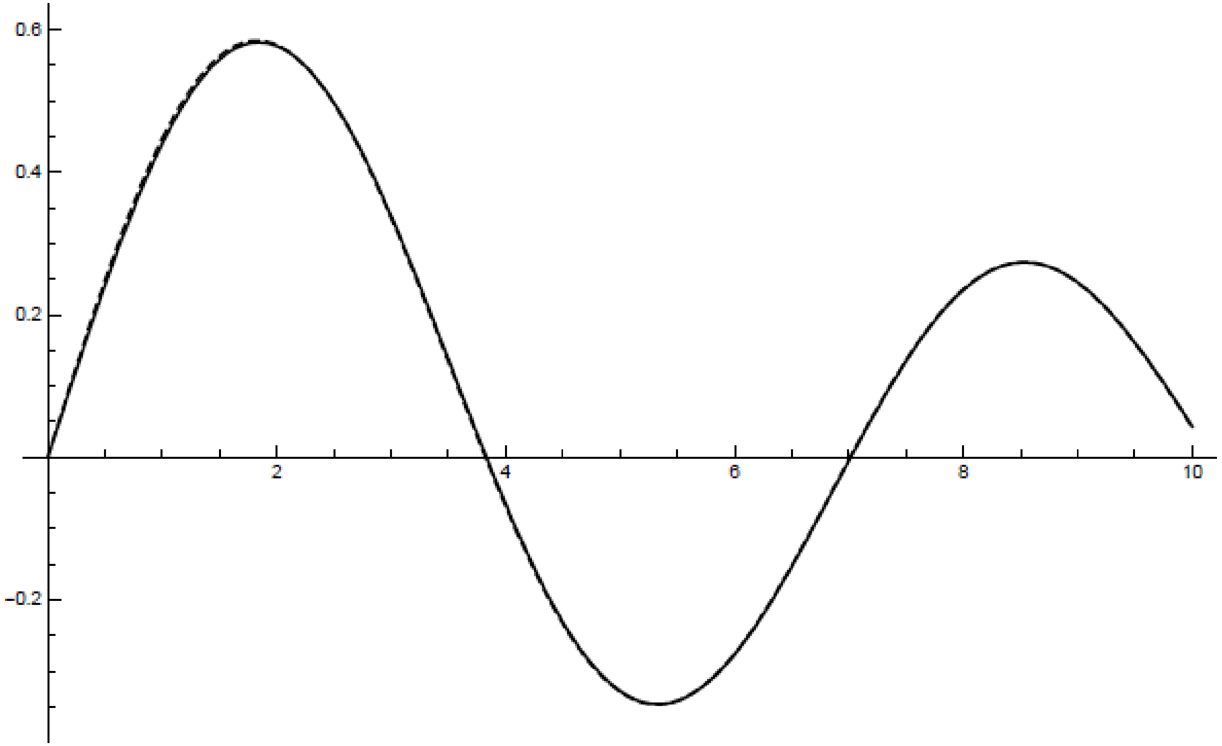
$$\tau^{2/3}(\alpha(x)) := -\left(\frac{\gamma}{2}\ln\frac{\gamma+\sqrt{\gamma^2-x^2}}{\gamma-\sqrt{\gamma^2-x^2}} - \sqrt{\gamma^2-x^2}\right)^{2/3}$$

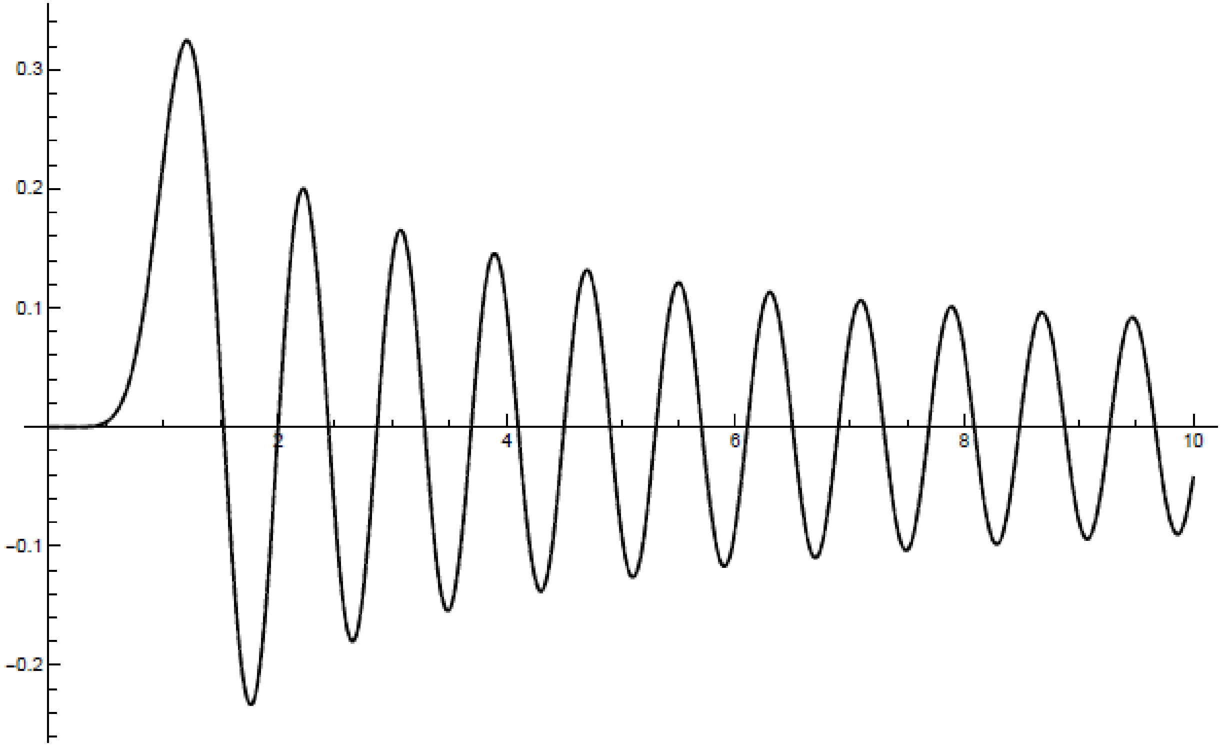
$$\left(\frac{\tau(\alpha(x))}{\alpha^3(x)}\right)^{1/6} := \left(\frac{\frac{\gamma}{2}\ln\frac{\gamma+\sqrt{\gamma^2-x^2}}{\gamma-\sqrt{\gamma^2-x^2}} - \sqrt{\gamma^2-x^2}}{(\gamma^2-x^2)^{3/2}}\right)^{1/6}$$

$x < \gamma$

Numerical comparison

- Computations for $\nu = 1$ (left) and $\nu = 8$ (right)

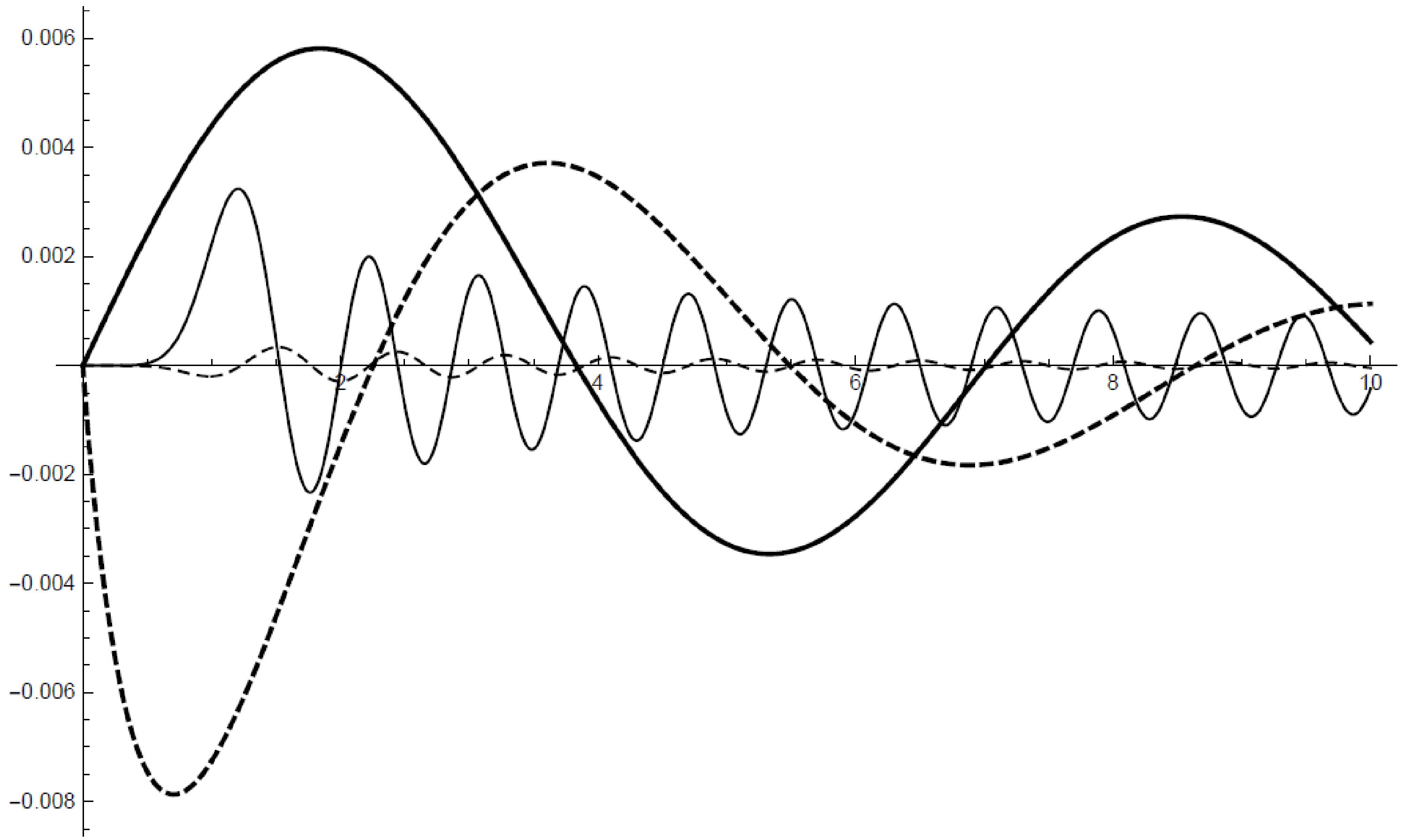




- (solid line: $\mathbf{J}_\nu(\nu x)$; dotted line, canonical operator (Airy) approximation)
- Difference almost invisible; hence the next graph

Approximation error under microscope

- Error magnified by a factor of 100



- Solid lines: $\mathbf{J}_\nu(\nu x)$ ($\nu = 1$, 8). Dotted lines: approximation error x 100

Asymptotics in a Neighborhood of a Standard Cusp Point

Lagrangian Manifold

- Standard Lagrangian singularity A_3 : Lagrangian manifold $L_\pm \subset \mathbb{R}^4_{(x,p)}$

$$x = X_\pm(\alpha) = (\mp 4\alpha_1^3 - 2\alpha_1\alpha_2, \alpha_2), \qquad p = P_\pm(\alpha) = (\alpha_1, \alpha_1^2) \qquad (\alpha_1, \alpha_2) = (p_1, x_2) \in \mathbb{R}^2$$

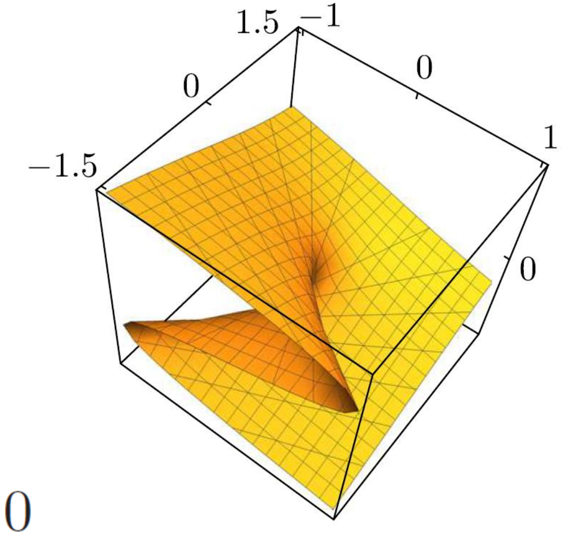
- Can be given by the generating function $\mathcal{S}_\pm(p_1, x_2) = \pm p_1^4 + x_2 p_1^2$

$$x_1 = -\frac{\partial \mathcal{S}_\pm}{\partial p_1}(p_1, x_2) = \mp 4p_1^3 - 2x_2 p_1,$$

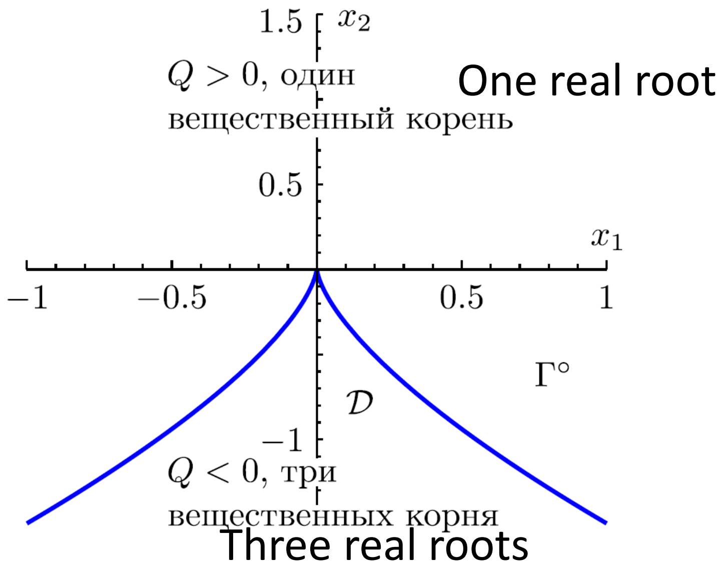
$$p_2 = \frac{\partial \mathcal{S}_\pm}{\partial x_2}(p_1, x_2) = p_1^2$$

- Equation of focal points

$$J_\pm = \det \frac{\partial X_\pm}{\partial \alpha}(\alpha_1, \alpha_2) \equiv \mp 12\alpha_1^2 - 2\alpha_2 = 0$$



$$Q(x) = \left(\frac{x_2}{6}\right)^3 + \left(\frac{x_1}{8}\right)^2$$



caustic

Canonical operator

- General formula (measure $d\mu = d\alpha_1 \wedge d\alpha_2$)

$$[K^h_{L_\pm} A](x) = \frac{e^{i\pi/4}}{(2\pi h)^{1/2}} \int_{-\infty}^{\infty} e^{(i/h)(\pm p_1^4 + x_2 p_1^2 + x_1 p_1)} A(p_1, x_2)\, dp_1$$

$$[K^h_{L_-} A](x) = i\overline{[K^h_{L_+} B](-x)},$$

$$B(p_1, x_2) = \overline{A(p_1, -x_2)}$$

- For the unit amplitude we obtain the Pearcey function

$$[K^h_{L_+} 1](x) = \frac{e^{i\pi/4}}{(2\pi)^{1/2} h^{1/4}} \mathrm{P}\left(\frac{x_2}{h^{1/2}}, \frac{x_1}{h^{3/4}}\right),$$

$$\mathrm{P}(y,z) = \int_{-\infty}^{\infty} e^{i(t^4 + yt^2 + zt)}\, dt$$

- For a general amplitude $A(p_1, x_2)$: the first equation of the manifold L_+ has the form

$$\Pi(p_1, x) = p_1^3 + \frac{1}{2} x_2 p_1 + \frac{1}{4} x_1 = 0,$$

hence

$$A(p_1, x_2) = \Pi(p_1, x) g(p_1, x) + \sum_{j=0}^{2} \rho_j(x) p_1^j$$

Malgrange's preparation thm

$$[K^h_{L_+} A](x) = \frac{e^{i\pi/4}}{(2\pi)^{1/2} h^{1/4}} \Big[\rho_0(x) \mathrm{P}\left(\frac{x_2}{h^{1/2}}, \frac{x_1}{h^{3/4}}\right) - i h^{1/4} \rho_1(x) \frac{\partial \mathrm{P}}{\partial z}\left(\frac{x_2}{h^{1/2}}, \frac{x_1}{h^{3/4}}\right) - i h^{1/2} \rho_2(x) \frac{\partial \mathrm{P}}{\partial y}\left(\frac{x_2}{h^{1/2}}, \frac{x_1}{h^{3/4}}\right)\Big] + O_{L_+}(h)$$

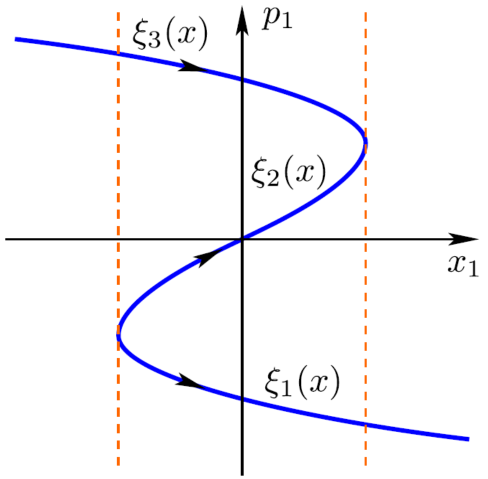
How to compute the functions $\rho_j(x)$ explicitly?

Roots of the cubic polynomial $\Pi(p_1, x)$

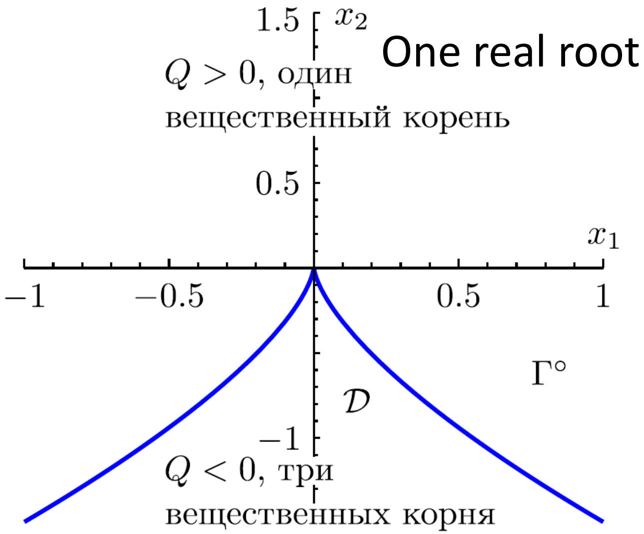
- In $\mathscr{D}$ there are 3 roots $\xi_1(x) < \xi_2(x) < \xi_3(x)$

$$\xi_k(x) = \left(-\frac{2}{3}x_2\right)^{1/2} \cos\varphi_k,$$

$$\varphi_{1,2} = \frac{2\pi}{3} \pm \varphi_3, \quad \varphi_3 = \frac{1}{3}\arccos\frac{3^{3/2}x_1}{(-2x_2)^{3/2}}$$



$x_2 = \text{const} < 0$



One real root

Three real roots

- In $\mathbb{R}^2 \setminus \overline{\mathscr{D}}$ there is one real root $\xi_0(x)$ and two complex conjugate roots $\eta(x)$, $\overline{\eta}(x)$

$$\xi_0(x) = \left(Q^{1/2}(x) - \frac{x_1}{8}\right)^{1/3} - \left(Q^{1/2}(x) + \frac{x_1}{8}\right)^{1/3}$$

$$\eta(x) = -\frac{1}{2}\xi_0(x) + \frac{i\sqrt{3}}{2}\left[\left(Q^{1/2}(x) - \frac{x_1}{8}\right)^{1/3} + \left(Q^{1/2}(x) + \frac{x_1}{8}\right)^{1/3}\right]$$

(Cardano's formulas)

In the domain $\mathscr{D}$

- Since $A(p_1, x_2) = \Pi(p_1, x)g(p_1, x) + \sum_{j=0}^{2} \rho_j(x) p_1^j$, we have $A(\xi_k(x), x_2) = \sum_{j=0}^{2} \rho_j(x) \xi_k^j(x)$ for $\xi_k(x) \in \mathbb{R}$.
- Hence in the domain $\mathscr{D}$ (all three roots are real) we have $R(x, \xi) = \sum_{j=0}^{2} \rho_j(x) \xi^j$

  This is an interpolation polynomial for $A(\xi, x_2)$ on the points $\xi_k(x)$. Hence

$$\rho_0(x) = R(x,0) = \sum_{k=1}^{3} \frac{A(\xi_k(x), x_2)\xi_l(x)\xi_m(x)}{W_k(\xi_k(x), x)},$$

$$\rho_1(x) = \frac{\partial R}{\partial \xi}(x,0) = -\sum_{k=1}^{3} \frac{A(\xi_k(x), x_2)(\xi_l(x) + \xi_m(x))}{W_k(\xi_k(x), x)},$$

$$\rho_2(x) = \frac{1}{2}\frac{\partial^2 R}{\partial \xi^2}(x,0) = \sum_{k=1}^{3} \frac{A(\xi_k(x), x_2)}{W_k(\xi_k(x), x)}, \qquad W_k(\xi, x) = \frac{\prod_{l=1}^{3}(\xi - \xi_l(x))}{\xi - \xi_k(x)}$$

The triple (k, l, m) is a cyclic permutation of the triple $(1, 2, 3)$

In the domain $\mathbb{R}^2 \setminus \overline{\mathscr{D}}$

- Task: continue the functions $\rho_j(x)$ into $\mathbb{R}^2 \setminus \overline{\mathscr{D}}$ as smooth functions with the preservation of the identity

$$A(p_1, x_2) = \Pi(p_1, x) g(p_1, x) + \sum_{j=0}^{2} \rho_j(x) p_1^j$$

- Here the analytic continuation is (a) nonconstructive; (b) impossible if the amplitude is nonanalytic; (c) is not required, because the following theorem holds:

Theorem: It suffices to continue the functions $\rho_j(x)$, *j=1,2,* into $\mathbb{R}^2 \setminus \overline{\mathscr{D}}$ *in an arbitrary smooth way* and then set

$$\rho_0(x) = A(\xi_0(x), x_2) - \rho_1(x)\xi_0(x) - \rho_2(x)\xi_0^2(x)$$

We know (by Malgrange's theorem), that there exists at least one smooth continuation.

How to construct it explicitly? The continuation through a smooth curve is trivial.

However, the caustic has a cusp!

Constructive continuation of the functions $\rho_j(x)$, $j=1,2$

- Consider the projection

$$\pi : L_+ \to \mathbb{R}^2 \ni (x_1, x_2) \qquad x = X(\alpha) = (-4\alpha_1^3 - 2\alpha_1\alpha_2, \alpha_2)$$

- Use the projection to lift the function $\rho(x)$ = $\rho_j(x)$, $j=1,2$, from the domain $\mathscr{D}$ to the domain

$$\pi^{-1}(\mathscr{D}) = \{(\alpha_1, \alpha_2) : \alpha_2 < -6\alpha_1^2\} \subset L_+$$

- The lift $R(\alpha) = \rho(\pi(\alpha))$ admits a smooth continuation $\widetilde{R}(\alpha)$ to the entire L_+
- The desired continuation of $\rho(x)$ has the form

$$\widetilde{\rho}(x) = \begin{cases} \rho(x), & x \in \mathscr{D}, \\ \widetilde{R}(\xi_0(x), x_2), & x \in \mathbb{R}^2 \setminus \mathscr{D}. \end{cases}$$

- A constructive extension (finite smoothness, but we do not need more) is given by

$$\widetilde{R}(\alpha_1, \alpha_2) = \sum_{k=0}^{n-1} C_n^{k+1} (-1)^k R(\alpha_1, -6\alpha_1^2 - k(\alpha_2 + 6\alpha_1^2)), \qquad \alpha_2 \geqslant -6\alpha_1^2,$$

$$C_n^k := \binom{n}{k}$$

Asymptotics in a Neighborhood of a Generic A_3 Cusp

Reduction to Normal Form

- Lagrangian equivalence: a symplectic diffeomorphism $g\colon T^*\mathbb{R}^2 \to T^*\mathbb{R}^2$ commuting with the projection $T^*\mathbb{R}^2_x \to \mathbb{R}^2_x$, i.e., taking fibers to fibers.
- Any Lagrangian equivalence $g\colon T^*\mathbb{R}^2 \to T^*\mathbb{R}^2,\ (y,q) \mapsto (x,p)$, has the form

$$x = f(y), \qquad p = {}^t\left(\frac{\partial f}{\partial y}(y)\right)^{-1} q + \frac{\partial \Phi}{\partial x}(f(y))$$

- If a Lagrangian equivalence takes $L_+ \subset T^*\mathbb{R}^2_y$ to $\Lambda \subset T^*\mathbb{R}^2_x$, then

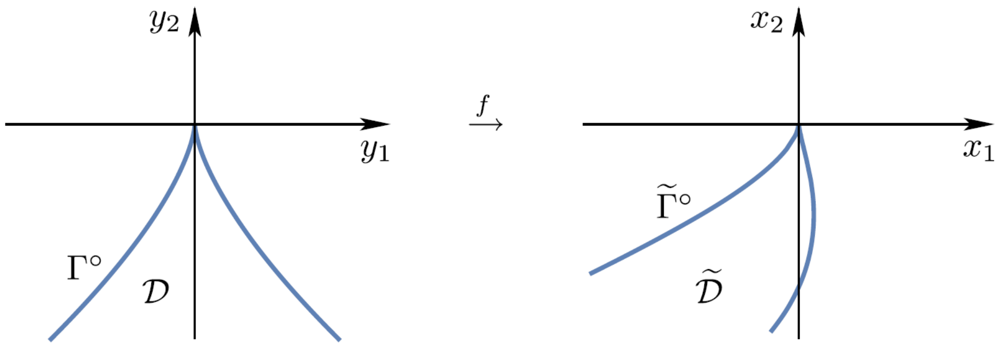
$$[K^h_\Lambda A](x) = e^{(i/h)\Phi(x)}[K^h_{L_+} A](y(x))\sqrt{\det \frac{\partial y}{\partial x}(x)} + O_\Lambda(h),$$

where $y = y(x)$ is the inverse function of $x = f(y)$

There exists a Lagrangian equivalence of this kind by the theorem on the reduction to normal form due to [Arnold, Varchenko, Gussein-zade]

Computation of the transformation producing the reduction

L_+



Λ

- Action functions $\tau = \int q\,dy, \qquad S = \int p\,dx$
- In the nonsingular charts corresponding to each other, we have $S(x) = \tau(f^{-1}(x)) + \Phi(x).$
- In the domains $\mathscr{D}$ and $\widetilde{\mathscr{D}}$ there exist three branches of the action function

$$\tau_1(y) < \tau_2(y) < \tau_3(y), \quad y \in \mathscr{D}; \qquad S_1(x) < S_2(x) < S_3(x), \quad x \in \widetilde{\mathscr{D}}.$$

$$S_j(x) = \tau_j(f^{-1}(x)) + \Phi(x), \qquad j = 1, 2, 3.$$

How to find f and Φ from these data

- Set

$$\begin{aligned}\delta(y) &= (\delta_2(y), \delta_3(y)), \quad \text{где} \quad \delta_j(y) = \tau_j(y) - \tau_1(y), \\ \widetilde{\delta}(x) &= (\widetilde{\delta}_2(x), \widetilde{\delta}_3(x)), \quad \text{где} \quad \widetilde{\delta}_j(x) = S_j(x) - S_1(x),\end{aligned} \qquad j = 2, 3.$$

- Then

$$\delta(y) = \widetilde{\delta}(f(y)).$$

- Hence in $\widetilde{\mathscr{D}}$ we can set

$$f^{-1} = \delta^{-1} \circ \widetilde{\delta}$$

Φ is found from any of the equations

$$S_j(x) = \tau_j(f^{-1}(x)) + \Phi(x)$$

- Outside $\widetilde{\mathscr{D}}$ we extend f to a diffeomorphism $\widehat{f}$ of full neighborhoods and set

$$\widehat{\Phi}(x) = \begin{cases} \Phi(x) & \text{in } \mathcal{D} \\ S(x) - \tau(\widehat{f}^{-1}(x)) & \text{outside } \mathcal{D} \end{cases}$$

Example of extension of the canonical operator: *Punctured (pseudohomogeneous) Lagrangian manifolds*

Model problem

Potential motion of a liquid in the uniform field of gravity in a horizontally infinite basin of finite depth

Parameters:

- Basin depth d
- Characteristic basin horizontal size L
- Initial perturbation horizontal size l
- Initial perturbation vertical amplitude A

Assumptions:

$$\mu = d/L, \quad l/L, \quad A/d \ll 1$$

$$d \sim l$$ (strong dispersion case)

Equation for the free surface elevation:

$$\mu^2 \frac{\partial^2 \eta}{\partial t^2} + \mathcal{H}\eta = 0, \qquad \eta|_{t=0} = V\left(\frac{x}{\mu}\right), \qquad \eta_t|_{t=0} = 0$$

$$\mathcal{H} = \mathcal{H}\left(\overset{2}{x}, -i\mu\frac{\overset{1}{\partial}}{\partial x}, \mu\right), \quad \mathcal{H}(x,p,\mu) = H(x,p) - \frac{ih}{2}\sum_{j=1}^{2} H_{x_j p_j}(x,p) + O(\mu^2),$$

$$H(x,p) = |p|\tanh(D(x)|p|)$$

Dobrokhotov, Zhevandrov, Funct. Anal. Appl. 1985

Cauchy problem

Evolution equation with small parameter $\mu > 0$

$$\mathcal{H}\left(\overset{2}{t}, \overset{2}{x}, -i\mu\overset{1}{\frac{\partial}{\partial t}}, -i\mu\overset{1}{\frac{\partial}{\partial x}}, \mu\right)\vec{\Psi}(x,t,\mu) = 0, \quad t \in [0,T], \quad x \in \mathbb{R}^n,$$

where the symbol $\mathcal{H}(t, x, p_0, p, \mu)$ is an $r \times r$ matrix and a polynomial of degree l in p_0

Localized initial conditions

$$\frac{\partial^j \vec{\Psi}}{\partial t^j}(0, x, \mu) = \vec{V}_j\left(\frac{x' - f(x'')}{\mu}, x''\right), \quad j = 0, 1, \ldots, l-1, \qquad \vec{V}_j(y', x'') \text{ decay as } y' \to \infty$$

near the submanifold $Y = \{x' = f(x'')\} \subset \mathbb{R}^n, \quad x' = (x_1, \ldots, x_k), \quad x'' = (x_{k+1}, \ldots, x_n)$

Task: construct an asymptotic solution as $\mu \to 0$

(under certain conditions on the symbol $\mathcal{H}$)

Representation of localized initial data

- Simplest case: $Y = \{x_0\}$ is a point

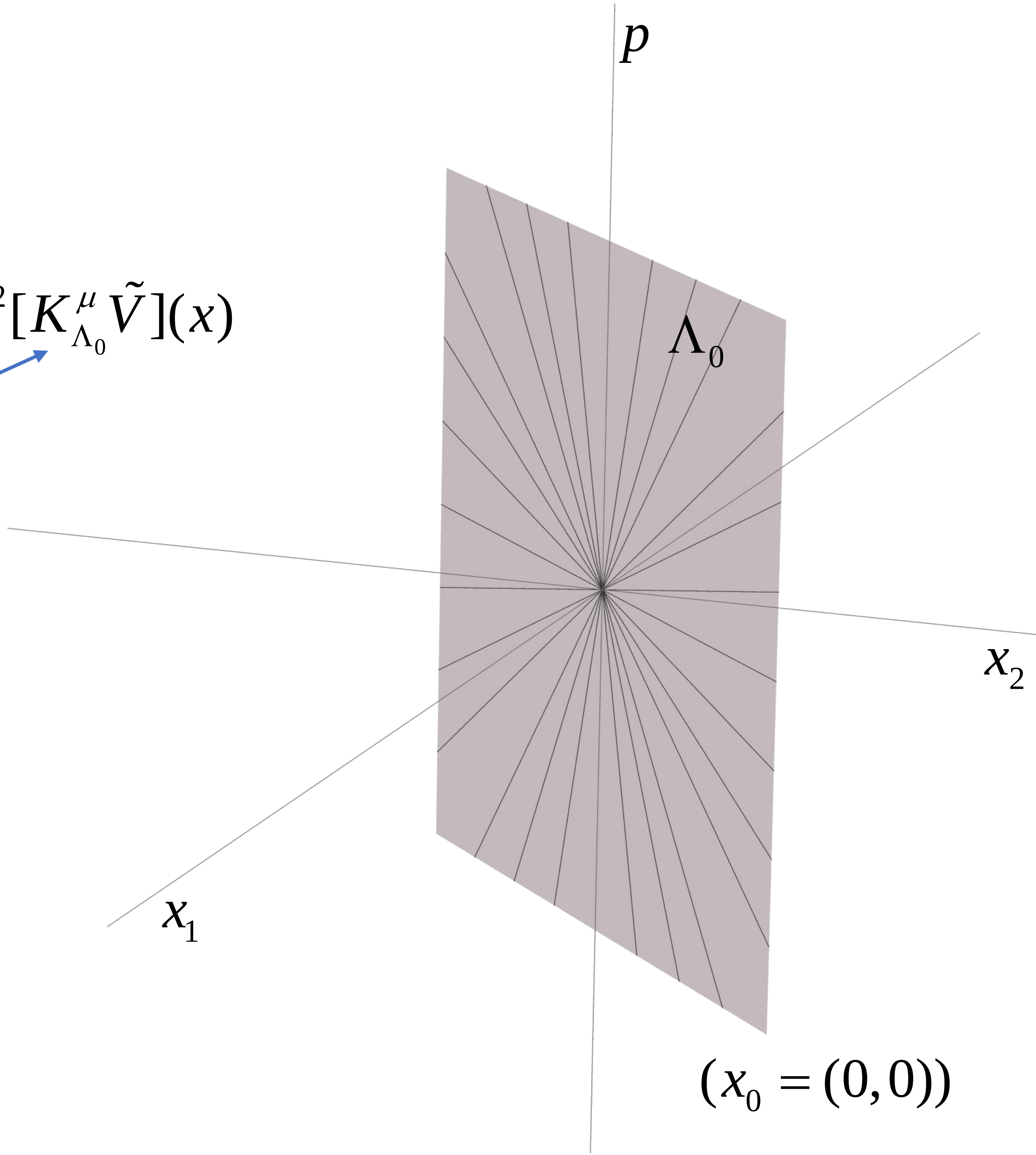
$$V\left(\frac{x-x_0}{\mu}\right) = \left(\frac{i}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{\frac{i}{\mu}\langle p,x\rangle} \tilde{V}(p)\,dp = \mu^{n/2}[K^{\mu}_{\Lambda_0}\tilde{V}](x)$$

Maslov's canonical operator
on the Lagrangian manifold

$$\Lambda_0 = \{(x,p) \in \mathbb{R}^{2n} : x = x_0\}$$

with the measure $d\sigma = dp_1 \wedge \ldots \wedge dp_n$

$\tilde{V}$ is the Fourier transform of V



Representation of localized initial data (continued)

- General case: Y is a submanifold of codimension $\operatorname{codim} Y = k$

$$V\left(\frac{x' - f(x'')}{\mu}, x''\right) = \left(\frac{i}{2\pi}\right)^{k/2} \int_{\mathbb{R}^k} e^{\frac{i}{\mu}\langle p', x' - f(x'')\rangle} \tilde{V}(p', x'')\, dp' = \mu^{k/2}[K^{\mu}_{\Lambda_0} \tilde{V}](x)$$

partial Fourier transform

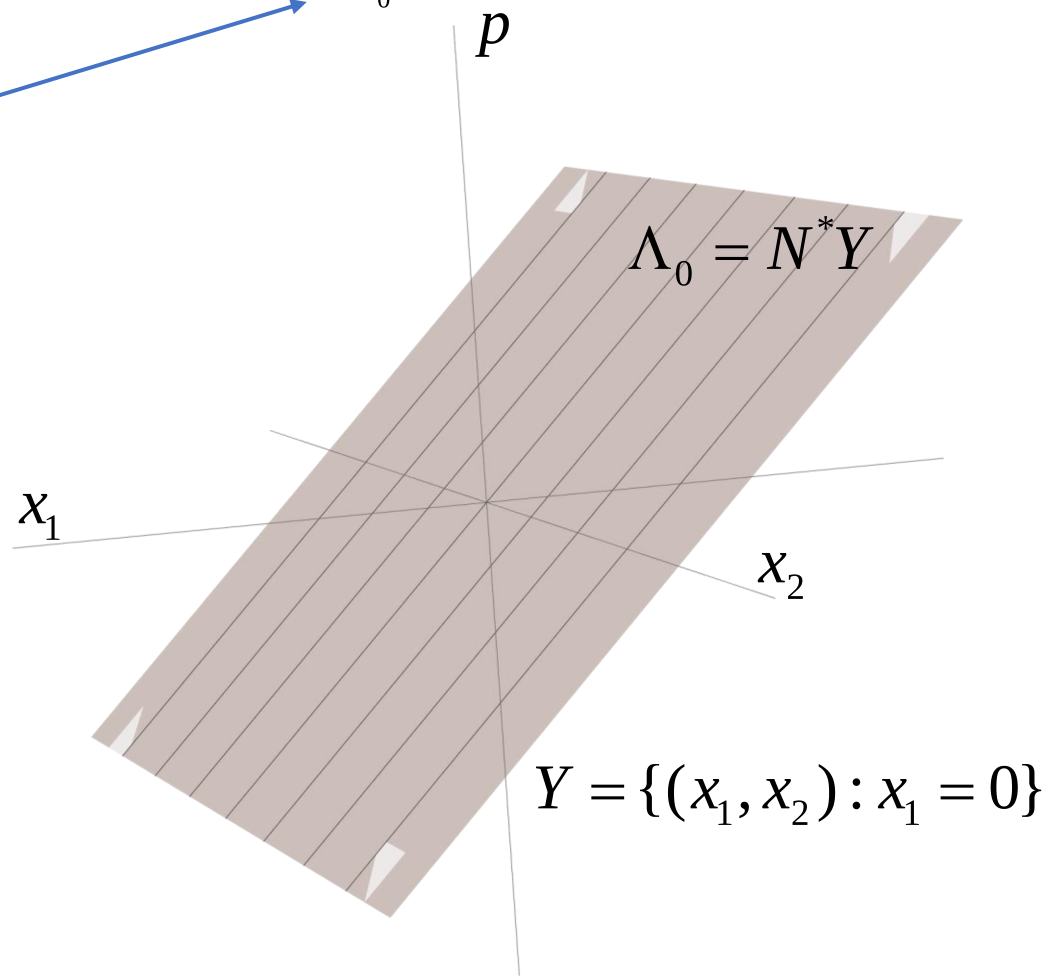
Maslov's canonical operator on the Lagrangian manifold

$$\Lambda_0 = \{(x,p) \in \mathbb{R}^{2n} : x \in Y, \langle p, \xi\rangle = 0\ \forall \xi \in T_x Y\}$$

with the measure

$$d\sigma = dp_1 \wedge \ldots \wedge dp_k \wedge dx_{k+1} \wedge \ldots \wedge dx_n$$

$\Lambda_0 = N^* Y$ is the conormal bundle of Y



General scheme for constructing semiclassical asymptotics in the Cauchy problem

$$\mathcal{H}\left(\overset{2}{t},\overset{2}{x},-i\mu\overset{1}{\frac{\partial}{\partial t}},-i\mu\overset{1}{\frac{\partial}{\partial x}},\mu\right)\vec{\Psi}(x,t,\mu)=0,$$

$\mathcal{H}(t,x,p_0,p,\mu)$ is an $r\times r$ matrix and a polynomial of degree l in p_0

$$\frac{\partial^j\vec{\Psi}}{\partial t^j}(0,x,\mu)=[K^{\mu}_{\Lambda_0}\vec{A}_j](x),\quad j=0,1,\ldots,l-1,$$

Λ_0 is a Lagrangian manifold in $\mathbb{R}^{2n}_{(x,p)}$

$K^{\mu}_{\Lambda_0}$ is Maslov's canonical operator

Solution: *effective Hamiltonians* $\lambda_s(t,x,p)$

(roots of the equation $\det\mathcal{H}(t,x,\lambda,p,0)=0$)

$$\Lambda^s_t=g^t_{\lambda_s}(\Lambda_0)$$

shifts along the trajectories of Hamiltonian systems

$$\vec{\Psi}(x,t,\mu)=\sum_s[K^{\mu}_{\Lambda^s_t}\vec{A}_s(t)](x)$$

Difficulties in specific problems. Examples of effective Hamiltonians

The general scheme works "as is" if the roots are real and smooth (of constant multiplicity) at least in a neighborhood of the Lagrangian manifolds etc. etc. etc.

The roots of the equation $\det \mathcal{H}(t,x,\lambda,p,0)=0$ are branches of an algebraic function of the coefficients $\Longrightarrow$ nonsmoothness (at the branching points)

One often has nonsmoothness at $p=0$

Examples:

- Petrovsky hyperbolic systems: $\lambda_s(t,x,p)$ are homogeneous functions of p
- Wave equation: just the same, $\lambda_\pm(x,p)=\pm c(x)|p|$
- Water wave equation with dispersion: $\lambda_\pm(x,p)=\pm\sqrt{|p|\tanh(|p|D(x))}$
- Wave equation on a 2D lattice: $\lambda_\pm(x,p)=\pm c(x)\sqrt{\sin^2 p_1+\sin^2 p_2}$

A class of effective Hamiltonians: – *pseudo-homogeneous Hamiltonians*

That is, Hamiltonians admitting **expansion in homogeneous functions**:

$$\lambda(x,p)=\lambda^{(1)}(x,p)+\lambda^{(2)}(x,p)+\ldots+\lambda^{(N)}(x,p)+O(|p|^{N+1}),\quad p\to 0,\quad \forall N$$

$$\lambda^{(j)}(x,\tau p)=\tau^{j}\lambda(x,p),\quad \tau>0$$

smooth for $p\neq 0$

or, equivalently,

$\lambda(x,r\omega)=:\lambda(x,r,\omega)$ is a smooth functions of $x\in\mathbb{R}^n,\ r\in[0,\infty),\ \omega\in\mathbb{S}^{n-1}$

$$\lambda(x,0,\omega)=0$$

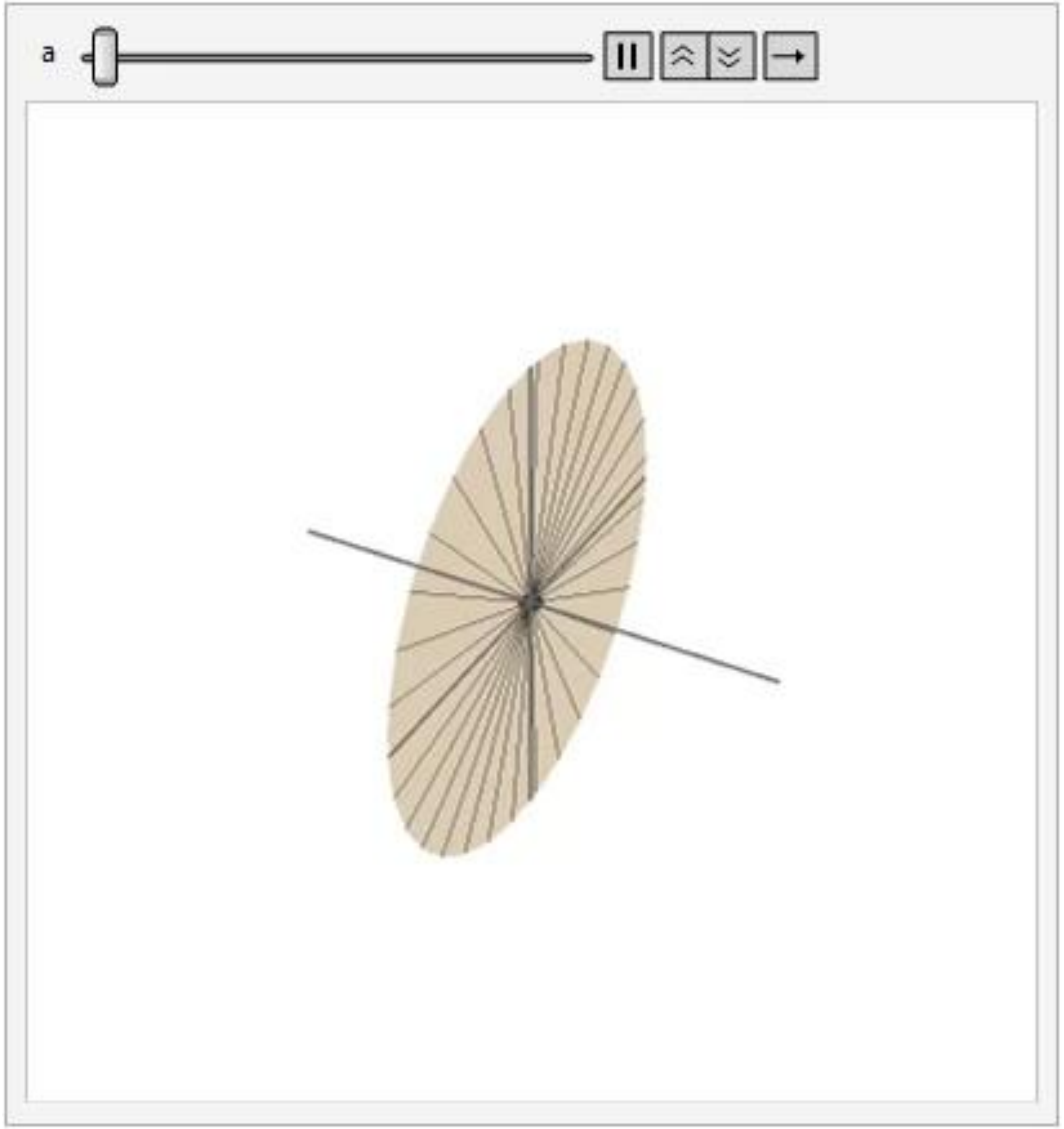
The substitution

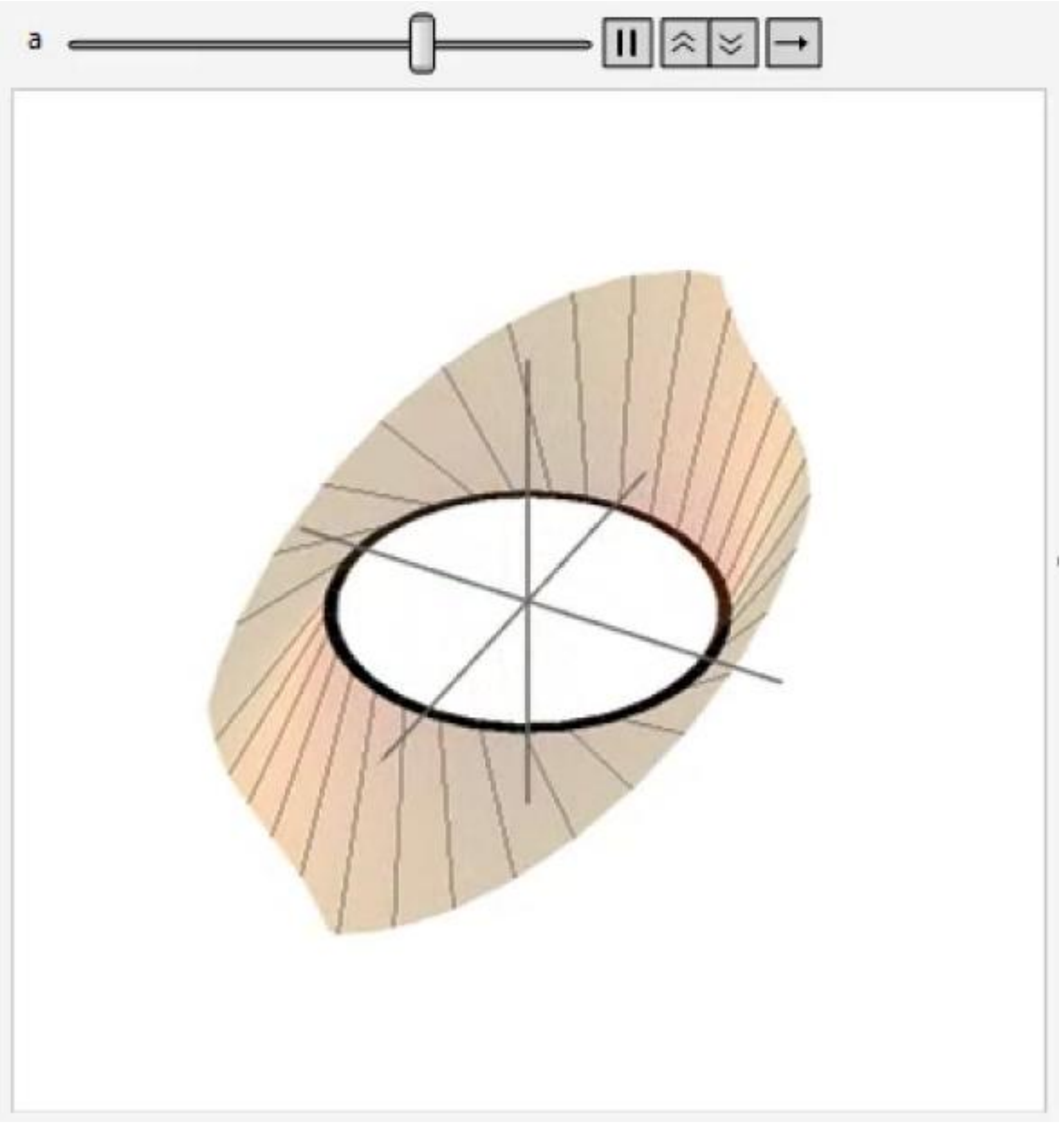
$$p=r\omega,\quad r\in[0,\infty),\ \omega\in\mathbb{S}^{n-1}$$

is very convenient

What does the Hamiltonian flow do with the Lagrangian manifold?

"Punctured" manifold





Punctured Lagrangian manifolds

Λ is a smooth n-dimensional manifold with boundary $\partial\Lambda$

$$\Lambda \ni \alpha = (\text{near the boundary}) = (\phi, \rho), \quad \rho \geq 0, \quad \partial\Lambda = \{\rho = 0\}$$

Definition

A punctured Lagrangian manifold := a smooth mapping $\jmath : \Lambda \to \mathbb{R}^{2n}$,

$$\alpha \mapsto (X(\alpha), P(\alpha))$$

- One has $P_\alpha^T X_\alpha - X_\alpha^T P_\alpha = 0 \quad (\jmath^*(dp \wedge dx) = 0)$ (Lagrangian property)
- For $\alpha \notin \partial\Lambda$: $P(\alpha) \neq 0, \quad \operatorname{rank}\begin{pmatrix} X_\alpha \\ P_\alpha \end{pmatrix} = n$ immersion
- For $\alpha = (\phi, 0) \in \partial\Lambda$: $\operatorname{rank}\begin{pmatrix} X_\phi(\phi,0) & 0 \\ P_{\phi\rho}(\phi,0) & P_\rho(\phi,0) \end{pmatrix} = n$

The last condition $\Leftrightarrow \mathbf{\Lambda} := \partial\Lambda \times \mathbb{R}_+ \ni (\phi, \tau) \mapsto (X(\phi,0), \lim_{\varepsilon\to 0} \varepsilon^{-1} P(\phi, \varepsilon\tau))$

is a homogeneous Lagrangian manifold

Invariance with respect to the Hamiltonian flow

Theorem

The class of punctured Lagrangian manifolds is invariant under shifts along Hamiltonian vector fields corresponding to pseudo-homogeneous Hamiltonians.

Statement of the problem

On punctured Lagrangian manifolds, define a canonical operator

$$K_\Lambda^\mu : C_0^\infty(\Lambda) \longrightarrow H_\mu^\infty(\mathbb{R}^n)$$

$$H_\mu^\infty(\mathbb{R}^n) = \bigcap_{s=0}^{\infty} H_\mu^s(\mathbb{R}^n), \qquad \|u\|_s = \sup_{\mu\in(0,1)} \left\|(1-\mu^2\Delta)^{s/2}u\right\|_{L^2(\mathbb{R}^n)} .$$

On functions in $C_0^\infty(\Lambda^\circ) \subset C_0^\infty(\Lambda)$, where $\Lambda^\circ = \Lambda \setminus \partial\Lambda$, it must

coincide with the "standard" Maslov canonical operator.

Microlocal definition of the standard canonical operator

following [Izv17]

Nondegenerate phase function: $\Phi(x,\xi)$ is defined on an open set $V \subset \mathbb{R}^n_x \times \mathbb{R}^m_\xi$

The differentials $d(\Phi_{\xi_1}), \dots, d(\Phi_{\xi_m})$ are linearly independent on

$$C_\Phi = \{(x,\xi) \in V : \Phi_\xi(x,\xi) = 0\}$$

$\Lambda_\Phi = \jmath_\Phi(C_\Phi)$ is a Lagrangian mfld

$$\jmath_\Phi : (x,\xi) \mapsto (x, \Phi_x(x,\xi))$$

$\Lambda_\Phi \subset \Lambda$ $\quad$ Φ-chart on Λ

Pre-canonical operator in the Φ-chart:

$$[K^\mu_\Phi A](x,\mu) = \frac{e^{\frac{\pi i m}{4}}}{(2\pi\mu)^{m/2}} \int_V e^{\frac{i}{\mu}\Phi(x,\xi)} a(x,\xi)\, d\xi_1 \cdots d\xi_m$$

cf. Hörmander's Fourier integral distributions

$$A \in C_0^\infty(\Lambda) \qquad a|_{C_\Phi} = \jmath_\Phi^*(A)\sqrt{F_{\Phi,d\sigma}},$$

$$F_{\Phi,d\sigma} = \frac{\jmath_\Phi^*(d\sigma) \wedge d(-\Phi_{\xi_1}) \wedge \dots \wedge d(-\Phi_{\xi_m})}{dx_1 \wedge \dots \wedge dx_n \wedge d\xi_1 \wedge \dots \wedge d\xi_m}$$

$d\sigma$ is a measure (volume form) on Λ

New phase functions

We need a new notion of phase function to describe a punctured Lagrangian manifold near the boundary

$V \subset \mathbb{R}^n_x \times \mathbb{R}^{m-1}_\theta \times \overline{\mathbb{R}}_{+\rho}$ is a (relatively) open set

$\Phi(x,\theta,\rho)$ is a smooth real function on V, $\Phi(x,\theta,0) = 0$

Define

$$\Psi(x,\theta,\rho) = \frac{1}{\rho}\Phi(x,\theta,\rho)$$

The following conditions hold on the set

$$C_\Phi = \{(x,\theta,\rho) \in V : \Psi_\theta(x,\theta,\rho) = 0, \quad \Psi(x,\theta,\rho) + \rho\Psi_\rho(x,\theta,\rho) = 0\}$$

$\rho > 0$	$\rho = 0$
(i) $\Phi_x \neq 0$ (ii) $\operatorname{rank}\begin{pmatrix}\Phi_{\xi x}(x,\xi) & \Phi_{\xi\xi}(x,\xi)\end{pmatrix} = m$ $\xi = (\theta,\rho)$	$\operatorname{rank}\begin{pmatrix}\Psi_{\theta x}(x,\theta,0) & \Psi_{\theta\theta}(x,\theta,0) \\ \Psi_x(x,\theta,0) & 0\end{pmatrix} = m$

<u>New phase functions</u> (continued)

<u>Теорема</u>

Punctured Lagrangian manifolds = Lagrangian manifolds locally described by phase functions of this form via the mapping

$$j_{\Phi} : (x, \xi) \mapsto (x, \Phi_x(x, \xi))$$

What about oscillatory integrals?

First, consider the initial function

$$I(x,\mu)=\left(\frac{i}{2\pi\mu}\right)^{k/2}\int_{\mathbb{R}^k} e^{\frac{i}{\mu}\langle p',x'-f(x'')\rangle}\tilde{V}(p',x'')dp',$$

localized on a submanifold Y, $\operatorname{codim} Y = k$.
The corresponding Lagrangian manifold is the conormal bundle $\Lambda_0 = N^*Y$
with natural coordinates $(p',x'')=(p_1,\dots,p_k,x_{k+1},\dots,x_n)$ and with
measure $d\sigma = dp_1\wedge\dots\wedge dp_k\wedge dx_{k+1}\wedge\dots\wedge dx_n$ homogeneous of degree k in p'.

In the polar coordinates:

$$I(x,\mu)=\left(\frac{i}{2\pi\mu}\right)^{k/2}\int_0^\infty d\rho\int d\theta e^{\frac{i\rho}{\mu}\langle\theta,x'-f(x'')\rangle}\rho^{k-1}\tilde{V}(\rho\theta,x''),\ \ \theta\rho=p'$$

The degree of homogeneity of the measure=codimension

What about oscillatory integrals? (continued)

Pre-canonical operator in the Φ-chart

$$\Phi = \rho\Psi$$

$$[K_\Phi^\mu A](x,\mu) = \frac{e^{\frac{\pi i m}{4}}}{(2\pi\mu)^{m/2}} \int_0^\infty d\rho \int d\theta\, e^{\frac{i\rho}{\mu}\Psi(x,\theta,\rho)} a(x,\theta,\rho), \qquad A \in C_0^\infty(\Lambda),$$

$$a|_{C_\Phi} = j_\Phi^*(A)\sqrt{F_{\Phi,d\sigma}}.$$

$$F_{(\Phi,d\sigma)} = \frac{j_\Phi^*(d\sigma)\wedge d(-\Phi_{\theta_1})\wedge\cdots\wedge d(-\Phi_{\theta_{m-1}})\wedge d(-\Phi_\rho)}{dx_1\wedge\cdots\wedge dx_m\wedge d\theta_1\wedge\ldots d\theta_{m-1}\wedge d\rho} \qquad \operatorname{ord} F_{(\Phi,d\sigma)} = k+m-2$$

Hence $a(x,\theta,\rho) = \rho^{\frac{k+m}{2}-1} b(x,\theta,\rho), \quad b \in C_0^\infty.$

$\Psi^\rho(x,\theta) = \Psi(x,\theta,\rho)$ is a nondegenerate phase function on its own

$$[K_\Phi^\mu A](x,\mu) = \frac{e^{\frac{\pi i m}{4}}}{(2\pi\mu)^{m/2}} \int_0^\infty \rho^{\frac{k+m}{2}-1} [K_{\Lambda_{\Psi^\rho}}^{\mu/\rho} B^\rho]\left(x, \frac{\mu}{\rho}\right) d\rho, \quad B^\rho \in C_0^\infty(\Lambda_{\Psi^\rho}),$$

$h = \dfrac{\mu}{\rho}$ new small parameter

What about oscillatory integrals? (still continued)

Thus, what we have is this:

$$[K_\Phi^\mu A](x,\mu) = \frac{e^{\frac{\pi i m}{4}}}{(2\pi\mu)^{m/2}} \int_0^\infty \rho^{\frac{k+m}{2}-1} [K_{\Lambda_{\Psi^\rho}}^{\mu/\rho} B^\rho]\left(x, \frac{\mu}{\rho}\right) d\rho, \quad B^\rho \in C_0^\infty(\Lambda_{\Psi^\rho}),$$

$h = \dfrac{\mu}{\rho}$ new small parameter

If we wish to consider lower-order terms (and we must!), we need

(i) $B^\rho = B^\rho(\cdot, h) = B_0^\rho + hB_1^\rho + h^2 B_2^\rho + \ldots$ (ii) replace the integration interval by $[c, +\infty), \quad c > 0.$

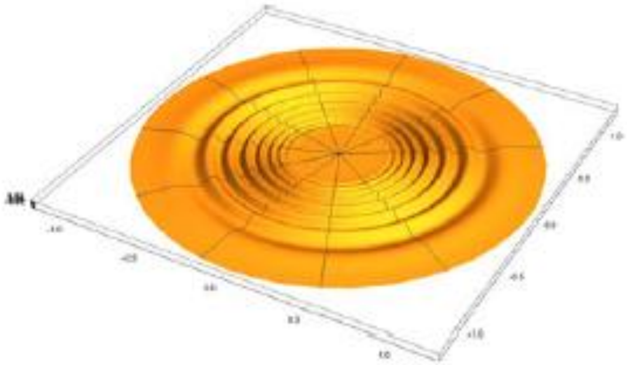
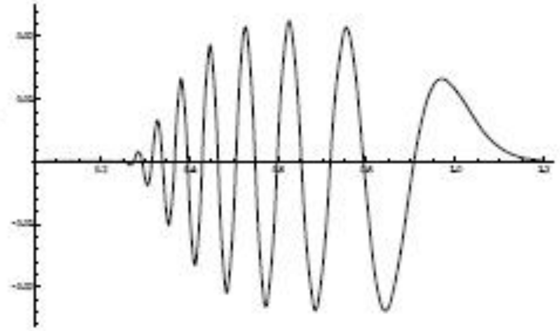
Typical asymptotic expansion: $$\frac{e^{\frac{\pi i m}{4}}}{(2\pi\mu)^{m/2}} \sum_{j=1}^\infty \int_c^\infty \rho^{\frac{k+m}{2}-1} \left(\frac{\mu}{\rho}\right)^j [K_{\Lambda_{\Psi^\rho}}^{\mu/\rho} B_j^\rho]\left(x, \frac{\mu}{\rho}\right) d\rho, \quad B_j^\rho \in C_0^\infty(\Lambda_{\Psi^\rho})$$

Maximum possible accuracy: $$R_{(c,d)j} = \frac{1}{\mu^{m/2}} \int_{c\mu}^{d\mu} \rho^{\frac{k+m}{2}-1} \left(\frac{\mu}{\rho}\right)^j f_j(x,\rho)\, d\rho$$

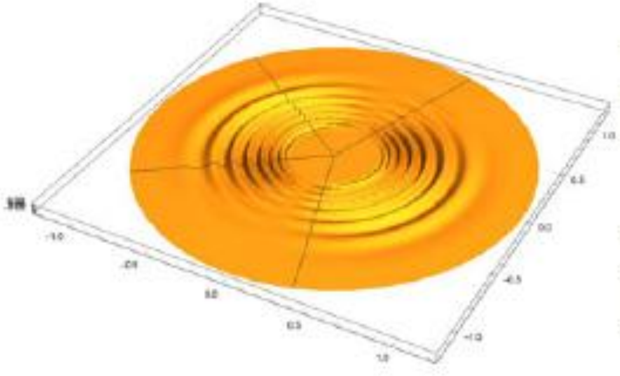
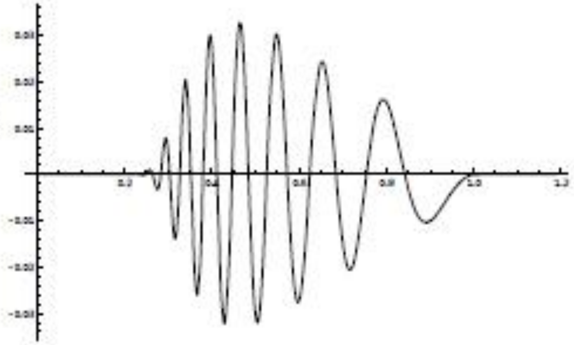
$$\| \nabla^\beta R_{(c,d)j} \| \le C\mu^{k/2}$$

Examples of solution of the original problem

$$\mu^2 \frac{\partial^2 \eta}{\partial t^2} + \mathcal{H}\eta = 0, \qquad \eta\,|_{t=0} = V\left(\frac{x}{\mu}\right), \qquad \eta_t\,|_{t=0} = 0, \qquad H(x,p) = |p|\tanh(D(x)\,|p|)$$

$$t = 1, \quad \mu = 0.02, \quad V(y) = e^{-|y|^2/2}$$

$$t = 1, \quad \mu = 0.02, \quad V(y) = y_1 e^{-|y|^2/2}$$

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Thank you!