Bernoulli problem

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Hereinafter we assume that
$\Omega \subset \mathbb{R}^2$ is a bounded domain with C^∞ boundary $\partial\Omega$;
an interface (free boundary) $\Gamma \Subset \Omega$ is a Jordan C^1 curve. It splits Ω into two parts:
a simply-connected domain $\Omega^+ \Subset \Omega$ (inclusion or void) with $\partial\Omega^+ = \Gamma$;
the annulus $\Omega^+ = \Omega \setminus \Omega^+$.
We denote by $\mathbf{n}$ the inward normal vector to $\partial\Omega^+$

Bernoulli Problem

Beurling, Alt & Cafarelli, Bucur & Trebesch, Crank, Danilyuk, Demidov, Flucher & Rumpf, Hamilton, Harbrecht, Henrot & Onodera.
Applications: electrochemical machining, potential flow in fluid mechanics, tumor growth, optimal insulation, plasma equilibrium, gravity and centrifugal water waves.

To find Ω^- (equivalently Γ) and a potential $u : \Omega^- \to \mathbb{R}$ satisfying the equations

$$\begin{aligned} \Delta u = 0 \quad \text{in} \quad \Omega^-, \quad u = h \quad \text{on} \quad \partial\Omega, \qquad u &= 0 \quad \text{on} \quad \Gamma, \\ \partial_n u &= -Q \quad \text{on} \quad \Gamma, \end{aligned}$$

where $h : \partial\Omega \to \mathbb{R}$ and $Q : \Omega \to \mathbb{R}$ are smooth positive functions,

$$h \geq c > 0, \quad Q > c > 0.$$

Methods

Upper and lower function. Monotonicity. Beurling, Cafarelli and others.
Various variational principles. Alt & Cafarelli, Danilyuk and others.

The Alt-Cafarelly variational problem

$$\min J(u), \quad J(u) = \frac{1}{2}\int_{\Omega} |\nabla u|^2 dx + \int_{u>0} Q^2\, dx.$$

Here the minimum is taken over the class of functions $u \in W^{1,2}(\Omega)$. satisfying the boundary condition $u = h$ on $\partial\Omega$. Note that

$$\Omega^- = \{u > 0\}, \quad \Omega^+ = \Omega \setminus \overline{\Omega^-}.$$

Bernoulli functional

$$J(u) = \frac{1}{2} \int_{\Omega^-} (|\nabla u|^2 + Q^2)\, dx.$$

where

$$\Delta u = 0 \ \text{ in } \ \Omega^-, \quad u = h \ \text{ on } \ \partial\Omega, \qquad u = 0 \ \text{ on } \ \Gamma.$$

Regularization

Usually, shape optimization problems are ill-posed and have no regular solutions. In practice, only regularized problems are considered. The latter means that the objective function J is replaced by

$$\mathcal{E} + J,$$

where $\mathcal{E}$ is a regularizing term. Below is the list of such terms

Historically, the first regularizer is the capillary energy, which equals the length $\mathcal{E}_p$ of Γ (perimeter of Ω^+)

$$\mathcal{E}_p = \int_\Gamma ds.$$

It was proposed by Mumford and Ambrosio.

The second is the elastic energy (1D Willmore functional) $\mathcal{E}_e$ of Γ (proposed by Mumford) which is defined by the equality

$$\mathcal{E}_e = \frac{1}{2} \int_\Gamma |\mathbf{k}|^2 \, ds,$$

where $\mathbf{k}$ is the curvature of Γ.

The third is the M öbius energy $\mathcal{E}_m$ introduced by O'Hara. It is defined as follows Let $\Gamma = f(\mathbb{S}^1)$. Then

$$\mathcal{E}_m(\Gamma) = \int_0^{2\pi}\int_0^{2\pi}\Big(\frac{1}{|f(\theta)-f(\sigma)|^2}-\frac{1}{D(f(\theta),f(\sigma))^2}\Big)\,|f'(\theta)||f'(\sigma)|d\theta d\sigma.$$

Here $D(f(\theta), f(\sigma))$ denotes the minimum length of subarcs of Γ with endpoints $f(\theta)$ and $f(\sigma)$.

Each regularization method has its own advantages and disadvantages.
The boundedness of the capillary energy guarantees the boundedness of the curve diameter, but does not guarantee its smoothness.
The boundedness of the elastic energy implies $C^{1+\alpha}$ smoothness of the curve and prevents the curve from being collapsed into a point. On the other hand, it does not provide an upper bound for the curve diameter.
The boundedness of the Mobius energy prevents the appearance of double points and self-intersections.
Therefore, it is advisable to use a linear combination of all three types of energy as a regularizer with general form of the regularized functional

$$\mathcal{E} + J = \varepsilon_e \mathcal{E}_e(\Gamma) + \varepsilon_m \mathcal{E}_m(\Gamma) + \varepsilon_p \mathcal{E}_p(\Gamma) + J, \quad \varepsilon_i \geq 0.$$

Curves or immersions?

A Jordan curve is a locus of points on a plane or simply a compact subset of the plane. On the other hand, a parametrized curve (immersion) is a mapping of the unit circle into a plane. Thus, curves and immersions belong to two different classes of mathematical objects and correspond to different mathematical theories.

All the functionals mentioned above are defined on the manifold of Jordan curves and are invariant with respect to the choice of parametrizations of these curves. Therefore, this manifold is a natural choice for the definition set of functionals in shape optimization theory.

However, in this paper we will consider curves as immersions. There are at least two important reasons for this.

The first is related to the numerical construction of approximate solutions using finite-dimensional approximation.
Approximation of a smooth curve using the moving grid method is an expensive and time-consuming procedure. On the other hand, the approximation of immersion leads to an easily solved problem of approximating a periodic function.
The second reason is due to the fact that the equations of gradient flows that arise when applying the steepest descent method are historically always formulated in terms of immersions.

Shape calculus. Gradients

Notation

An immersion $f : \mathbb{S}^1 \to \mathbb{R}^d$ is a C^1 mapping satisfying the condition

$$0 < f^- \leq |\partial f(\theta)| \leq f^+ < \infty, \quad \partial \equiv \partial_\theta.$$

The arc-length variable s on $\Gamma = f(\mathbb{S}^1)$ and the length element ds of Γ are defined by

$$s(\theta) = \int_0^\theta \sqrt{g(\sigma)}\, d\sigma, \quad ds = \sqrt{g(\theta)}\, d\theta, \quad g = |\partial f|^2.$$

We have

$$\partial_s = |\partial f|^{-1} \partial$$

Notation

The tangent vector $\boldsymbol{\tau}$ to Γ and the curvature vector $\mathbf{k}$ are given by

$$\boldsymbol{\tau}(\theta) = \partial_s f(\theta) := |\partial f|^{-1}\,\partial_\theta f(\theta), \quad \mathbf{k}(\theta) = \partial_s \boldsymbol{\tau}(\theta) = \partial_s^2 f(\theta).$$

Denotations ∇_s, ∇_t stand for the normal connections

$$\nabla_s\,\phi = \partial_s\phi - (\partial_s\phi \cdot \boldsymbol{\tau})\,\boldsymbol{\tau}, \quad \nabla_t\,\phi = \partial_t\phi - (\partial_t\phi \cdot \boldsymbol{\tau})\,\boldsymbol{\tau},$$

or equivalently

$$\nabla_s\,\phi = \Pi\partial_s\phi, \quad \nabla_t\,\phi = \Pi\partial_t\phi, \quad \Pi\phi = \phi - (\phi \cdot \boldsymbol{\tau})\,\boldsymbol{\tau}.$$

Hadamard gradient

A vector field

$$d\mathcal{J}(f) = \mathscr{B}\mathbf{n} : \mathbb{S}^1 \to \mathbb{R}^2, \quad \mathscr{B} \in L^1(\mathbb{S}^1)$$

is said to be the *Hadamard gradient* of $\mathcal{J}$ at f, if the integral identity

$$\lim_{t\to 0} \frac{1}{t}\big(\mathcal{J}(f+t\delta f)-\mathcal{J}(f)\big) = \int_\Gamma \mathscr{B}\mathbf{n}\cdot\delta f\, ds \equiv \int_0^{2\pi} \sqrt{g}\,\mathscr{B}\mathbf{n}\cdot\delta f\, d\theta \tag{1}$$

holds for every smooth vector field $\delta f : \mathbb{S}^1 \to \mathbb{R}^2$.

$$\nabla\mathcal{J} = \sqrt{g}d\mathcal{J} \text{ is the true } L^2(\mathbb{S}^1) - \text{gradient of } \mathcal{J}$$

The Hadamard gradient of the Bernoully functional J_b is given by

$$dJ = \big(\, Q^2 - |\nabla u|^2 \,\big)\, \mathbf{n},$$

The gradient dJ also can be regarded as a nonlinear operator acting on $f : \mathbb{S}^1 \to \mathbb{R}^2$. We will denote this operator by

$$\mathcal{B}[f] := dJ[f] = \mathscr{B}\, \mathbf{n}.$$

The Hadamard gradient of the length functional $\mathcal{E}_p$ is given by

$$d\mathcal{E}_p = -\mathbf{k},$$

The associated nonlinear operator

$$\mathcal{A}_p(f) := -\mathbf{k}.$$

The Hadamard gradient of the elastic energy $\mathcal{E}_e$ is given by

$$d\mathcal{E}_e = \nabla_s \nabla_s \, \mathbf{k} + \frac{1}{2} |\mathbf{k}|^2 \, \mathbf{k},$$

The associated nonlinear operator

$$\mathcal{A}_e(f) := d\mathcal{E}_e.$$

The Hadamard gradient of the Möbius energy $\mathcal{E}_m$ is given by

$$d\mathcal{E}_m =$$
$$2\int_0^{2\pi}\Big(\frac{2\Pi(\theta)(f(\sigma)-f(\theta))}{|f(\sigma)-f(\theta)|^2}-\mathbf{k}(\theta)\Big)\frac{|f'(\sigma)|d\sigma}{|f(\sigma)-f(\theta)|^2}$$

The associated nonlinear operator

$$\mathcal{A}_m(f) := d\mathcal{E}_m.$$

Hessians

Let

$$\mathcal{A} = d\mathcal{E}$$

We will write

$$\delta\mathcal{A} = \lim_{\sigma\to 0}\frac{1}{\sigma}\big(\mathcal{A}[f+\sigma\delta f] - \mathcal{A}[f]\big) \equiv \mathcal{A}'[f]\delta f.$$

Let an immersion $f : \mathbb{S}^1 \to \mathbb{R}^d$, $d \geq 2$, satisfy the conditions

$$f \in C^7(\mathbb{S}^1), \quad 0 < c^{-1} \leq |\partial_\theta f(\theta)| \leq c.$$

The moving frame coordinates (ω, ς) in the space of vector fields δf are defined by the equalities

$$\delta f = \omega\, \mathbf{n} + \varsigma\, \boldsymbol{\tau}.$$

Let

$$d\mathcal{E} = \mathcal{A}[f] = \mathscr{A}[f]\,\mathbf{n}.$$

Assume that the scalar function $\mathscr{A}$ is invariant with respect to the choice of parametrization $\theta \to f(\theta)$. This means that

$$\mathscr{A}[f_\phi] = \mathscr{A}[f] \circ \phi, \quad f_\phi = f \circ \phi,$$

for any smooth diffeomorphism $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ and for any smooth perturbation δf. Then

$$\mathscr{A}'[f]\,(\varsigma\boldsymbol{\tau}) = \frac{1}{\sqrt{g}}\,(\partial_\theta\,\mathscr{A}[f])\,\varsigma \ \text{ for all } \ \varsigma \in C^\infty(\mathbf{S}^1).$$

$$
\mathcal{A}'[f]\delta f = \begin{pmatrix} \mathbf{A}_{11}, & \dfrac{1}{\sqrt{g}}\,\partial_\theta\,\mathscr{A} \\ -\dfrac{1}{\sqrt{g}}\,\mathscr{A}\,\partial_\theta, & -\mathscr{A}K \end{pmatrix} \begin{pmatrix} \omega \\ \varsigma \end{pmatrix}.
$$

$$\mathcal{A}'_\alpha[f]\delta f = \begin{pmatrix} \mathbf{A}^\alpha_{11}, & \dfrac{1}{\sqrt{g}}\,\partial_\theta\,\mathscr{A}_\alpha \\ -\dfrac{1}{\sqrt{g}}\,\mathscr{A}_\alpha\,\partial_\theta, & -\mathscr{A}_\alpha K \end{pmatrix} \begin{pmatrix} \omega \\ \varsigma \end{pmatrix}.$$

$$dJ = \mathcal{B} = \mathscr{B}\,\mathbf{n}, \quad \mathscr{B} = Q^2 - |\nabla u|^2$$

$$\mathcal{B}'[f]\delta f = \begin{pmatrix} \mathbf{B}_{11}, & \dfrac{1}{\sqrt{g}}\,\partial_\theta\,\mathscr{B} \\ -\dfrac{1}{\sqrt{g}}\,\mathscr{B}\,\partial_\theta, & -\mathscr{B}K \end{pmatrix} \begin{pmatrix} \omega \\ \varsigma \end{pmatrix},$$

$$\begin{aligned}
\mathbf{A}^e_{11}\omega &= \partial_s^4\omega + \frac{5}{2}K\partial_s(K\partial_s\omega) + (3K\mathscr{A}_e + 3(\partial_s K)^2 + K\partial_s^2 K)\,\omega, \\
A^p_{11}\omega &= -\partial_s^2\omega - K^2\omega
\end{aligned}$$

$$u = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{in\theta}\, u_n, \quad u_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-in\theta}\, u(\theta)\, d\theta,$$

The Hilbert transform $\mathbf{H}$,

$$(\mathbf{H}u)_n = -i \ \mathrm{sgn}\ n\, u_n, \quad n \in \mathbb{Z}. \tag{2}$$

The Dirichlet to Neumann operator $\mathbf{N}$,

$$\mathbf{N} \equiv |\partial| \equiv (-\partial^2)^{1/2} = \partial\, \mathbf{H} = \mathbf{H}\partial, \tag{3}$$

$$(\mathbf{N}u)_n = |n|\, u_n, \quad n \in \mathbb{Z}.$$

$$
\begin{aligned}
\sqrt{g}\,\mathbf{A}^m_{11}\,\omega &= -\boldsymbol{\alpha}_3\partial_\theta^2|\partial_\theta|\omega - \boldsymbol{\alpha}_2\partial_\theta|\partial_\theta|\omega - \boldsymbol{\alpha}_1|\partial_\theta|\omega - \boldsymbol{\alpha}_0\mathbf{H}\omega \\
&+ \partial_\theta(\boldsymbol{\beta}_1\partial_\theta\omega) + \boldsymbol{\beta}_0\omega + \mathbf{R}\omega,
\end{aligned}
$$

where $\mathbf{R}$ is unessential smoothing reminder. The principle coefficient $\boldsymbol{\alpha}_3$ equals $2/(3g)$. The differential operator

$$
\boldsymbol{\alpha}_3\,\partial_\theta^3 + \boldsymbol{\alpha}_2\,\partial_\theta^2 + \boldsymbol{\alpha}_1\,\partial_\theta + \boldsymbol{\alpha}_0
$$

is skew-symmetric, i. e.,

$$
\boldsymbol{\alpha}_2 = \frac{3}{2}\partial_\theta\,\boldsymbol{\alpha}_3, \quad 2\,\boldsymbol{\alpha}_0 = \partial_\theta\,\boldsymbol{\alpha}_1 - \frac{1}{2}\partial_\theta^3\,\boldsymbol{\alpha}_3.
$$

$$\frac{1}{2}\mathbf{B}_{11}[f]\,\omega = \frac{1}{\sqrt{g}}\,\partial_n u\,|\partial_\theta|\,\big(\,\partial_n u\,\omega\,\big) - \big(Q\,\partial_n Q + K\,|\partial_n u|^2\,\big)\;\omega + \mathbf{R}\omega.$$

Gradient flow

Let us consider the general regularized problem

$$\partial_t f + \mathcal{A}[f] + \mathcal{B}[f] = 0 \;\text{ for }\; t \in (0, T), \quad f\Big|_{t=0} = f_0.$$

Here regularization term $\mathcal{A}$ admits the representation

$$\mathcal{A} = \varepsilon_e\, \mathcal{A}_e + \varepsilon_m\, \mathcal{A}_m + \varepsilon_p\, \mathcal{A}_p, \quad \varepsilon_e > 0, \quad \varepsilon_m,\, \varepsilon_p \geq 0.$$

Example. Elastic flow. Straightening equation

$$\partial_t f + \mathcal{A}_e(f) + \lambda \mathcal{A}_p(f) = 0 \quad \text{in} \quad (0, \infty), \quad f(0) = f_0.$$

or equivalently

$$\partial_t f + \nabla_s \nabla_s \mathbf{k} + \frac{1}{2} |\mathbf{k}|^2 \mathbf{k} - \lambda \mathbf{k} = 0 \text{ for } t \in (0, \infty),$$

$$f(0) = f_0.$$

Important paper: Dziuk, Kuwert, and Schatzle (2002)

Koiso (1992), Wen (1995), Polden (1996), Lin (2012),Wheeler (2012), Dall' Acqua&Pozzi (2014), Mantegazza & Posetta (2020), Rupp & Spener (2020)

Example. Möbius flow

$$\partial_t f + \mathcal{A}_m(f) = 0 \quad \text{in} \quad (0,\infty), \quad f(0) = f_0.$$

or equivalently

$$\partial_t f =$$

$$-2\int_0^{2\pi}\Big(\frac{2\Pi(\theta)(f(\sigma)-f(\theta))}{|f(\sigma)-f(\theta)|^2} - \mathbf{k}(\theta)\Big)\frac{|f'(\sigma)|d\sigma}{|f(\sigma)-f(\theta)|^2},$$

$$f(0) = f_0.$$

Important papers: He (2000), Blatt (2012,2016)

Contrary to popular belief, none of the considered gradient flows defines a parabolic system of equations. These flows can be considered as systems of integro-differential equations of composite type for the joint determination of the shape of the desired curve and its parametrization. This leads to the fact that in the process of local research we encounter the phenomenon of loss of smoothness at iterations.

Historically, three methods have emerged to overcome the difficulties associated with the phenomenon of loss of smoothness:

The Schauder "pseudo-linearization"method Schauder (1935), Sobolev (1940);

Regularization. Kohn, Nirenberg (1965,1967), Oleinik, Radkevich (1971, 1974-1978,)

Nash-Moser theory: Moser (1961,1966), Hamilton (1982), see also Nirenberg (1974), Hörmander (1997).

Application to the Bernoulli problem – Hamilton (1982), Henrot, Onodera (2020). Application to the Möbius flow – He (2000).

Local existence result

Theorem

Assume that an immersion $f_0 : \mathbb{S}^1 \to \mathbb{R}^2$ *defines the Jordan curve* $\Gamma_0 = f_0(\mathbb{S}^1) \Subset \Omega$ *and*

$$\|f_0\|_{W^{r+\lambda,2}(\mathbb{S}^1)} \leq c(\lambda), \quad r \geq 7, \ \lambda \geq 42r$$

Then there is $\varepsilon_0 > 0$*, dependent only on* f_0*, with the following property. For every* $0 < \varepsilon \leq \varepsilon_0$*, there is* $T > 0$ *such that the problem*

$$\partial_t f + \mathcal{A}(f) + \mathcal{B}(f) = 0 \ \ in \ \ \mathbb{S}^1 \times (0,T), \quad f\Big|_{t=0} = f_0 \ \ in \ \ \mathbb{S}^1.$$

has a solution $f \in W^{1,\infty}(0,T,W^{r,2}(\mathbb{S}^1))$ *with* $\|f(t) - f_0\|_{W^{r,2}(\mathbb{S}^1)} \leq \varepsilon$.

Problem

Let $\mathfrak{M}$ be a manifold of all $C^{1+\alpha}$ Jordan curves Γ such that $\Gamma \cup \partial\Omega = \emptyset$
Intuitively $\partial\mathfrak{M}$ consists of: single points, curves with double points, curves having common points with $\partial\Omega$.

$$\mathcal{E}[f] + J[f] \to \infty \quad \text{as} \quad \Gamma = f(\mathbb{S}^1) \to \partial\mathfrak{M}$$

Problem

Let $\gamma_0 = f_0(\mathbb{S}^1) \in \mathfrak{M}$ and f_0 is sufficiently smooth, say $f_0 \in W^{7,2}(\mathbb{S}^1)$. Then the problem

$$\partial_t f + \mathcal{A}(f) + \mathcal{B}(f) = 0 \ \text{ in } \ \mathbb{S}^1 \times (0,T), \quad f\Big|_{t=0} = f_0 \ \text{ in } \ \mathbb{S}^1.$$

has a unique solution $f \in C^{1,\infty}(0,\infty, W^{7,2}(\mathbb{S}^1))$ with such that

$$\Gamma(t) = f(t,\mathbb{S}^1) \to \Gamma_\infty \in \mathfrak{M}$$

uniformly with the accuracy up to reparametrization.