Homoclinic Chaos in Some Kinetic Models

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- Rössler attractor
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Mathematical model

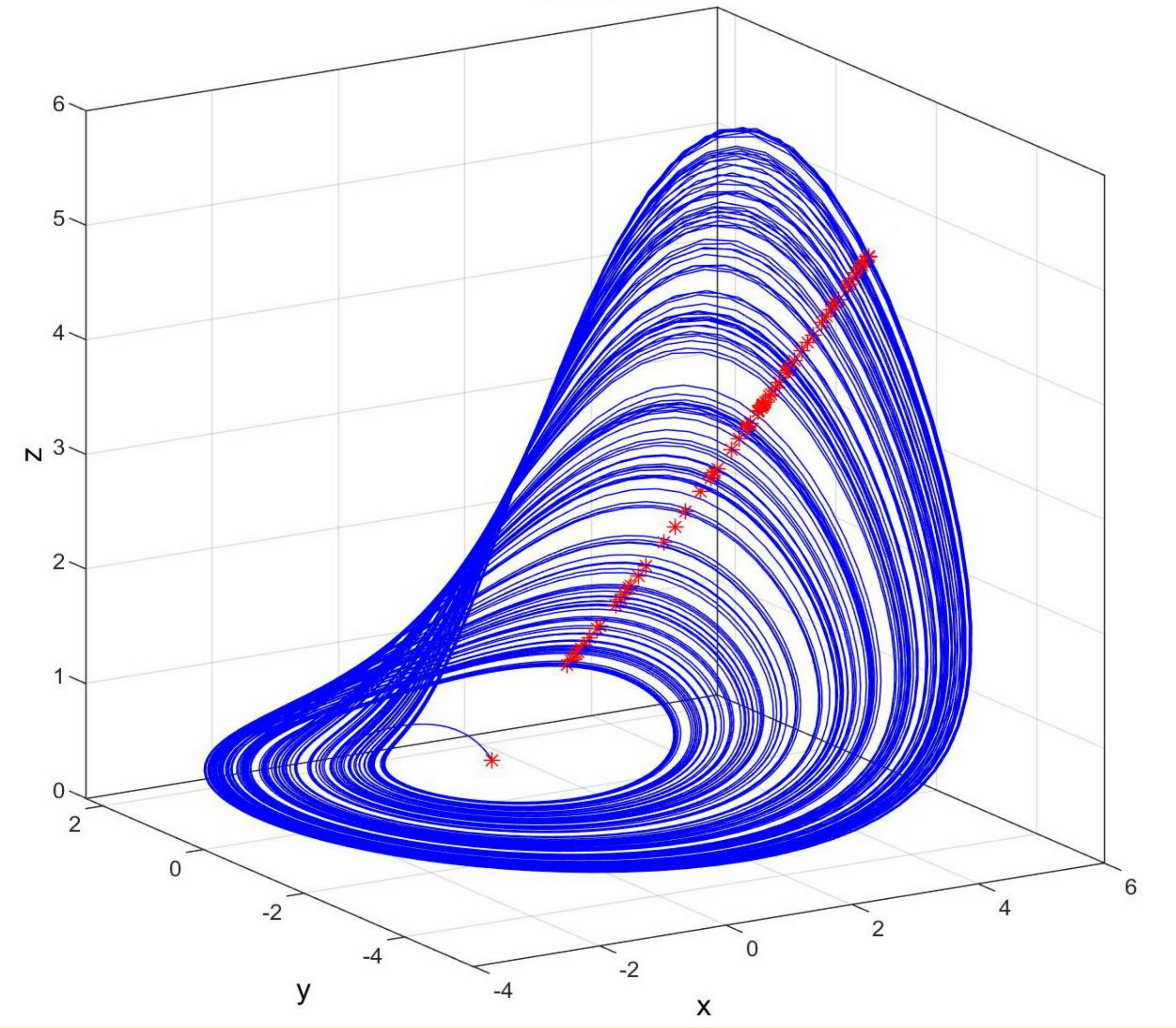
$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= b + z(x - c), \end{aligned} \qquad (1)$$

$$a = 0.398, \qquad b = 2, \qquad c = 4.$$

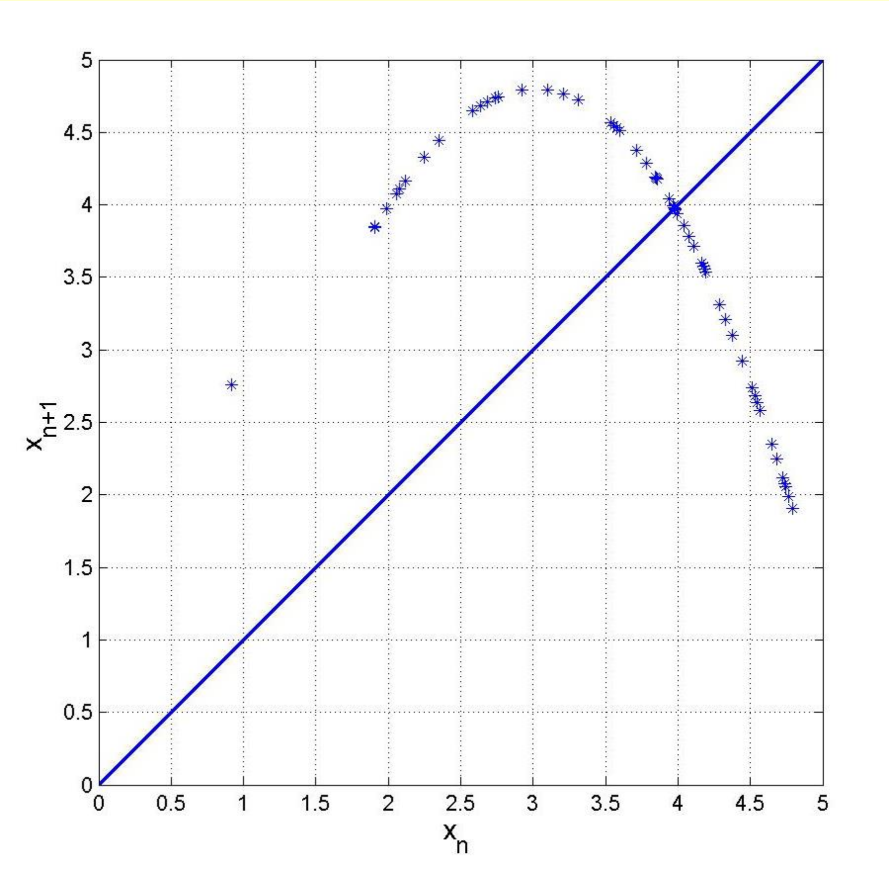
O.E. Rössler, *Chaotic behaviour in simple reaction system*. *Zeitschrift für Naturforschung*. **31**a, 259–264 (1976).

O.E. Rössler, *Continuous chaos*. In: Haken H. (eds), Synergetics (Springer, Heidelberg, 1977).

Rössler attractor and Global Poincarè Cross Section



Poincare map $F(x_n)$



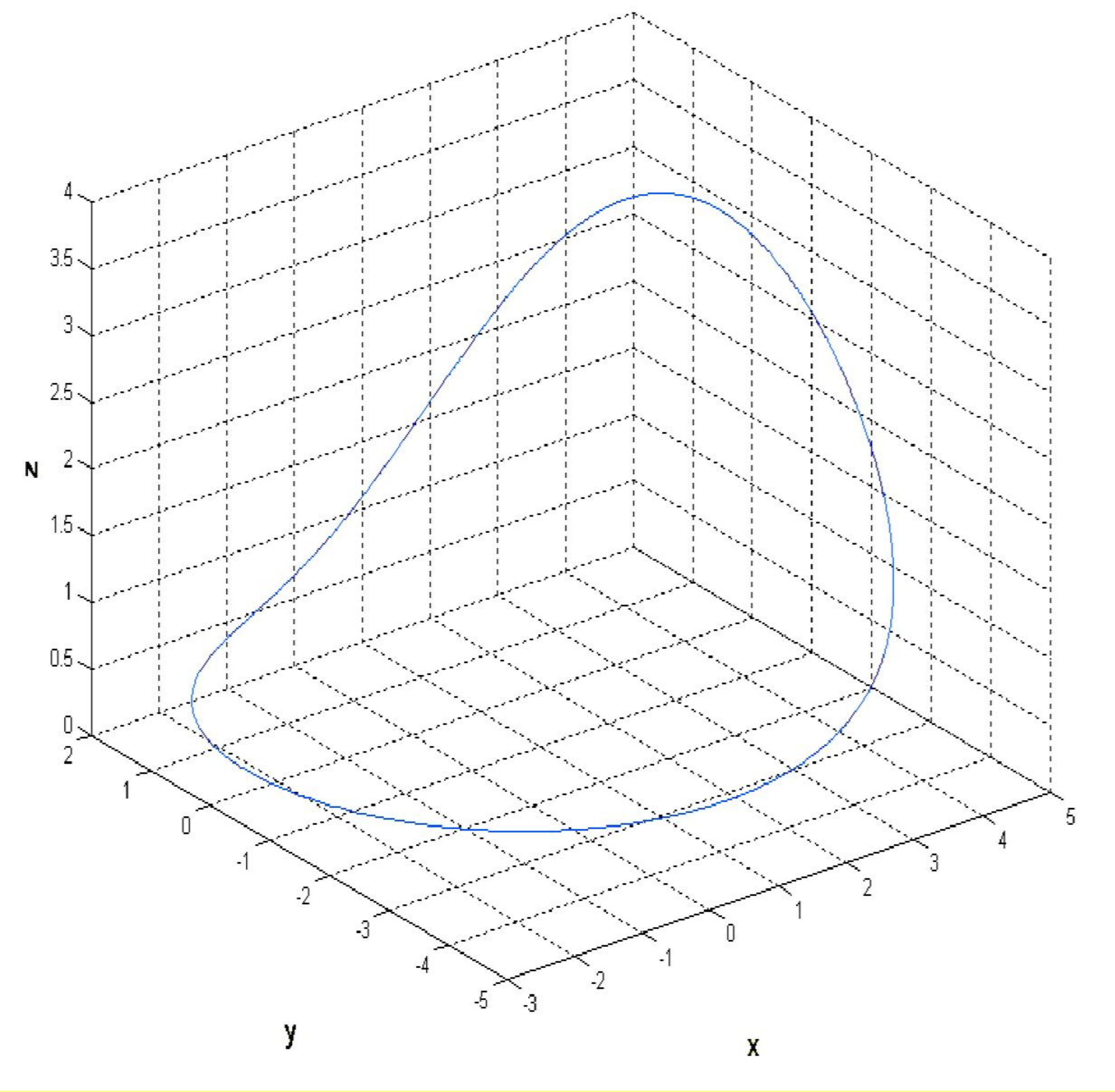
A remark about the relationship of the Poincarè map near a fixed point with the eigenvalue $\nu < -1$ to the continuous flow of the Rössler model around the corresponding periodic trajectory.

The points of the orbits given by the Poincarè map alternate from one side of the fixed point to the other along the direction of the eigenvector related to ν.

Therefore, the two-dimensional unstable manifold of the limit cycle is a Möbius strip with a middle line being the unstable periodic motion from which the representing point will "unwind".

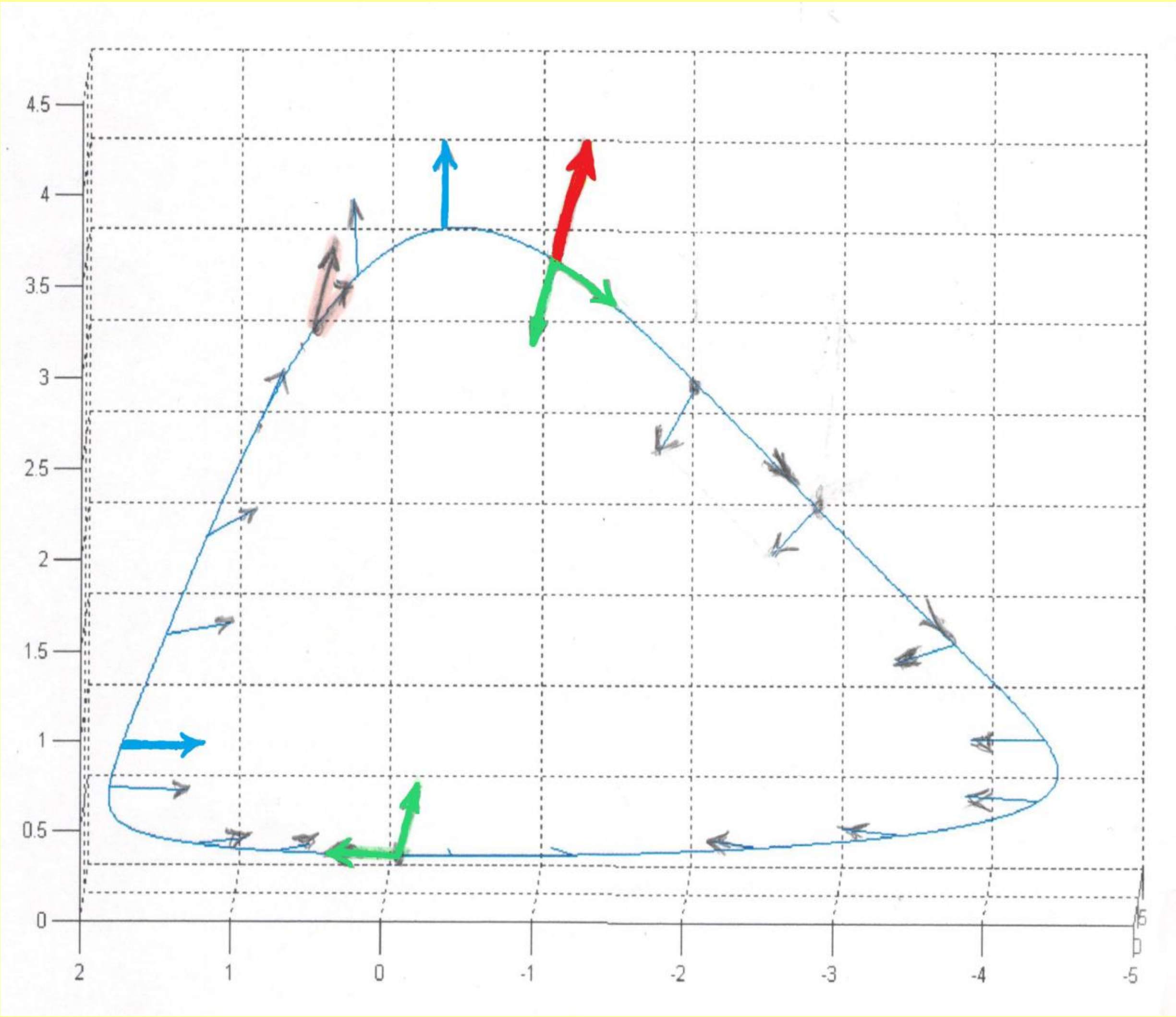
Cascade of period-doubling bifurcations

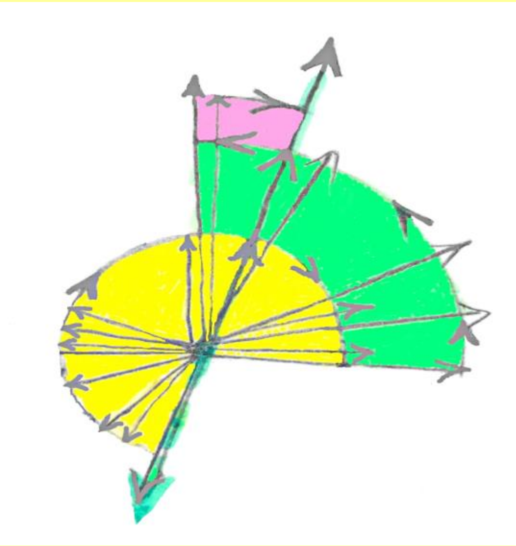
The role
of successive
period-doubling
bifurcations
in creation
of chaotic dynamics.



This T-periodic orbit
gives rise to the
Feigenbaum cascade
of flip bifurcations.

T-periodic Möbius orbit and Rotation of a non-orientable 2-D manifold



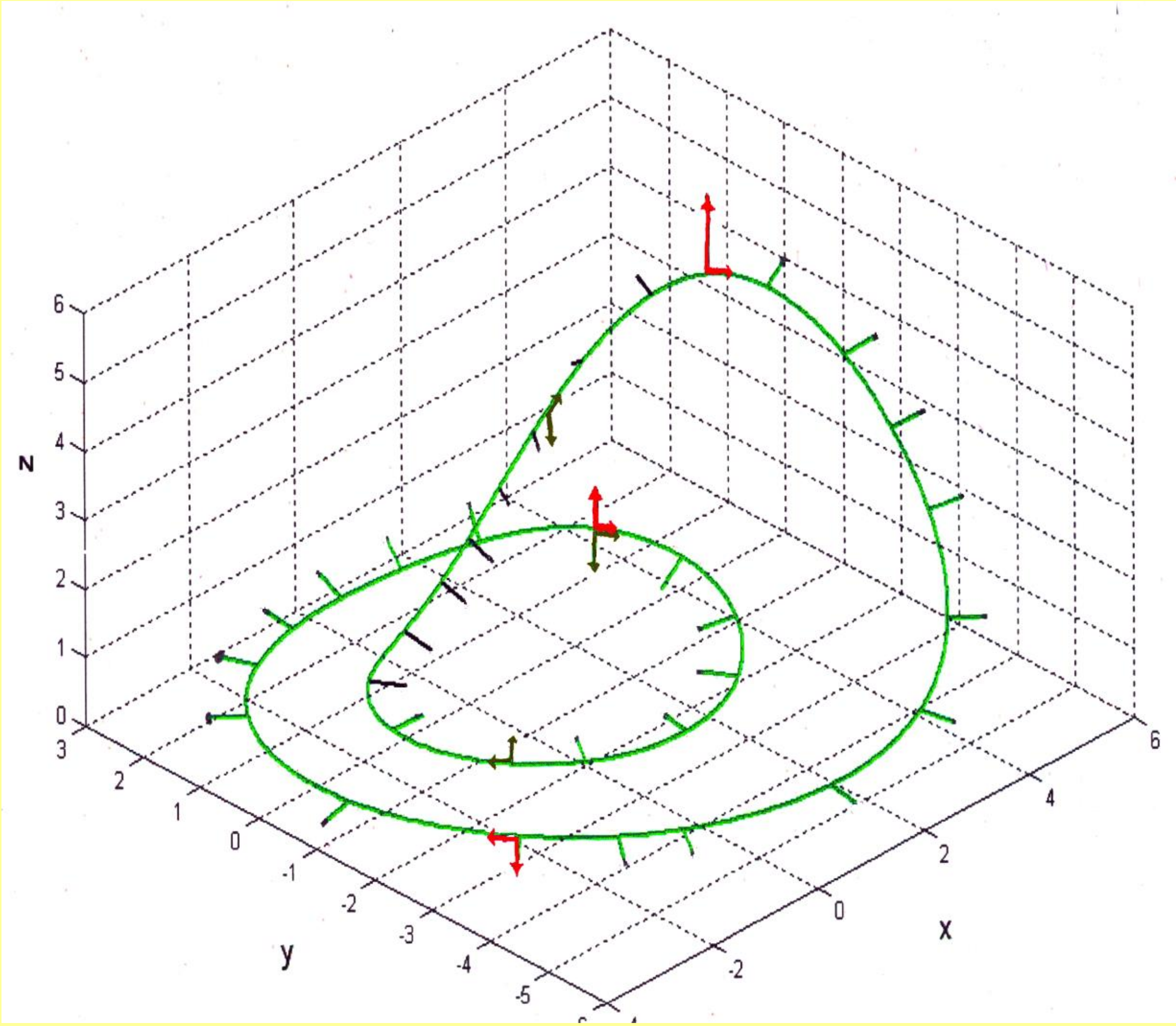


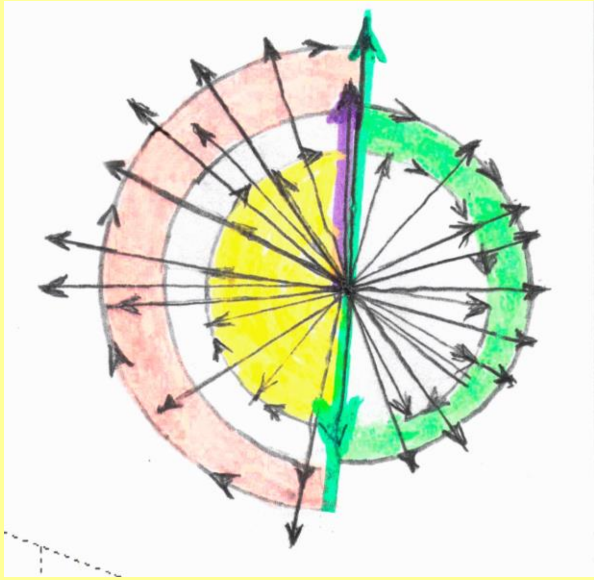
We have unstable manifold with a half-twist.

Period $T = 6.229$

Multiplier $\nu = -1.8$

2T-periodic Möbius orbit and Rotation of a non-orientable 2-D manifold



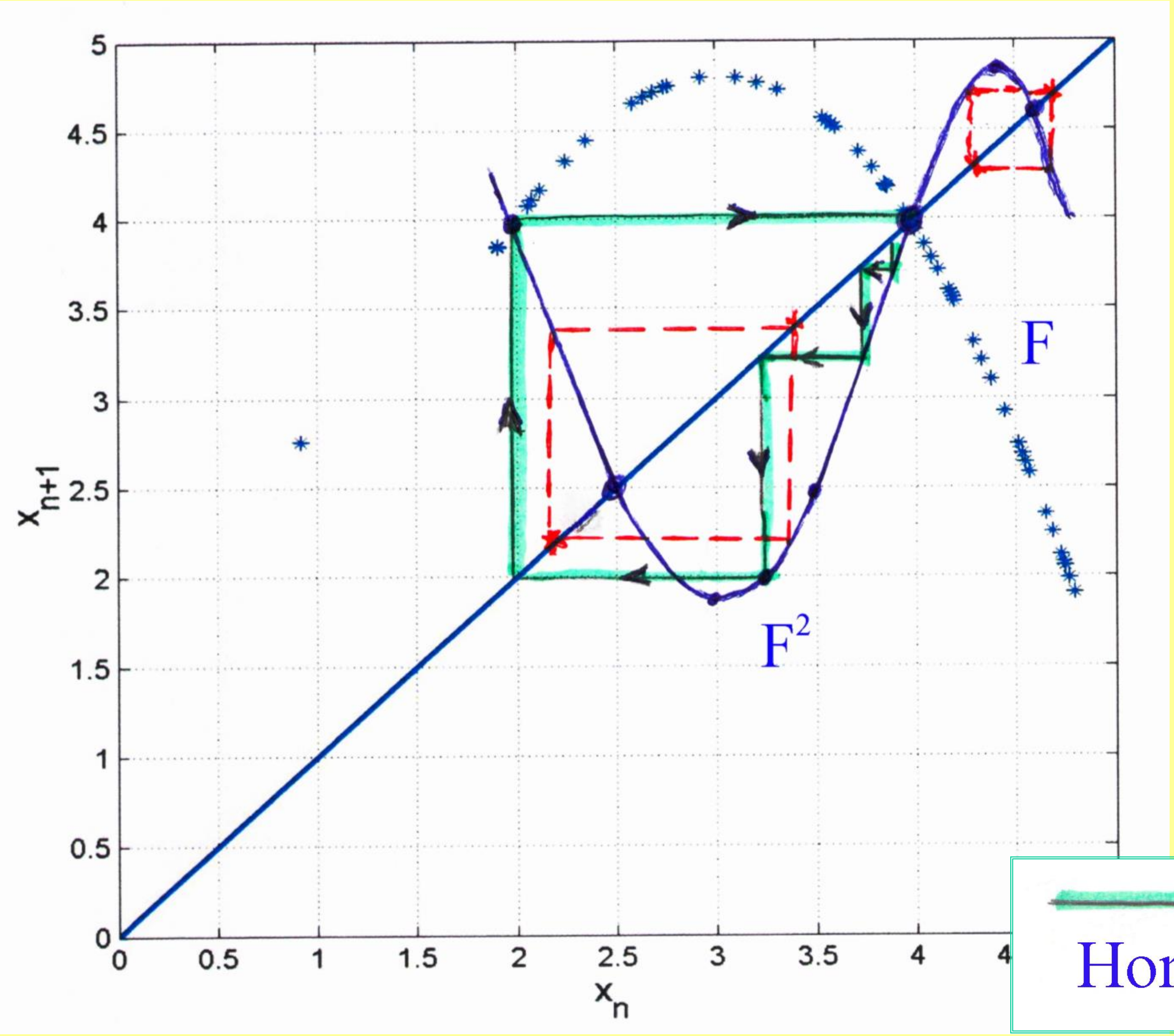


There is an unstable manifold
with five half-twists.

Period $T = 12.45$

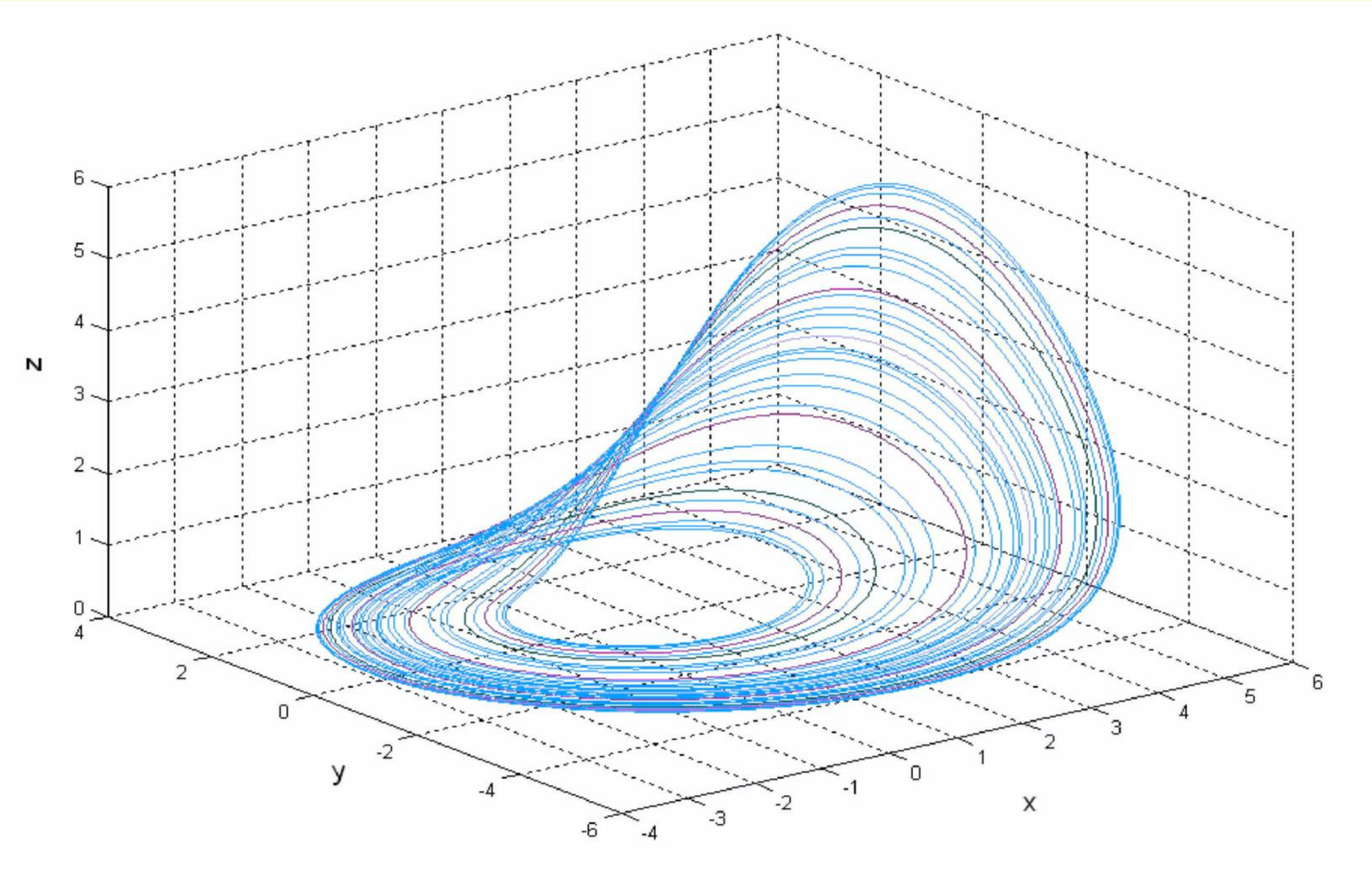
Multiplier $\nu = -2.3$

Poincare maps $F(x_n)$ and $F^2(x_n)$



The homoclinic orbit to a T-periodic solution is the topological limit of 2^kT-periodic Möbius orbits.

Rössler attractor and Möbius orbits



Basis sets of Möbius orbits on attractor are the minimal sets of periodic orbits that force the existence of all other in the flow. In this figure are shown Möbius orbits of periods T and 2^kT for $k = 1, 2, 3$, and 4.

3D mathematical model of a catalytic H_2 oxidation reaction

$$\dot{x} = K_1(1-x-y)^2 - K_{-1}x^2 - 2k_3(y)\cdot x^2 y,$$

$$\dot{y} = K_2(1-x-y)^2 - K_{-2}y^2 - k_4(y,z)\cdot y - k_3(y)\cdot x^2 y,$$

$$\dot{z} = \varepsilon\left[y(1-z) - \alpha z(1-x-y)\right],$$

where

$$0 \le x, \quad 0 \le y, \quad x + y \le 1, \quad 0 \le z \le 1,$$

$$k_3(y) = K_{30} exp(-\mu_3 y), \quad k_4(y,z) = K_{40} exp(-\mu_4 y + \mu_5 z).$$

Structure of the mathematical model

$$x' = \mu \cdot f(x,y), \quad y' = g(x,y,z), \quad z' = \varepsilon \cdot h(x,y,z)$$

2-D subsystem with the parameter z, $0<z<1$:

$$x' = \mu \cdot f(x,y), \quad y' = g(x,y,z),$$

Important case:

$$\varepsilon < \mu$$

Defining chaos

- ***Chaos*** is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions

- ***Aperiodic long-term behavior*** means that there are trajectories which do not settle down to fixed points, periodic orbits or quasi-periodic orbits as t goes to infinity.

- ***Sensitive dependence on the initial conditions*** means that nearby trajectories separate exponentially fast, i.e. the system has a positive Liapunov exponent.

Attractor and Strange attractor

- An ***attractor A*** is an indecomposable closed invariant set:

  (1) any trajectory that starts in ***A*** stays in ***A*** for all time;

  (2) ***A*** attracts an open set **U** of initial conditions, the largest *U* is called the basin of attraction of ***A***.

- We shall call an attractor ***strange*** if it contains a transversal homoclinic orbit.

- A ***homoclinic orbit*** is a motion tending to a periodic orbit as time increases and also to that same periodic orbit as time recedes into the past.

Strange attractor

Subharmonic cascade
of period doubling bifurcations:

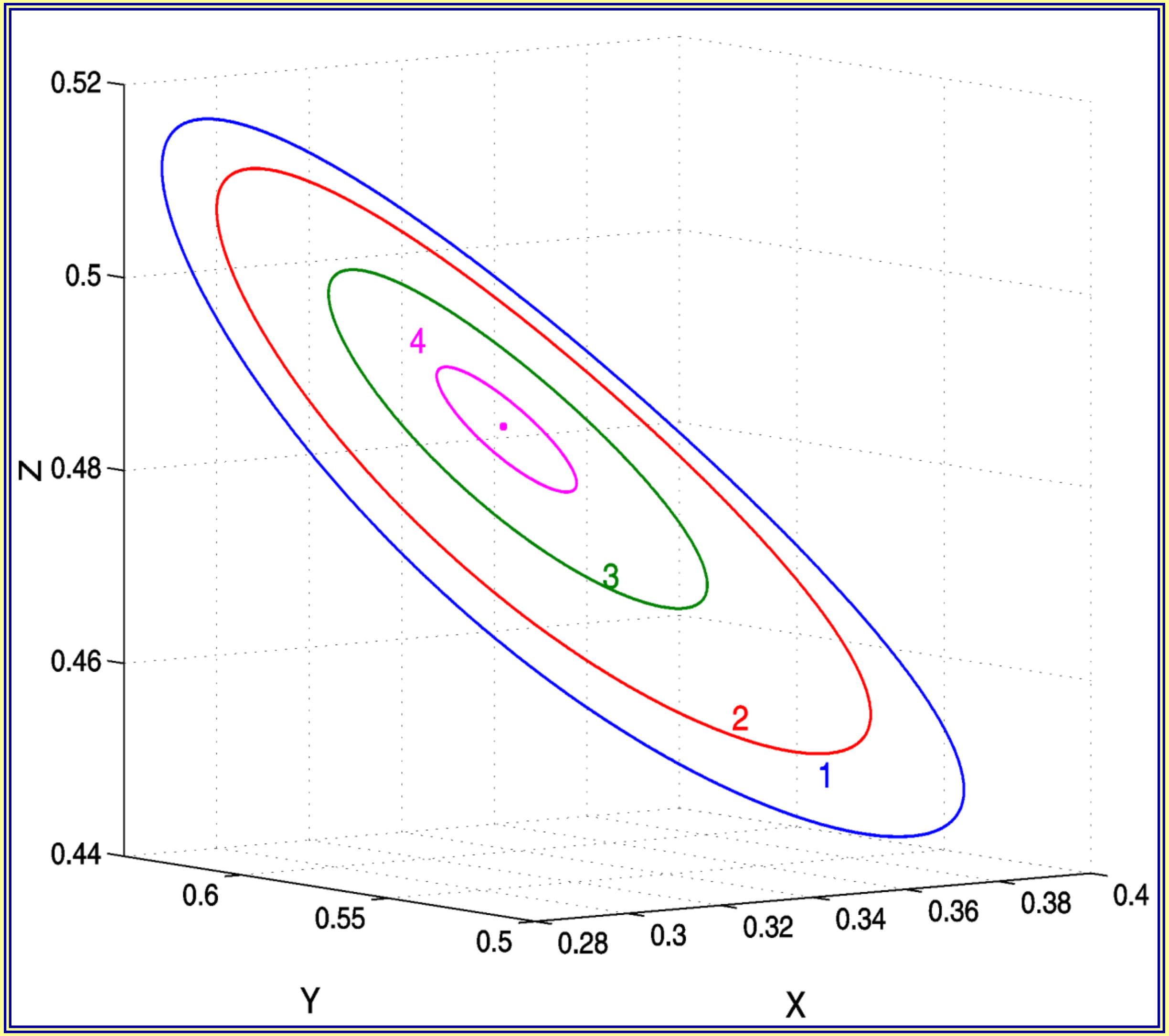
$\varepsilon < \mu$

$K_1=0.15$ $K_{-1}=0.008$ $K_2=20$ $K_{-2}=0$

$K_{30}=100$ $K_{40}=2$ $\alpha=7.88$ $\varepsilon=0.0024$

$\mu_3=30$ $\mu_4=12$ $\mu_5=10$

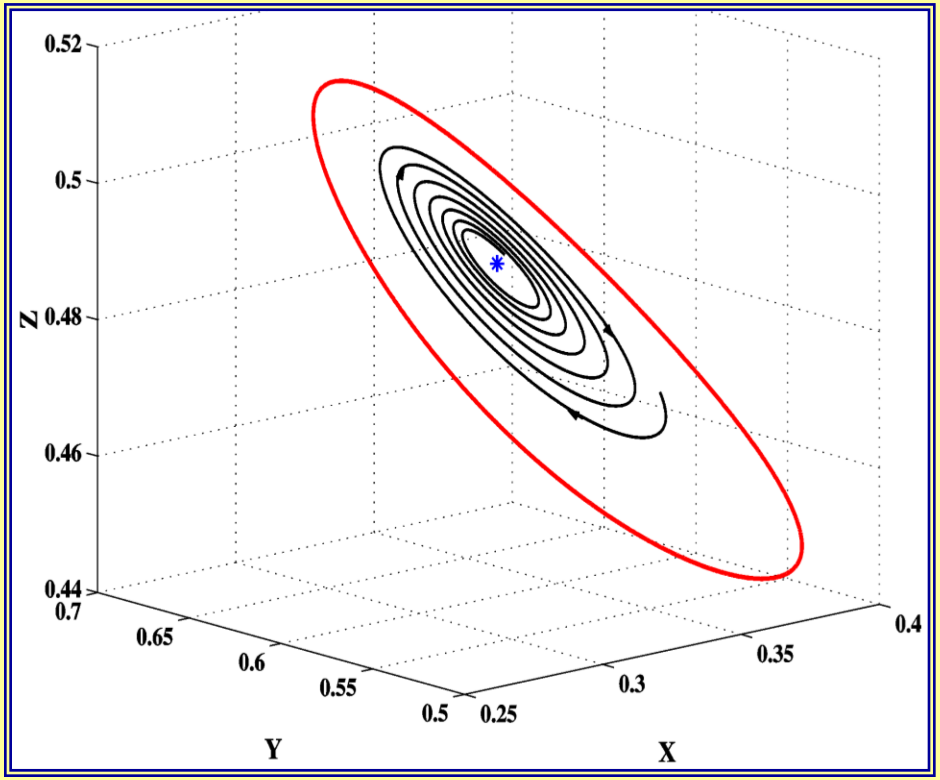
Center manifold of the Hopf bifurcation



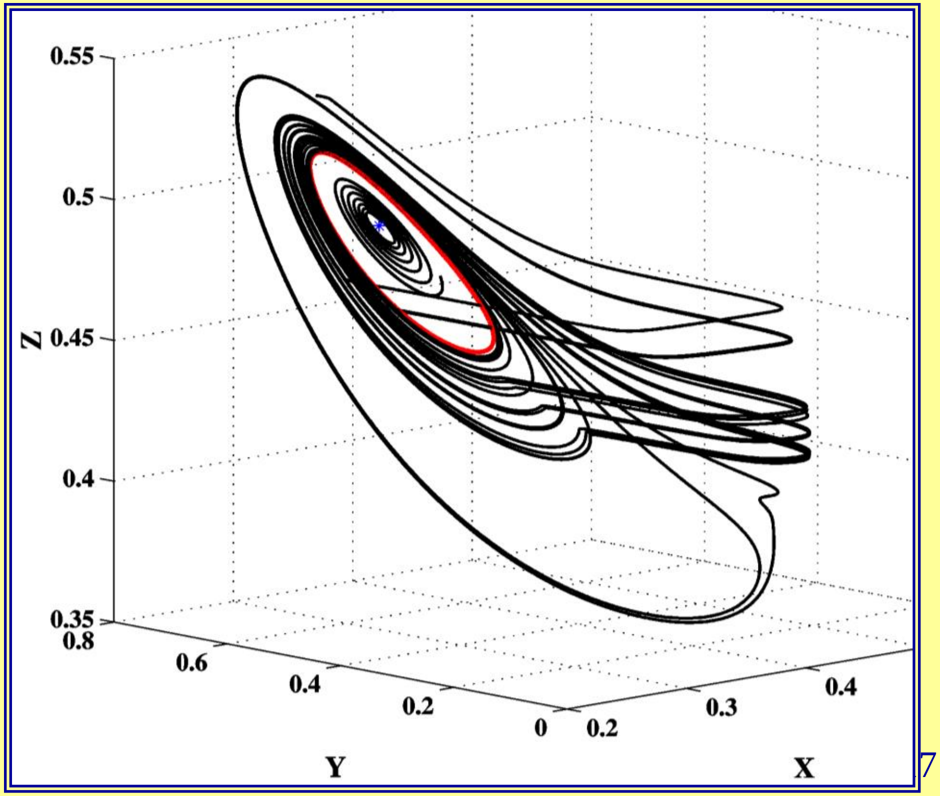
K_1	
0.135824260	(1)
0.1365	(2)
0.13749	(3)
0.13796	(4)

Multiplier :	
1.82	(1)
1.47	(2)
1.13	(3)
1.02	(4)

Unstable limit cycle with a trajectory inside it and a transversal homoclinic orbit

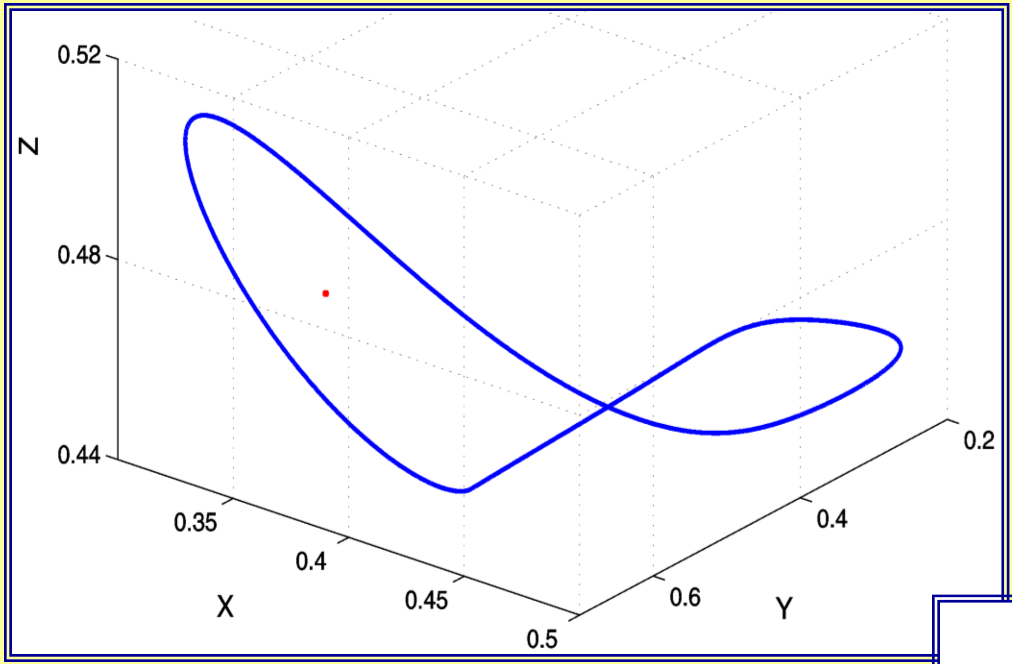


K_1=0.135824260



Period 1277.6
Multiplier 1.813

Cascade of period doubling bifurcations



T

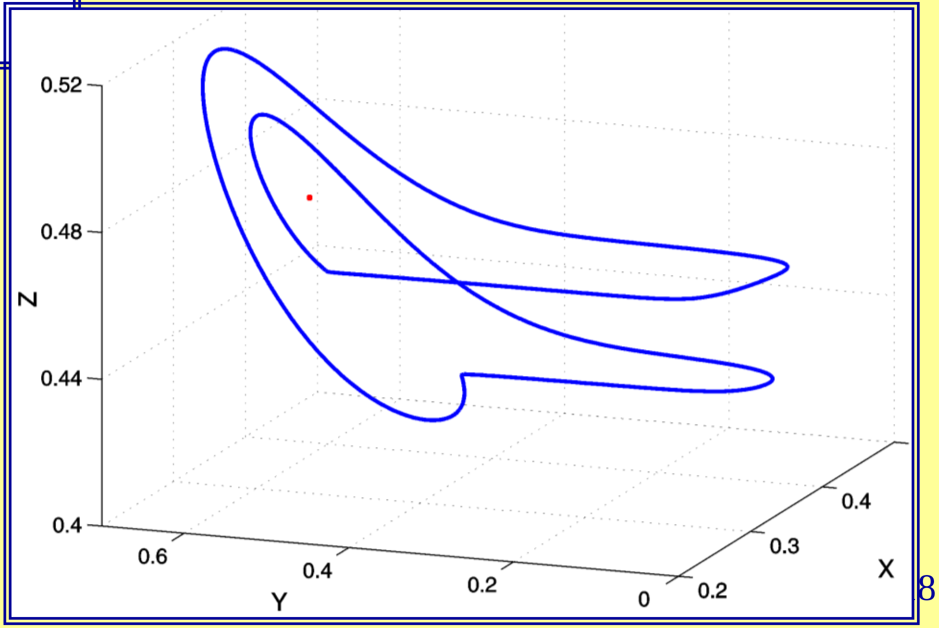
K_1=0.147045
K_{40}=1.9606

T= 970.4
λ= -1.143

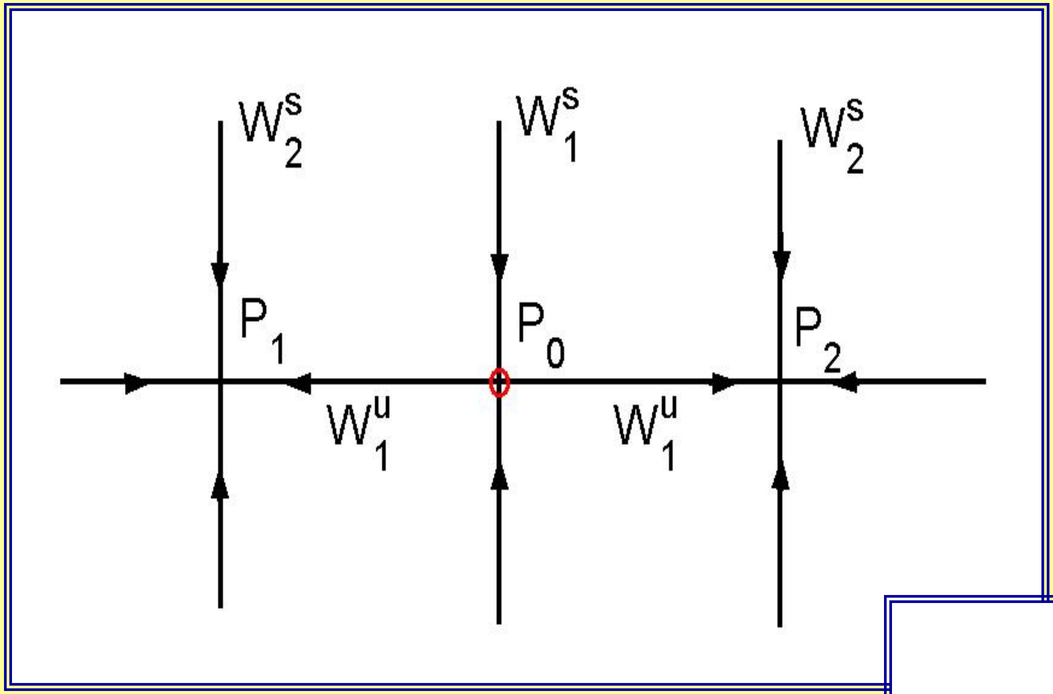
K_1= 0.146
K_{40}= 1.9466

T= 1931.1
λ= -0.18

2T

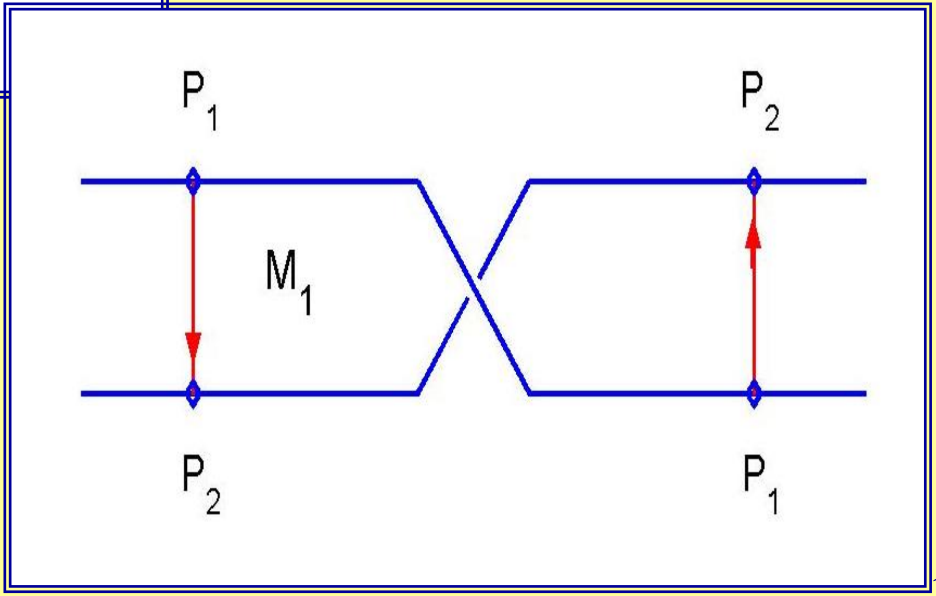


Möbius orbits

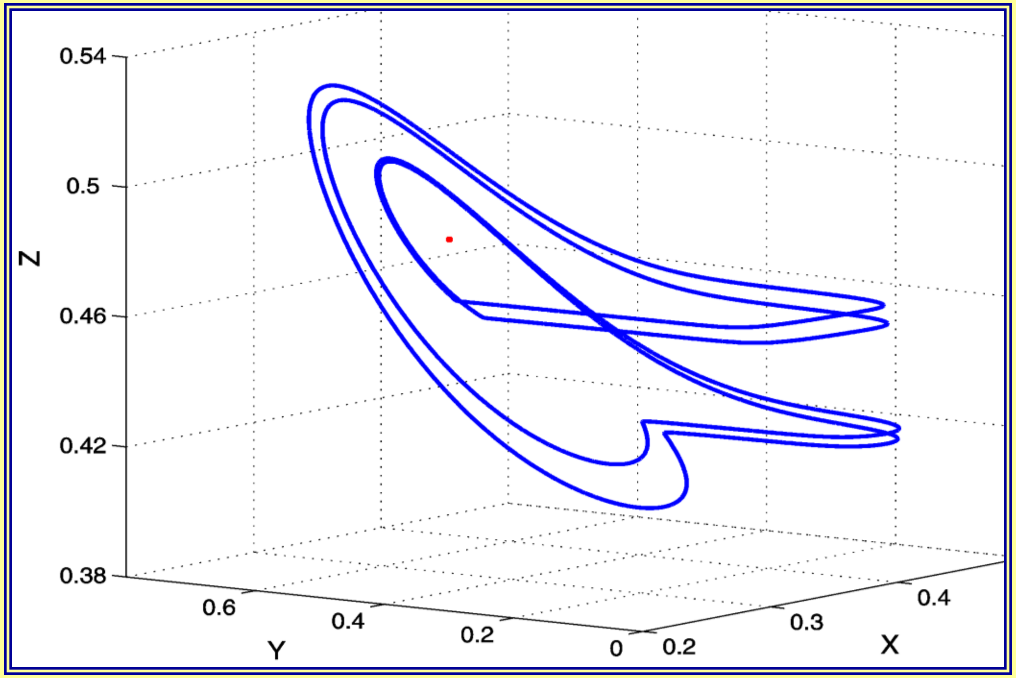


2D Poincare map near an attracting set $\Lambda_2 = M_1 \cup \varphi_2$

The topology of invariant set Λ_2



Cascade of period doubling bifurcations



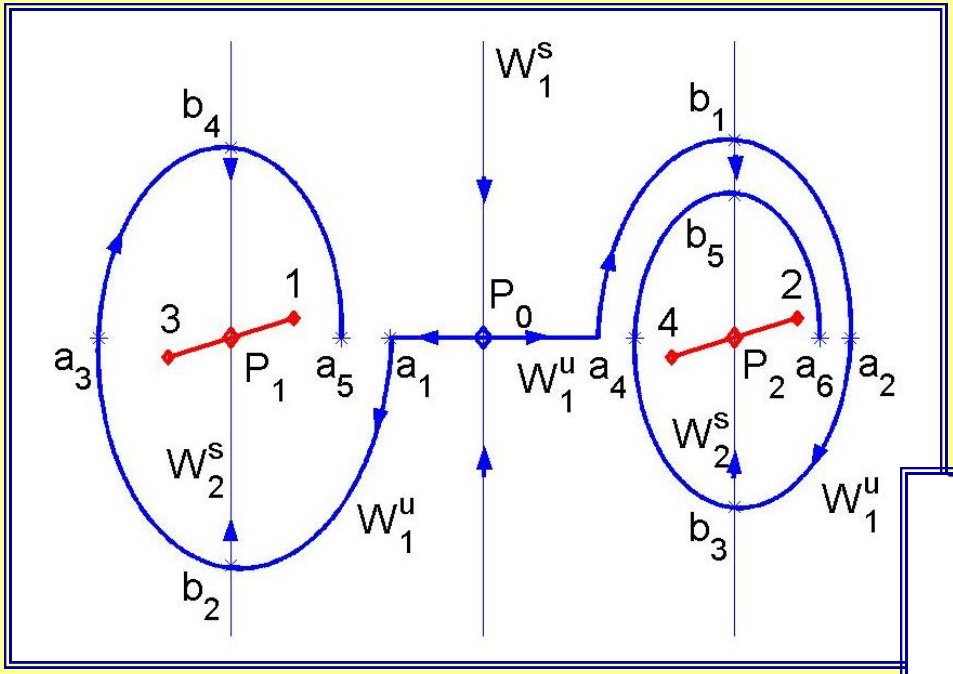
4T

K_1=0.145
K_{40}=1.9333

T= 3949.9
λ= -0.632

The 4T periodic orbit transforms from a simple closed loop into a *trefoil knot* in three-dimensional space because the period-doubling bifurcations in a chaotic system force the trajectory to twist and thread through itself according to the attractor's topological template.

Cascade of period doubling bifurcations II

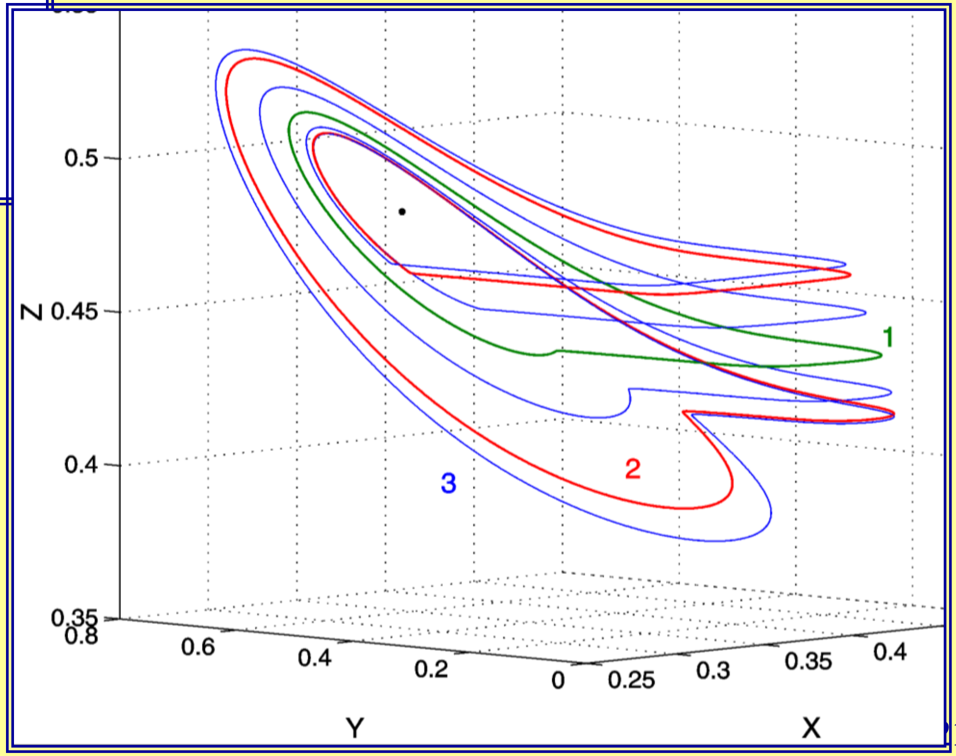


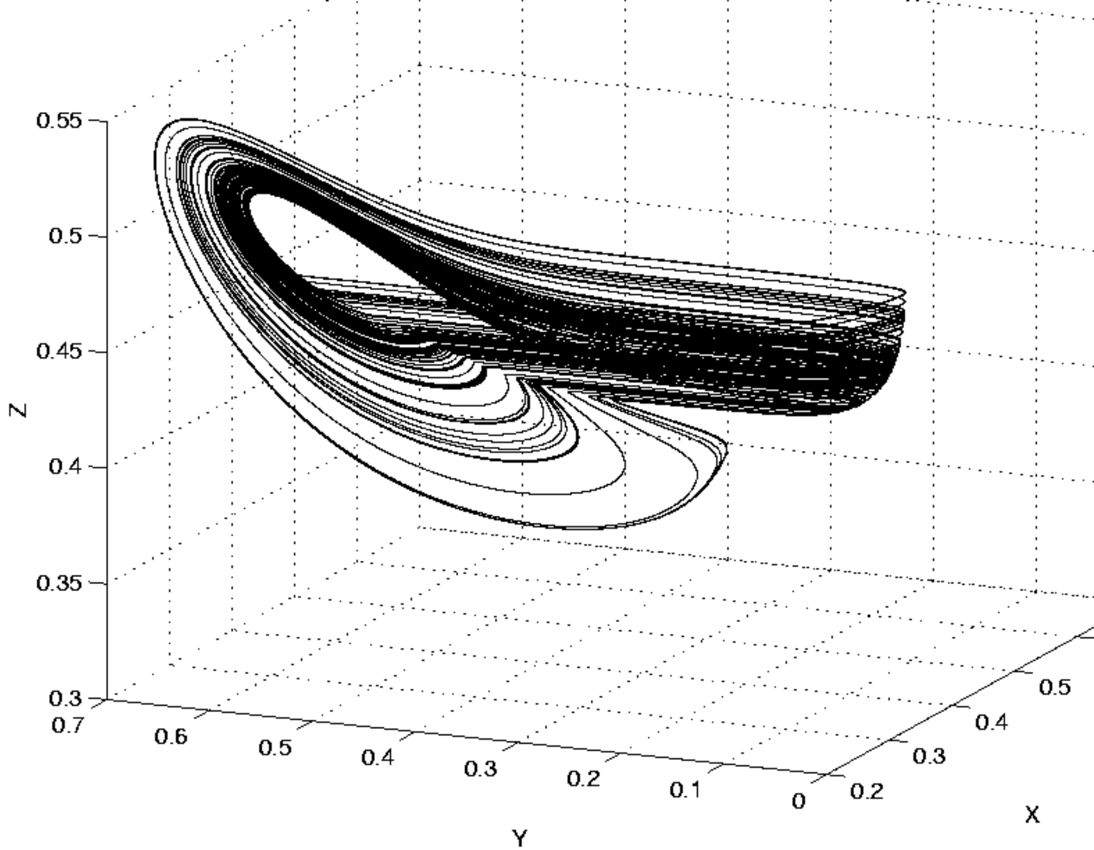
2D Poincare map
near an attracting set
$\Lambda_4 = M_1 \cup M_2 \cup \varphi_4$

Möbius orbits

$K_1=0.144$ $K_{40}=1.92$

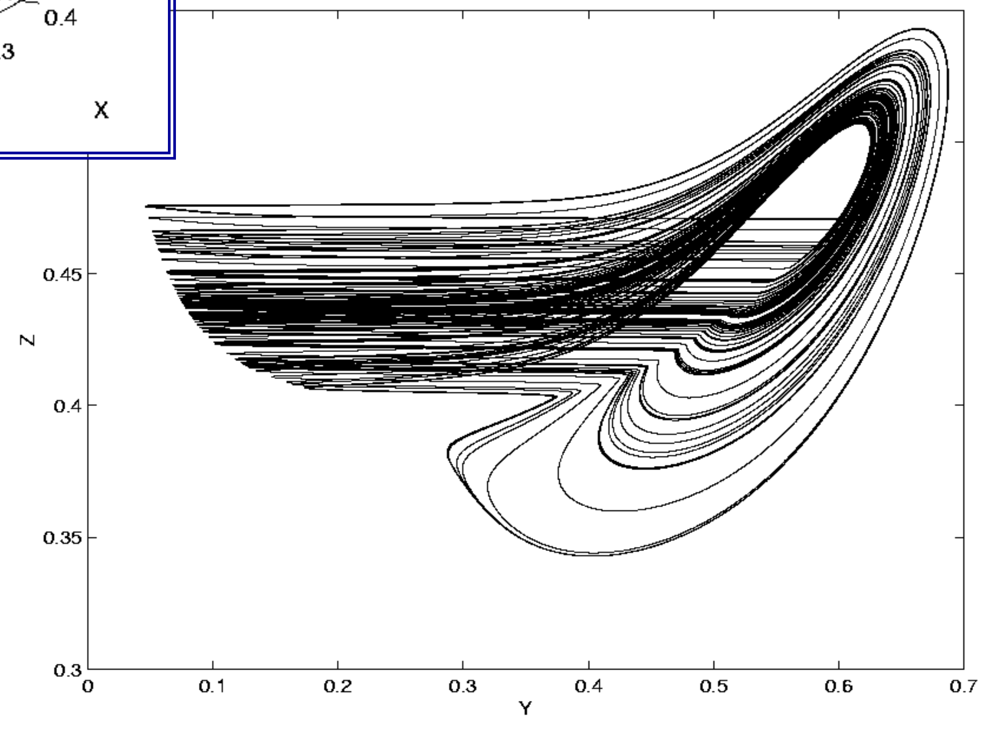
Periods:	Multipliers:
1047.9 (1)	-1.21 (1)
2008.0 (2)	-5.79 (2)
4057.8 (3)	-13.65 (3)





Strange attractor

$K_1=0.144, \quad K_{40}=1.92$

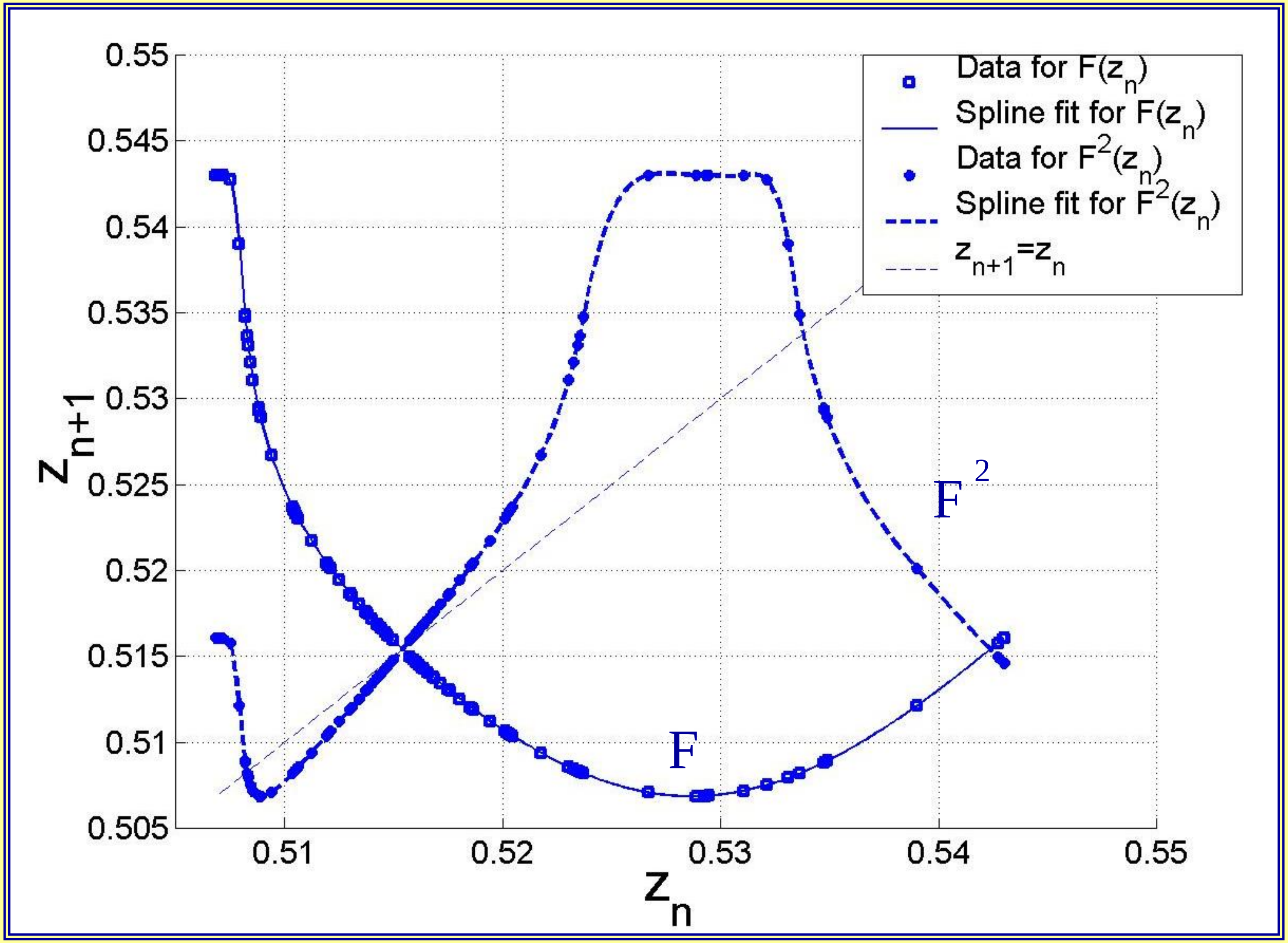


Liapunov dimension

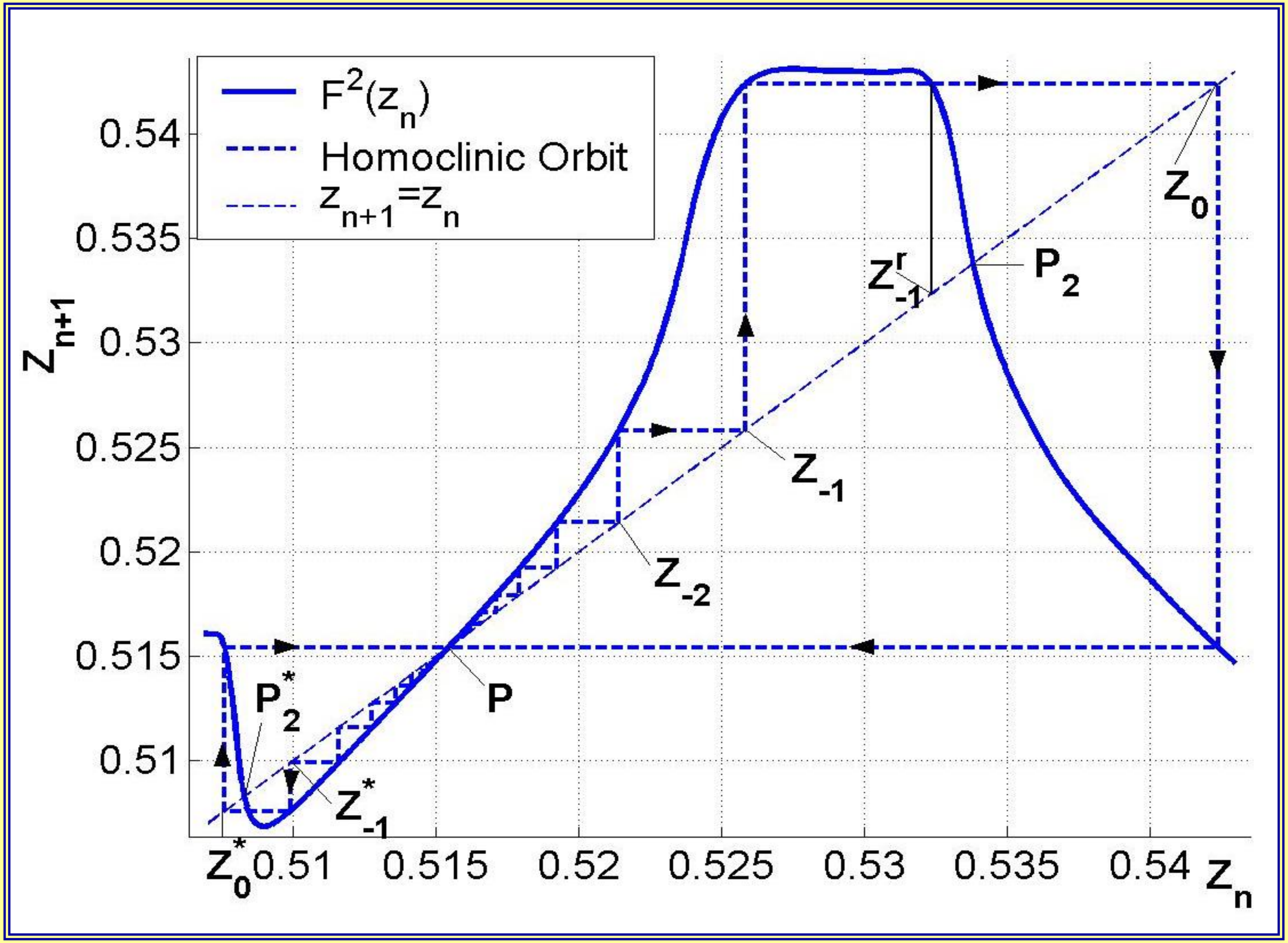
$$d_L = 2 - \sigma_1 / \sigma_3, \ \sigma_1 > 0 > \sigma_3, \ \sigma_2 = 0$$

$2.075838 < d_L(\Lambda) < 3$

Cascade of period doubling bifurcations: Poincare maps $F(z_n)$ and $F^2(z_n)$



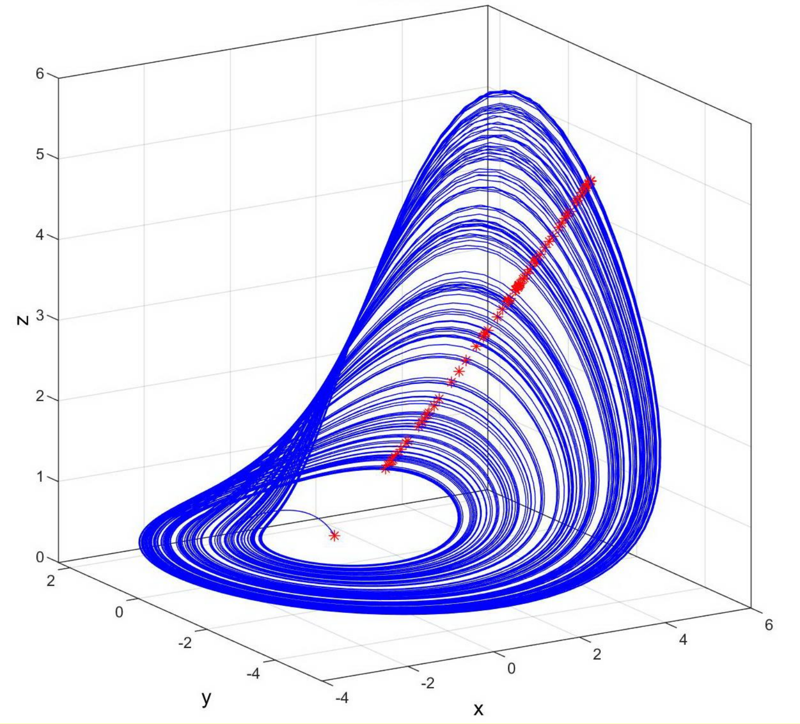
Cascade of period doubling bifurcations: Homoclinic orbit

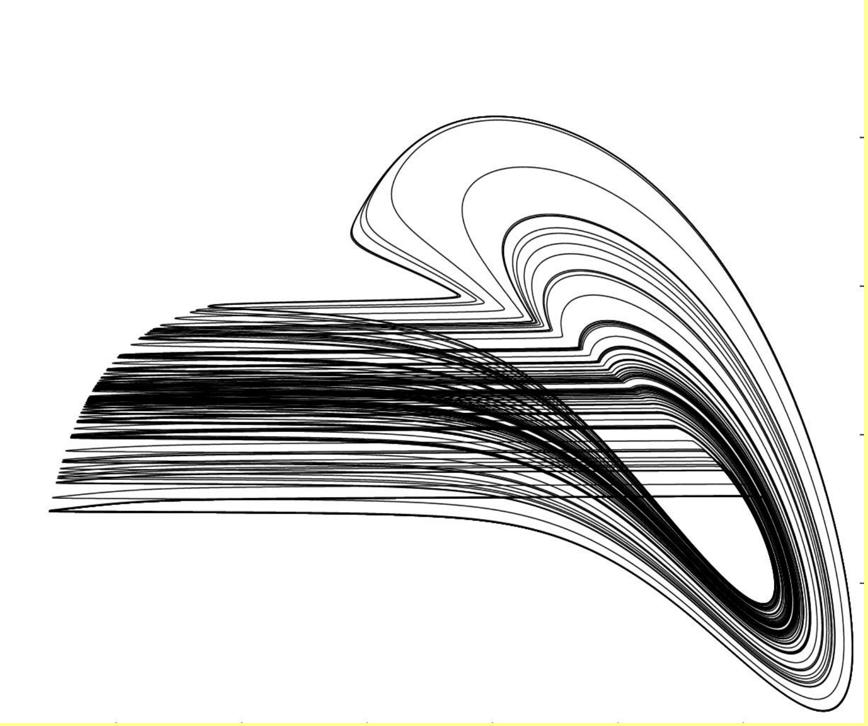


Conclusions

- In Rössler model, we considered the topological structure of an attractor that consists of infinite number of saddle periodic trajectories.
- Our analysis of chaos in the 3-D kinetic model of hydrogen oxidation has shown that there exists a very complicated dynamics.
- We have found successive period doubling bifurcations in which the flow becomes progressively more complex until the strange attractor appears for $\varepsilon < \mu$.
- We believe that the results obtained are of importance for understanding the reasons of chaotic dynamics in different heterogeneous catalytic systems.

Динамика подарит нам аттрактор!
В нем прорези для крыл, карман для звезд
и капюшон, чтоб не пожечь волос
об солнце, что у нас над головами.
И впереди теперь лишь путь познанья
Топчи его, топчи, как ветер знамя!





Спасибо за внимание!