Асимптотики линейных и некоторых нелинейных (псевдо)дифференциальных уравнений с пространство-локализованными правыми частями

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на основе совместных работ с
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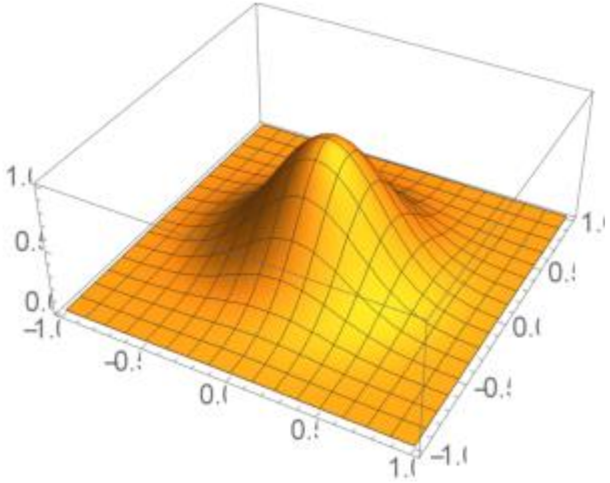
Formulation of the problem:

A time-harmonic spatially localized source

$$\frac{1}{c^2(x)}\frac{\partial^2 v}{\partial t^2} - \triangle v = e^{i\omega t}F(\frac{x-\xi}{h}), \quad x \in \mathbb{R}^n$$



$$v = e^{i\omega t}u(x) \quad \Longrightarrow$$

$$-\frac{1}{\omega^2}\triangle u - \frac{1}{c^2(x)}u = \frac{F}{\omega^2}(\frac{x-\xi}{h}) \quad \Longrightarrow$$

Far filed asymptotics

$$(-h^2\triangle u - V(x))u = f(\frac{x-\xi}{h}), \quad h \ll 1$$

More general problems:

Let $\widehat{\mathcal{H}} = \mathcal{H}(x, \hat{p}, h), \qquad x = (x_1, \ldots, x_n), \quad \hat{p} = (\hat{p}_1, \ldots, \hat{p}_n), \quad \hat{p}_j = -ih\frac{\partial}{\partial x_j}$

be a self adjoint operator with a real-valued symbol

$$\mathcal{H}(x, p, h) \in C^\infty([0,1]; S^\infty(\mathbb{R}^{2n}_{(x,p)})), \qquad \mathcal{H} = H(x,p) + hH_1(x,p) + \ldots .$$

h be a small positive parameter.

The equation:

$$\widehat{\mathcal{H}}u = f \equiv f(\frac{x-\xi}{h}),$$

here

$$f(y,h) = \Big(\frac{i}{2\pi h}\Big)^{n/2} \int_{\mathbb{R}^n} e^{ip\cdot y} \tilde{f}(p) dp_1 \ldots dp_1, \quad \arg i = \frac{\pi}{2}.$$

The function $A(p)$ is a smooth one.

If $\tilde{f} = 1$, than $f = \delta(x - \xi) \Longrightarrow \quad u$ is the Green function

Examples of operators

(1) The Helmholtz equation: $H(p,x) = p^2 - n^2(x)$.

(2) The generalized Helmholtz equation: $H(p,x,h) = \langle p, A(x)p \rangle - n^2(x)$, $n^2(x)$ be smooth, $n^2 \to \text{const} > 0$, as $|x| \to \infty$, $g(x)$ be a positive smooth matrix, $g \to g_0 = \text{const} > 0, \quad$ as $|x| \to \infty$.

(3) 3-D stationary Schrodinger equation with a Coulomb potential $H(p,x) = p^2 + \gamma/|x| - E$, E is the energy.

(4) The water wave equation: $H(p,x,h) = |p| \tanh(|p| D(x)| - \omega^2$, $D(x)$ is a smooth depth, ω is a frequency.

(5) The the fractional Laplace-type equation: $H(p,x,h) = b(x)|p|^\alpha - E$, $b(x) > 0$ is a smooth function, $E > 0$.

Much more general cases: systems of (pseudo)differential equations:

$\mathcal{H}(x,p,h)$ is $m \times m$ matrix- valued symbol,

$$\mathcal{H}(x,p,h) = \mathcal{H}_0(x,p) + h\mathcal{H}_1(x,p) + h^2\mathcal{H}_2(x,p) + \dots,$$

$f(y)$ is fast decaying $m-D$ vector function; all eigenvalues $H_1(x,p), \dots H_k, (x,p)$, $k \leq m$ are smooth functions,

$H_j \neq H_l$ (The effect of changing the multiplicities of terms is absent: $H_j \neq H_l$.

Examples: Maxwell and Dirac systems of equations, shallow water equation system

The Dirac equation in $\mathbb{R}^3_x$, m=4

$$\hat{\mathcal{H}} = c\boldsymbol{\alpha}\hat{\boldsymbol{\pi}} + \beta mc^2 + (e\Phi - \mathcal{E})\mathbf{E}, \qquad \mathcal{H} = \mathcal{H}^0 = c\boldsymbol{\alpha}\boldsymbol{\pi} + \beta mc^2 + (e\Phi - \mathcal{E})\mathbf{E}$$

$$\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \alpha_3\}, \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}.$$

Here
$\hat{\boldsymbol{\pi}} = \hat{\mathbf{p}} - (e/c)\mathbf{A}$, $\qquad \hat{\mathbf{p}} = -i\hbar\frac{\partial}{\partial x}$, $\qquad \boldsymbol{\pi} = p - (e/c)\mathbf{A}$

$(\mathbf{A}, \Phi(x))$ is the 4-D vector and scalar potentials of the electromagnetic field,

E is the 2×2 the unit matrix,

$\mathcal{E}$ is the energy,

$\hbar$, $e < 0$, m are the Plank constant, electron charge and mass,

σ_i, $i = 1, 2, 3$ –are the standard Pauli matrices satisfying the equalities
$\sigma_i\sigma_j = \delta_{ij}E + i\varepsilon_{ijk}\sigma_k$ with the absolutely antisymmetric tensor ε_{ijk}.

NONLINEAR EXAMPLE: NONLINEAR SYSTEM OF SHALLOW WATER EQUATIONS in the basin with a gentle shore:

$$\begin{cases} \dfrac{\partial \eta}{\partial t} + \langle \nabla, (\eta + D(x)\mathbf{u} \rangle = \operatorname{Re}\Big[f_1\left(\dfrac{x - x^0}{l}\right) e^{\frac{2\pi i}{T} t} \Big], \\ \dfrac{\partial \mathbf{u}}{\partial t} + \langle \mathbf{u}, \nabla \rangle \mathbf{u} + g\nabla\eta = \operatorname{Re}\Big[f_2\left(\dfrac{x - x^0}{l}\right) e^{\frac{2\pi i}{T} t} \Big]. \end{cases} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad t \in \mathbb{R}$$

x^0 is the source location point

$\eta(x,t)$ is a free elevation, $\mathbf{u}(x,t) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is the velocity , $D(x)$ is the depth

The basin without perturbation: $D(x) \geq 0$, the coastline $\Gamma = \{D(x) = 0\}$

The free boundary problem $\Omega_t = \{x : D(x) + \eta(x,t) \geqslant 0\}$

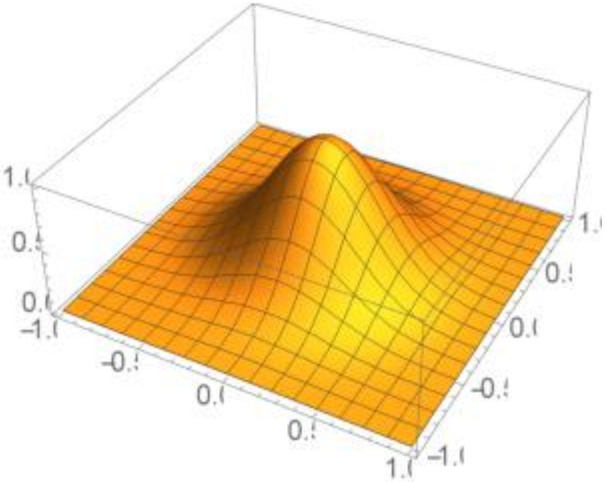
$f_j(z)$ are fast-decreasing (even finite) functions

l is the size of the source, T is the period, x^0 is the source location point

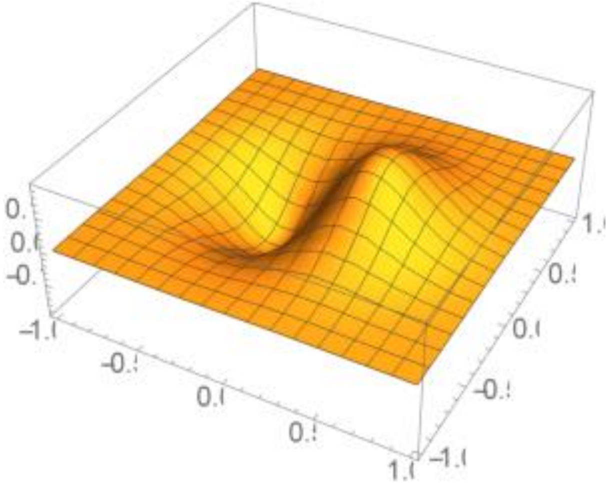
Examples of right hand side.

Directional radiation: let $n = 2$, $\rho = |p|$ and ψ be the polar angle of the momentum p,

$$(1) \qquad V(y) = \exp\left[-\frac{1}{2}\left(\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2}\right)\right] \qquad \text{then}$$



$$A(p) = ab\exp\left[-\frac{a^2p_1^2 + b^2p_2^2}{2}\right],$$



$$(2) \qquad (a_1y_1 + b_1y_2)\exp\left[-\frac{1}{2}\left(\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2}\right)\right] \qquad \text{then}$$

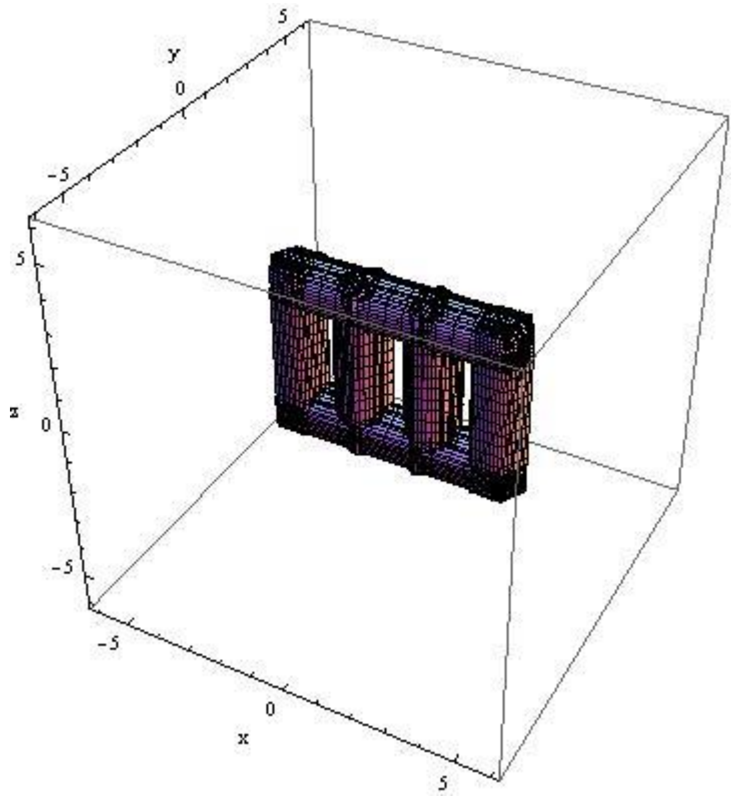
$$A(p) = iab(a^2a_1p_1 + b^2b_1p_2)\exp\left[-\frac{a^2p_1^2 + b^2p_2^2}{2}\right].$$

$$(3) \quad \text{antenna type} \qquad A(p) = ab\exp\left[-\frac{a^2p_1^2 + b^2p_2^2}{2}\right]\mathbf{e}(\psi), \quad \operatorname{supp}\mathbf{e}(\psi) \in [\psi^1, \psi^2].$$

The antenna

V.A. Kostin, D.V. Kostin, A.V. Kostin, some solutions and pictures D.S.Minenkov

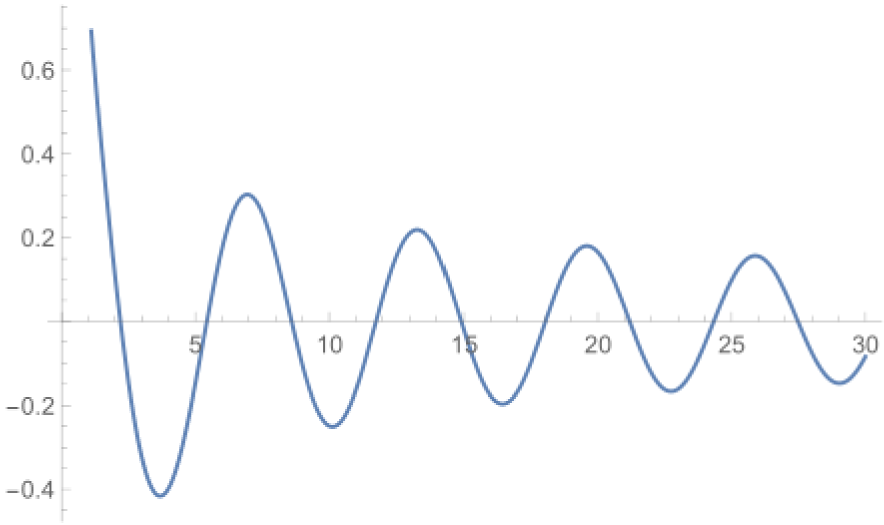
$f(y, h)$



Tubes are modeled by Gaussian exponentials and cutting functions along the axis based on the $\sin^2(\frac{y-a}{b})$ function + rotations and shifts

The Helmholtz equation with constant n and the Dirac delta function in the right hand side:

u is the Green function = the Hankel function $G(x,k) = \frac{i}{4}H_0^{(1)}(\frac{|x|}{h})$. The function has singularity in the point $x=0$ and has oscillations for large x.



The Helmholtz equation with variable coefficients:

J. B. Keller, 1962

V. M. Babich, 1964 (matching method , without focal points and caustics)

V. V. Kucherenko, 1968 (with focal points, the Maslov canonical operator)

Generalization:

L. Hörmander and J. J. Duistermaat, 1973, R. Melrose and G. A. Uhlmann, 1979 (pair of the manifolds, asymptotics with respect to smoothness)

B.Yu.Sternin, V.E.Shatalov, 1981 (some geometry)

A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, M. Rouleux 2017, (Doklady Mathematics, Solutions of stationary equations with the localized right hand side)

S. Yu. Dobrokhotov, V. E. Nazaikinskii, A. I. Shafarevich, 2016 (simplification of the canonical Maslov operator)

S. Yu. Dobrokhotov, D. S. Minenkov, M. Rouleux, 2018, (The Maupertuis-Jacobi correspondence in semiclassical stationary problems)

A. Yu. Anikin, S. Yu. Dobrokhotov, V. E. Nazaikinskii, M. Rouleux, Lagrangian manifolds and the construction of asymptotics for (pseudo)differential equations with localized right-hand sides, Theoret. and Math. Phys., 214:1 (2023), 1-23

LINEAR PROBLEMS

The usual additional conditions for the Helmholtz equation: the principle of limiting absorption $\sim$ to Sommerfeld condition

$$u = \lim_{\varepsilon \to +0} u_\varepsilon, \qquad (\widehat{\mathcal{H}} - i\varepsilon) u_\varepsilon(x, x_0) = f.$$

Additional conditions may be those that arise when constructing the Green function: the principle of limiting absorption $\sim$ to Sommerfeld condition for the Helmholtz equation.

We do not use such conditions and we speak about asymptotic solutions only.

The main idea (closed to idea of V.P.Maslov and V.V.Kucherenko) is based on consideration of the non stationary Cauchy problem

$$-ihv_t = \hat{\mathcal{H}}v, \quad v|_{t=0} = F(\frac{x-\xi}{h}),$$

and representation of solutions in the form of Duhamel integral

$$u = \frac{i}{h}\int_0^\infty v(x,\tau)d\tau$$

The aim: to construct asymptotic solution as $h \to +0$ and our conditions are related to assumptions that ensure (in a sense), **convergence of this integral**.

Нужные объекты (алгоритм построения асимптотики решения):

главный и субглавный символы (функции) $H(x,p), H_1(x,p)$ оператора $\widehat{\mathcal{H}}$

множество $L_0 : \{\xi, H(\xi,p) = 0\}$

семейство траекторий гамильтоновой системы с гамильтонианом $H(x,p)$, образующих лагранжево многообразие Λ

сужение преобразования Фурье от функции f на L_0 и ее продолжение на Λ,

канонический оператор Маслова на Λ +

его упрощение+ его глобальность и локальность

Reminder: WKB-asymptotics and Lagrangian manifolds:

$$\psi = a(x)e^{\frac{i}{h}S(x,t)}, \quad h \to +0$$

the (smooth) Lagrangian manifold in $2n - D$ phase space $\mathbb{R}^{2n}_{px}$

$$\Lambda^n = (p = \nabla S) = \{p = P(\alpha),\, x = X(\alpha), \quad \alpha = \alpha_1, \dots, \dots, \alpha_n\}$$

$$dS = P\,dX, \quad a = \frac{A(\alpha)}{\sqrt{J(\alpha)}}\Big|_{\alpha=\alpha(x)}.$$

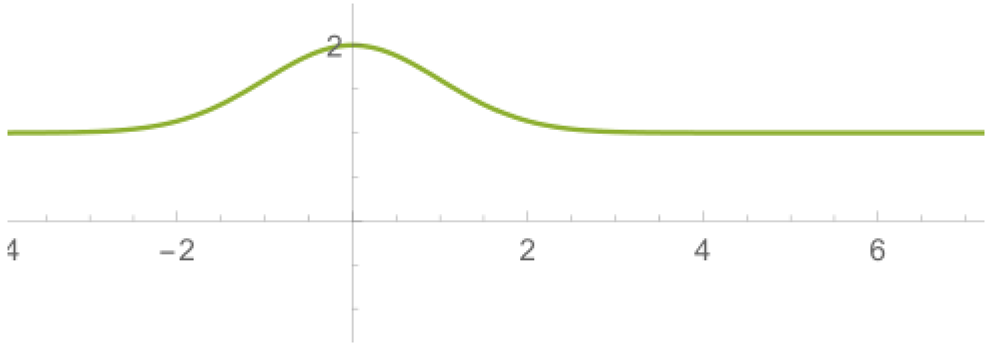
Here J is the Jacobian of projection $\Lambda^n \to \mathbb{R}^n_x$: $J(\alpha) = \det \frac{\partial X}{\partial \alpha}$ and $\alpha = \alpha(x)$ is the solution to the equations

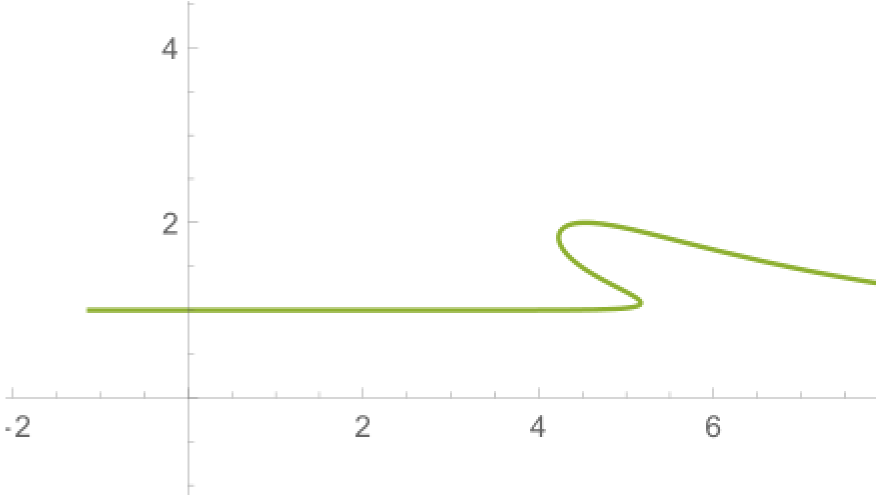
$$X(\alpha) = x \qquad \Longleftarrow \qquad \text{Parametric form}$$

the phase $S(x) = \int_{\alpha_0}^{\alpha(x,t)} PdX$,
(the solution to the Hamilton-Jacobi equation)

The Lagrangian singularities (focal points, caustics).

$$J(\alpha) = \det \frac{\partial X}{\partial \alpha} = 0$$





The catastrophe theory

(Thom, Poston, Stewart, Maslov, Arnold, Varchenko, Gusein-Zade, Lukin, Kryukovsky, Palkin, Rastyagaev, Il'in, Suleimanov ...)

The Maslov canonical operator:

WKB-type function near the regular points ($J(\alpha)=\det\frac{\partial X}{\partial\alpha}\neq 0$) and the integral representation near the focal points and caustics.

1-D case: the WKB representation in momentum coordinate+ inverse Fourier transform

$$\Psi(x)=\sqrt{\frac{i}{2\pi h}}e^{-i\frac{\pi}{2}m}\int\frac{A(\alpha)}{\sqrt{|\det\frac{\partial P(\alpha)}{\partial\alpha}|}}e^{\frac{i}{h}\left(\int_{\alpha_0}^{\alpha(p)}PdX-P(\alpha)(x-X(\alpha))\right)}\Big|_{\alpha=\alpha(p)}\quad dp,$$

here m is the Maslov index, $\alpha=\alpha(p)$ is the solution to equation:

$$P(\alpha)=p.$$

$$\Downarrow$$

two steps: 1) finding $\alpha(p)$ (could be not trivial), 2) integration over the momentum p

The main idea: let us pass to integration over α $\Longrightarrow$ the first step disappears

$$\Psi(x)=\sqrt{\frac{i}{2\pi h}}e^{-i\frac{\pi}{2}m}\int A(\alpha)\sqrt{|\det\frac{\partial P(\alpha)}{\partial\alpha}|}e^{\frac{i}{h}\left(\int_{\alpha_0}^{\alpha(p)}PdX-P(\alpha)(x-X(\alpha))\right)}\quad d\alpha,$$

n-D case:

mixed representation in the neighborhood of Lagrangian singularities.

Example $n = 2$. Canonical coordinates: (x_1, x_2), (x_1, p_2), (p_1, x_2), (p_1, p_2).

Th. *At least one of Jacobians* $\det \frac{\partial(X_1,X_2)}{\partial\alpha}$, $\det \frac{\partial(X_1,P_2)}{\partial\alpha}$, $\det \frac{\partial(P_1,X_2)}{\partial\alpha}$, $\det \frac{\partial(P_1,P_2)}{\partial\alpha}$ *is not equal to zero.*

Example of mixed representation in the case (p_1, x_2)

$$\Psi(x) = \sqrt{\frac{i}{2\pi h}} e^{-i\frac{\pi}{2}m} \int \frac{A(\alpha)\mathbf{e}(\alpha)}{\sqrt{|\det \frac{\partial(P_1(\alpha),X_2(\alpha))}{\partial\alpha}|}} e^{\frac{i}{h}\left(\int_{\alpha_0}^{\alpha}\langle P,dX\rangle - \langle P(\alpha),x-X(\alpha)\rangle\right)}\Big|_{\alpha=\alpha(p_1,x_2)}$$

here m is the Maslov index, $\alpha = \alpha(p)$ is the solution to equations:

$$P_1(\alpha) = p_1, \quad X_2(\alpha) = x_2.$$

two steps: 1) finding $\alpha(p_1, x_2)$ (could be not trivial),

2) integration over the momentum p_1

Generalization to n-D case: partial Fourier transform $(x,p) \rightarrow ((x^I, p^{\bar{I}}), (p^I, x^{\bar{I}}))$
Example $n = 3$:

$$(x_1, x_2, x_3), (p_1, x_2, x_3), (x_1, p_2, x_3), (x_1, x_2, p_3),$$
$$(p_1, p_2, x_3), (p_1, x_2, p_3), (x_1, p_2, p_3), (p_1, p_2, p_3)$$
$$+$$

passage from integration by $p^{\bar{I}}$ integration by some coordinates $\alpha_{\bar{I}}$ on Λ^n.

The Maslov canonical operator:

$$K_\Lambda^h A = \sum_{\Omega_I} \psi_I$$

Objects and assertions:

$\mathcal{H}(x,p,h)$, symbol

$H(x,p)=\mathcal{H}(x,p,0)$, principle symbol

$H_{\mathrm{sub}}(x,p)=\dfrac{\partial\mathcal{H}}{\partial h}(x,p,0)+\dfrac{i}{2}\sum_{i=1}^{n}\dfrac{\partial^2 H}{\partial x_j\partial p_j}(x,p)$ subprinciple symbol

Assertions

I. $|H(\xi,p)|\geqslant C|p|^{-N}$, $C,N>0$, $|p|$ is large

II. Solutions $(x,p)=(X(q,\tau),P(q,\tau))$ of the Cauchy problem for the Hamilton system

$$\dot{x}=H_p(x,p),\quad \dot{p}=-H_x(x,p),\quad x|_{\tau=0}=\xi,\quad p|_{\tau=0}=q\in\mathbb{R}^n,$$

defined for all $\tau\in[0,\infty)$ with $q\in L_0$, where $L_0=\{q\in\mathbb{R}^n|H(\xi,q)=0\}$.

III. For any $R>0$, there exists such a t_R that $|X(q,\tau)|>R$ for $\tau>t_R$ and $q\in L_0$.

Lagrangian manifolds Λ_0 and Λ

$$\Lambda_0 = \{x = \xi, p = q \in \mathbb{R}^n\} \in \mathbb{R}^{2n}_{x,p},$$

$$\Lambda = \{(x = X(q,\tau), t = \tau, p = P(q,\tau), E(q) - H(\xi, q))\} \in \mathbb{R}^{2n+2}_{x,t,p,E},$$

Expression of the right-hand side via the canonical operator on Λ_0

$$f\left(\frac{x-\xi}{h}\right) = \frac{e^{i\pi n/4}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle p, x-\xi\rangle} \widetilde{f}(p)\, dp = h^{n/2}[\mathcal{K}_0 \widetilde{f}](x,h),$$

where $\mathcal{K}_0$ is the Maslov canonical operator on the Lagrangian manifold Λ_0 with the measure $d\mu_0 = dq_1 \wedge \cdots \wedge dq_n$ and the Fourier transform

$$\widetilde{f}(p) = \frac{e^{i\pi n/4}}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\langle p, y\rangle} f(y)\, dy$$

Construction of an asymptotic solution

A partition of unity on Λ_0 permits one to write the solution as the sum of solutions of two such equations in one of which $\operatorname{supp} \widetilde{f} \cap L_0 = \varnothing$, while in the other the function $\widetilde{f}$ is compactly supported.

Theorem 1. ***(“Elliptical part” of the “near field”).*** *If* $\operatorname{supp} \widetilde{f} \cap L_0 = \varnothing$, *then there exists an asymptotic solution*

$$\psi(x,h) = h^{n/2}[\mathcal{K}_0 B](x,h)$$

where the amplitude B *is an element of the space* $\mathcal{S}(\Lambda_0; h)$ *of smooth functions on the parameter* $h \in [0,1]$ *ranging in the space* $\mathcal{S}(\Lambda_0)$, *and its leading term has the form*

$$B_0(q) \equiv B(q,0) = \frac{\widetilde{f}(q)}{H(\xi,q)}.$$

Lemma 1. *Let* $\widetilde{f}(q)$ *be a compactly supported function on* Λ_0*. Then on the manifold* Λ *there exists a smooth cutoff function* χ *such that*

(i) $\chi = 1$ *in some neighborhood of the set* $M = \big(L_0 \times [0,\infty)\big) \cup \big(\operatorname{supp} \widetilde{f} \times \{0\}\big)$.

(ii) The set $\pi_\Lambda^{-1}(K) \cap \operatorname{supp}\chi$ *is compact for any compact set* $K \subset \mathbb{R}^{n+1}_{x,t}$.

(iii) The set $\pi_\Lambda(\operatorname{supp}\chi)$ *has the following property: if* $(x,t) \in \pi_\Lambda(\operatorname{supp}\chi)$ *and* $|x| \le R$*, then* $-1 \le t \le \tau_{R+1} + 1$.

$$\Lambda = \{(x = X(q,\tau), t = \tau, p = P(q,\tau), E(q) - H(\xi, q))\} \in \mathbb{R}^{2n+2}_{x,t,p,E},$$

Let $\mathcal{K}\colon C^\infty_{pr}(\Lambda; h) \to \mathcal{F}^h(\mathbb{R}^{n+1}_{(x,t)})$ be the canonical operator on the Lagrangian manifold Λ with the measure $d\mu = d\mu_0(q) \wedge d\tau$.

Theorem 2. *(**The first main Theorem**) If $\widetilde{f}$ is compactly supported, then the initial equation has an asymptotic solution of the form*

$$\psi(x,h) = ih^{\frac{n}{2}-1} \int_0^\infty [\mathcal{K}(\chi A)](x,t,h)\, dt,$$

where χ is the function described in Lemma 1 and the leading term of the amplitude $A \in C^\infty(\Lambda \times [0,1])$ has the form

$$A_0(q,\tau) \equiv A(q,\tau,0) = e^{-i\int_0^\tau H_{\mathrm{sub}}(X(q,\tau'),P(q,\tau'))\,d\tau'} \widetilde{f}(q).$$

Theorem 3. *If the function $\widetilde{f}$ is compactly supported and* $\operatorname{supp} \widetilde{f} \cap L_0 = \varnothing$*, then the asymptotic solutions constructed in Theorems 1 and 2 coincide modulo $O(h^\infty)$.*

Principal type condition, the "main" Lagrangian manifold Λ_+, and the form of the asymptotic solution outside of the vicinity of the point $x = \xi$

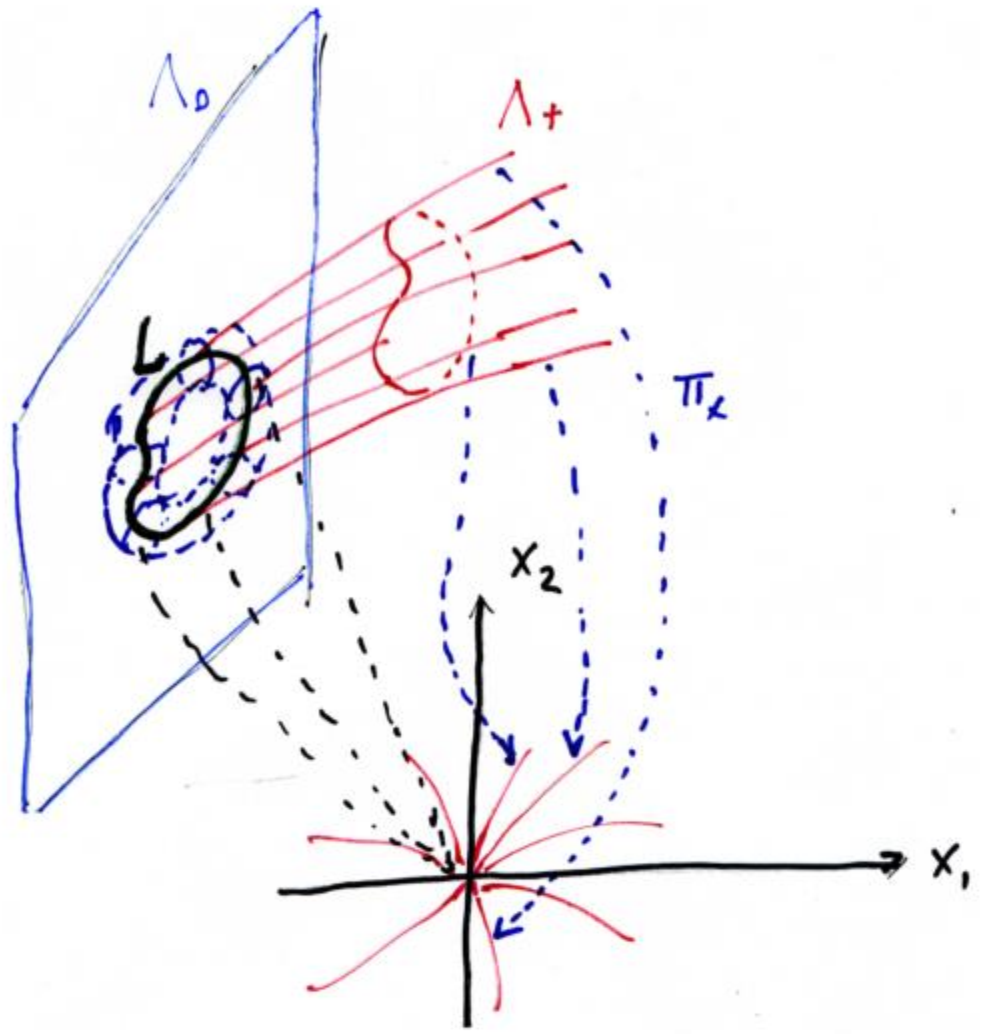
Assume that the following condition is satisfied:

IV. $H_p(\xi, p) \neq 0$ for $p \in L_0$.

Then L_0 is an $(n-1)$-dimensional submanifold in Λ_0. The points of L_0 will be denoted by α, and the corresponding values of $q \in \Lambda_0$, by $q(\alpha)$. We construct $n - D$ Lagrangian manifold

$$\Lambda_+ = \{x = X(q(\alpha), \tau), p = P(q(\alpha), \tau)\} \in \mathbb{R}^{2n}_{x,p}.$$

We equip Λ_+ with the measure $d\mu_+ = d\sigma_0(\alpha) \wedge d\tau$, where $d\sigma_0$ is the volume form on L_0 uniquely determined by the equality $d\mu_0 = d\sigma_0 \wedge dH$ on the tangent spaces to Λ_0 at the points of L_0.



Λ_0
Λ_+
L
π_x
x_2
x_1

Let

$$\mathcal{K}_+ \colon C^\infty_{pr}(\Lambda_+, \mathbb{R}^n_x \setminus \{\xi\}; h) \to \mathcal{F}^h(\mathbb{R}^n_x \setminus \{\xi\})$$

be the Maslov canonical operator on the manifold Λ_+ with the measure $d\mu_+$.

Theorem 4. *(**The main second theorem**). Assume that the function $\widetilde{f}$ is compactly supported and condition* (IV) *is satisfied. Then the asymptotic solution can be represented in the domain $\mathbb{R}^n_x \setminus \{\xi\}$ as*

$$\psi(x,h) = (2\pi)^{\frac{1}{2}} e^{\frac{\pi i}{4}} h^{\frac{n-1}{2}} [\mathcal{K}_+ Q](x,h) + O(h^\infty),$$

where the leading term of the amplitude $Q \in C^\infty_{pr}(\Lambda_+, \mathbb{R}^n_x \setminus \{\xi\}; h)$ has the form

$$Q_0(\alpha,\tau) \equiv Q(\alpha,\tau,0) = A_0(q(\alpha),\tau).$$

+ Simplifications, computational tricks and representations in the form of special functions of a complex argument

Wave part outside the focal points

$$u(x,h) = \sqrt{2\pi} e^{\frac{i\pi}{4}} h^{\frac{n-1}{2}} \sum_j e^{\frac{i}{h}S(\alpha_j,\tau_j) - \frac{i\pi}{2}m_j} \frac{A_0(q(\alpha_j),\tau_j)}{\sqrt{|J(\alpha_j,\tau_j)|}}\Big|_{(\alpha_j,\tau_j)=(\alpha_j(x),\tau_j(x))},$$

$$S(\alpha,\tau) = \int_0^\tau p\cdot H_p\Big|_{x=X(q(\alpha),\eta),p=P(q(\alpha),\eta)} d\eta, \qquad J = \det(\frac{\partial X}{\partial(\alpha,\tau)}),$$

the corresponding Maslov index is

$$m_j = -\frac{n+1}{2} + \frac{1}{\pi}\lim_{\varepsilon\to+0} \operatorname{Im}\int_0^{\tau_j} \frac{1}{J_\varepsilon}\frac{\partial J_\varepsilon}{\partial\tau}(\alpha_j,\eta)\, d\eta,$$

$$J_\varepsilon = \frac{\partial(X(q(\alpha),\tau) - i\varepsilon P(q(\alpha),\tau))}{\partial(\alpha,\tau)},$$

$(\alpha_j(x),\tau_j(x))$ is (are) the solution(s) to equations $X(q(\alpha),\eta) = x$

Directional pattern of generated waves

One has in regular points the WKB-amplitude $\mathcal{A}$ of the asymptotic solution:

$$\mathcal{A}(x) = \frac{\tilde{f}(q(\alpha))}{\sqrt{|J(q(\alpha), \tau)|}}\Big|_{\alpha=\alpha(x),\tau=\tau(x)}, \quad q(\alpha) \in \{q(\alpha) : H_0(\xi, q(\alpha)) = 0\}$$

for the Helmholtz equation $H = p^2 - \mathbf{n}^2(x, \gamma)$ and

$L_0 = \{q(\alpha)^2 = \mathbf{n}^2(\xi, \gamma)\}$ is $n - D$ sphere,

$$q = \mathbf{n}(\xi, \gamma)\,(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)^T$$

γ is a parameter, e.g. could be frequency or energy

Therefore, the radiation pattern is determined by the projection of the Fourier transform onto the sphere $L_0(\gamma)$

The Maupertui-Jacobi correspondence: $H(x,p) \to |p|C(x), \quad S(\tau,\phi) \to \tau.$

The Hamiltonian for the Helmholtz equation

$$H = p^2 - V(x) = 0 \quad \Longrightarrow \quad \frac{p^2}{V(x)} = 1 \quad \Longrightarrow \quad \frac{|p|}{\sqrt{V(x)}} = 1 \quad \Longrightarrow$$

Fermat variational principle for constructions of trajectories

$$\int_\xi^x \sqrt{V} d\tau \to \quad \text{extremum}$$

S. Yu. Dobrokhotov, D. S. Minenkov, M. Rouleux, The Maupertuis-Jacobi Principle for Hamiltonians of the Form F(x,|p|) in Two-Dimensional Stationary Semiclassical Problems, Math. Notes, 97:1 (2015), 42-49

S. Yu. Dobrokhotov, I. A. Nosikov, A. A. Tolchennikov, The Jacobi-Maupertuis principle and Fermat variational principle in the problem of short-wave asymptotics in the solution of the Helmholtz equation with a localized source, Comput. Math. Math. Phys., 65:4 (2025), 739-753

Example: the inhomogeneous Schrödinger equation with the Coulomb repulsive potential and localised right hand side (the Green function type problem)

S. Yu. Dobrokhotov, A. A. Tolchennikov, Keplerian Trajectories and an Asymptotic Solution of the Schroedinger Equation with Repulsive Coulomb Potential and a Localized Right-Hand Side, Russ.J.Math. Phys., Vol. 29, No. 4 (2022), 456-466

$$\left(-h^2\Delta + \frac{\gamma}{|x|} - E\right)\psi(x) = F\left(\frac{x - x^0}{h}\right), \quad x \in \mathbb{R}^3,$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

γ, E, h are positive constants, $h \ll 1$,

$F(y)$ is a smooth fast-decreasing function

$$x_0 = \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix}, b > 0$$

+ absorption conditions at infinity

The Lagrangian manifold via Keplerian trajectories

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{\gamma^2}{4bE^2} \begin{pmatrix} (\operatorname{ch}\xi_0 + \operatorname{ch}\beta)(\operatorname{ch}\xi + \operatorname{ch}\beta) + \operatorname{sh}\xi_0 \operatorname{sh}\xi \operatorname{sh}^2\beta \\ [\operatorname{sh}\xi_0 \operatorname{sh}\beta(\operatorname{ch}\xi + \operatorname{ch}\beta) - \operatorname{sh}\xi \operatorname{sh}\beta(\operatorname{ch}\xi_0 + \operatorname{ch}\beta)]\cos\theta \\ [\operatorname{sh}\xi_0 \operatorname{sh}\beta(\operatorname{ch}\xi + \operatorname{ch}\beta) - \operatorname{sh}\xi \operatorname{sh}\beta(\operatorname{ch}\xi_0 + \operatorname{ch}\beta)]\sin\theta \end{pmatrix}$$

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \frac{\gamma}{2b\sqrt{E}} \frac{1}{\operatorname{ch}\beta \operatorname{ch}\xi + 1} \begin{pmatrix} (\operatorname{ch}\xi_0 + \operatorname{ch}\beta)\operatorname{sh}\xi + \operatorname{sh}\xi_0 \operatorname{ch}\xi \operatorname{sh}^2\beta \\ [\operatorname{sh}\xi_0 \operatorname{sh}\xi \operatorname{sh}\beta - (\operatorname{ch}\xi_0 + \operatorname{ch}\beta)\operatorname{ch}\xi \operatorname{sh}\beta]\cos\theta \\ [\operatorname{sh}\xi_0 \operatorname{sh}\xi \operatorname{sh}\beta - (\operatorname{ch}\xi_0 + \operatorname{ch}\beta)\operatorname{ch}\xi \operatorname{sh}\beta]\sin\theta \end{pmatrix}$$

$$t = q(\operatorname{ch}\beta \operatorname{sh}\xi + \xi - t_0) \qquad q = \tfrac{a^{\frac{3}{2}}}{\sqrt{2\gamma}},\ t_0 = \operatorname{ch}\beta \operatorname{sh}\xi_0 + \xi_0.$$

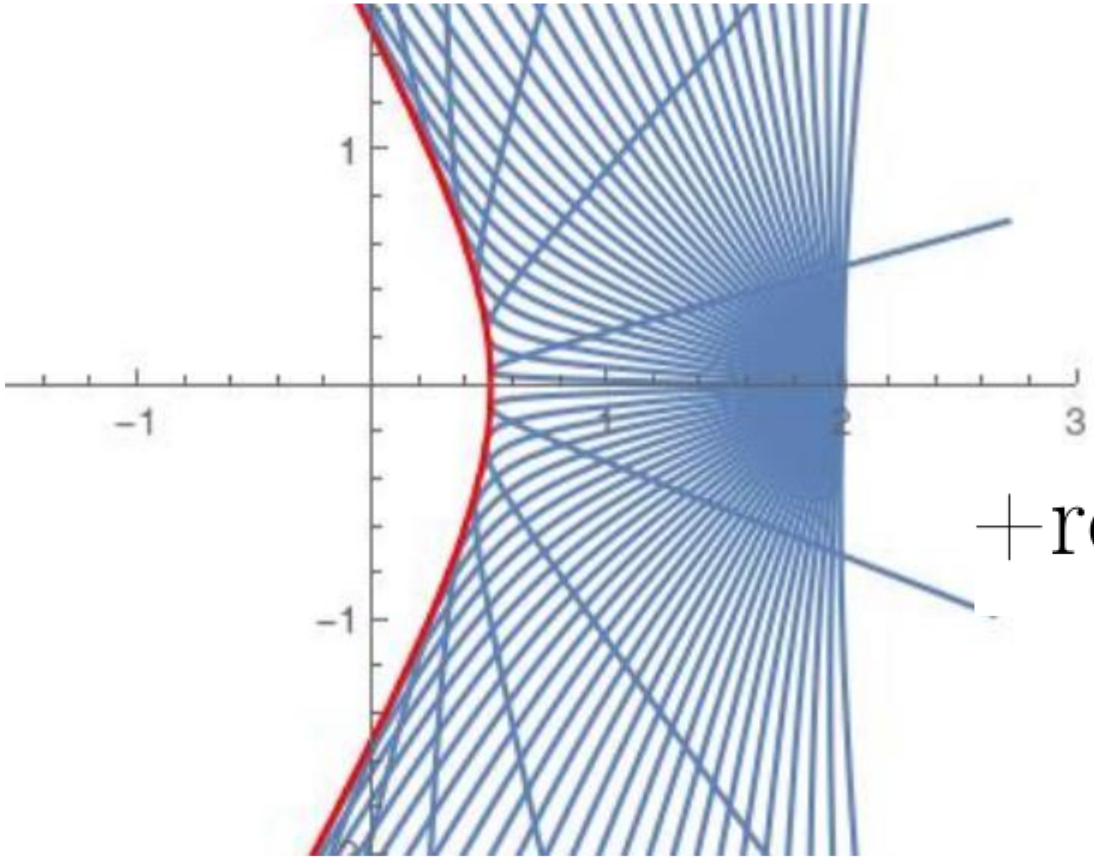
$$\operatorname{sh}\beta = -A\sin\psi,\ \operatorname{ch}\beta = \sqrt{1 + A^2\sin^2\psi}$$

$$A = \tfrac{2Fb\sqrt{E}}{\gamma}$$

$$\operatorname{sh}\xi_0 = \frac{A\cos\psi}{\sqrt{1 + A^2\sin^2\psi}},\ \operatorname{ch}\xi_0 = \frac{\sqrt{1 + A^2}}{\operatorname{ch}\beta}$$

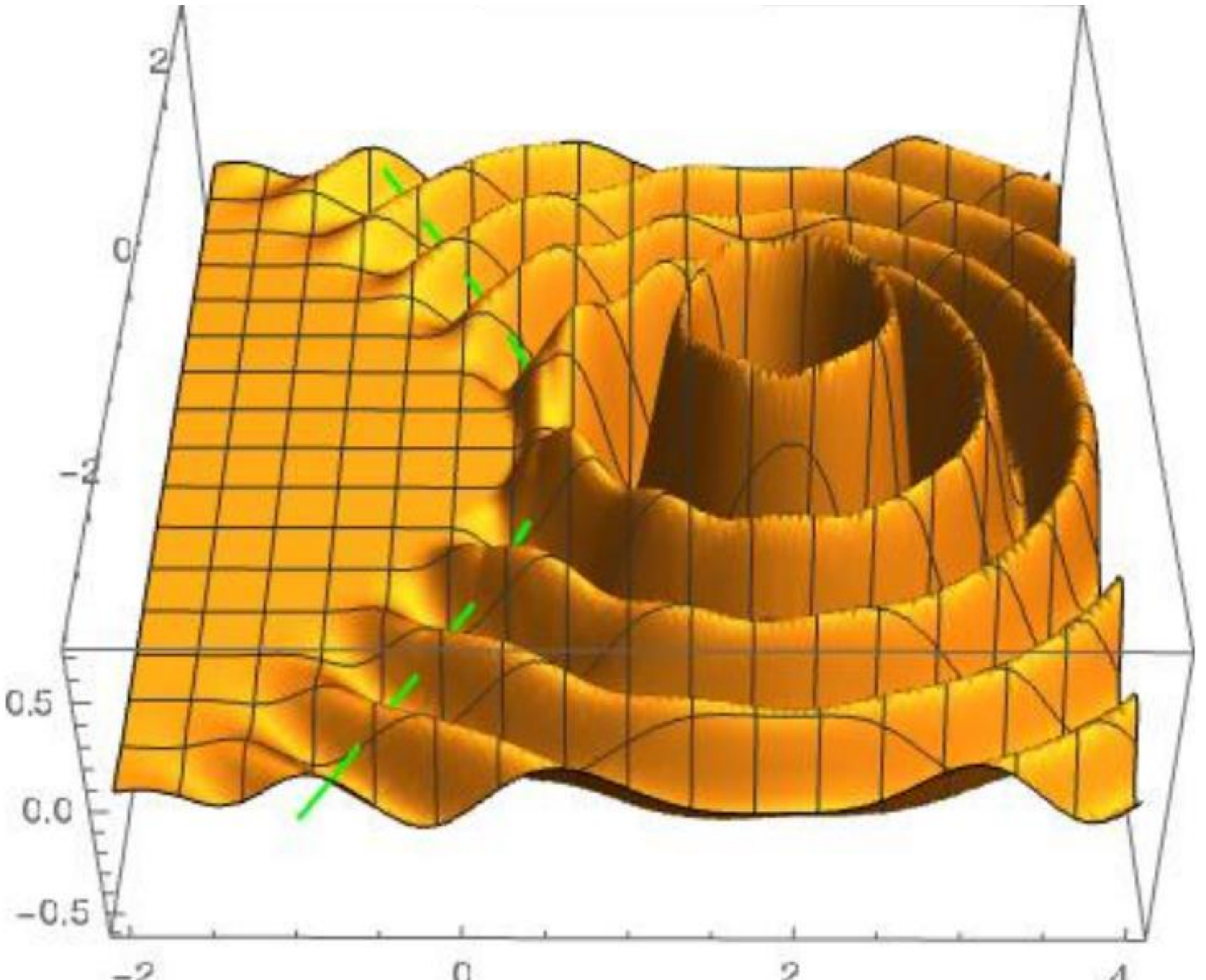
The caustic = the half of a bicuspid hyperboloid

$$\frac{\left(x_1 - \frac{b}{2}\right)^2}{\left(\frac{b}{2} - \frac{\gamma}{E}\right)^2} - \frac{x_2^2}{\left(\frac{b}{2}\right)^2 - \left(\frac{b}{2} - \frac{\gamma}{E}\right)^2} = 1$$



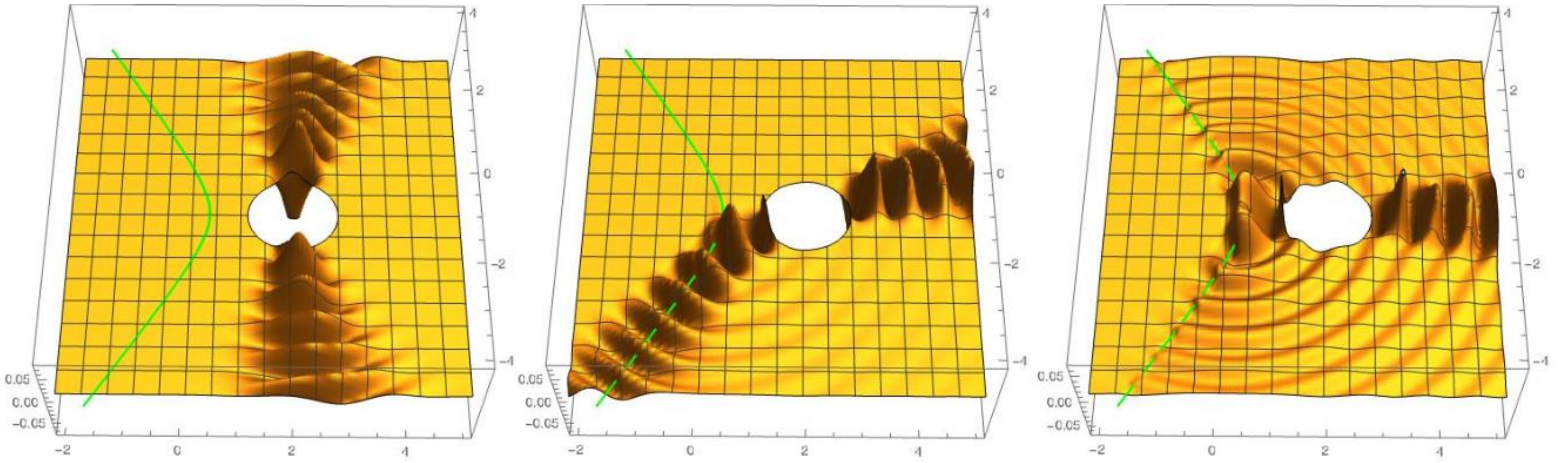
+rotation around the axis x_1

Example $V = e^{-|x|^2},\ b = 2,\ E = 2,\ \gamma = 1, x_3 = 0.$



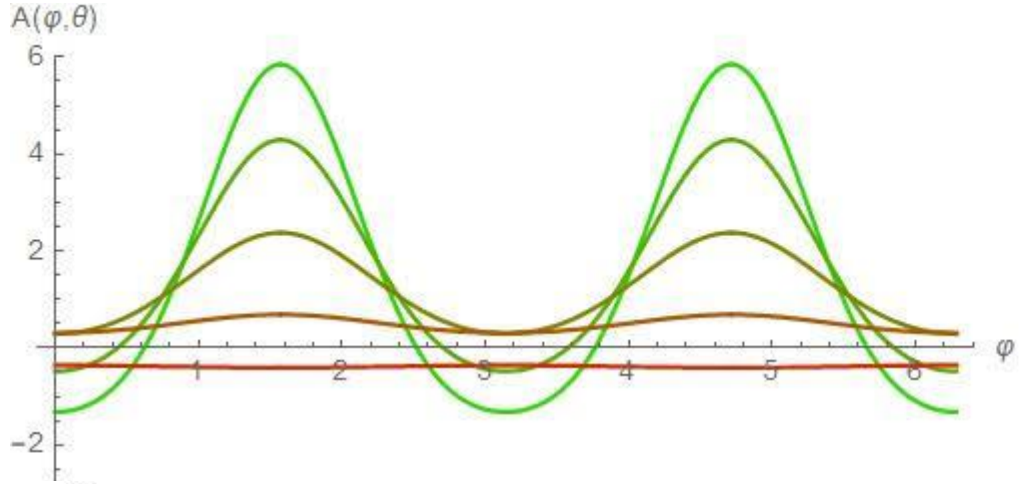
Example

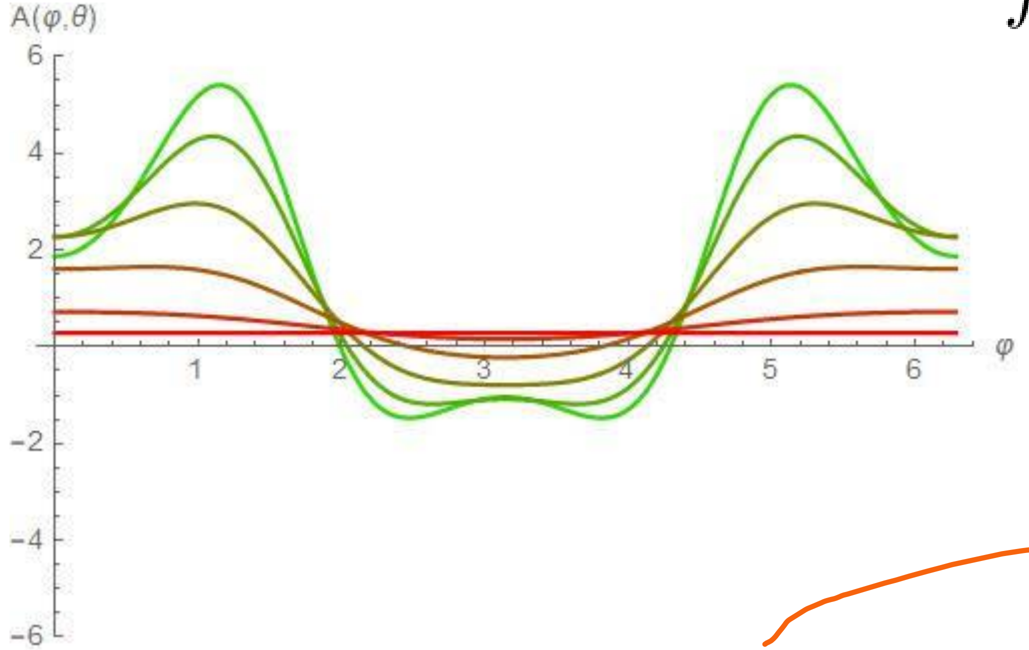
$$V(x) = e^{-\left(\frac{x_1 \cos\alpha + x_2 \sin\alpha}{a_1}\right)^2 - \left(\frac{-x_1 \sin\alpha + x_2 \cos\alpha}{a_2}\right)^2 - x_3^2},\ a_1/a_2 = 1/3,\ \alpha \in [0, 2\pi]$$

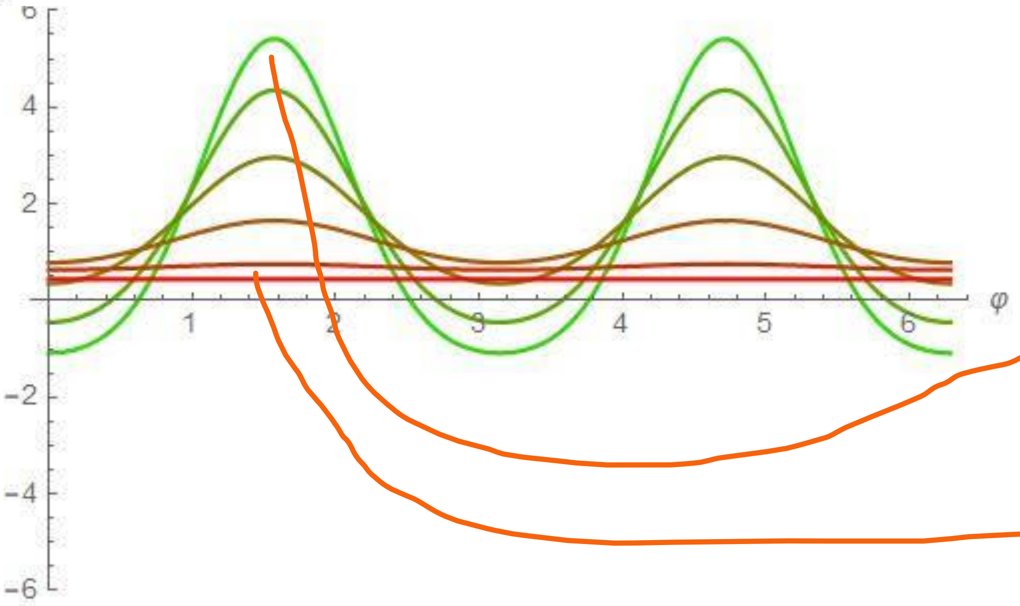


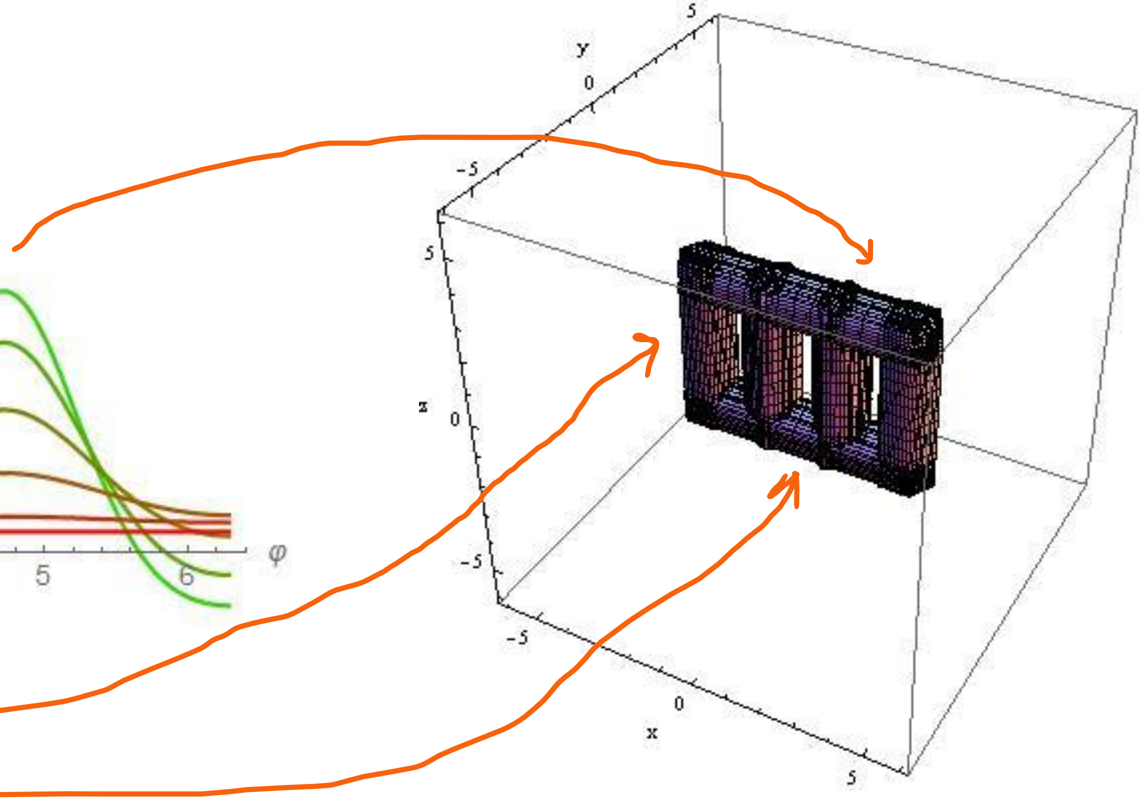
The antenna

$$\tilde{f}(q(\alpha))$$





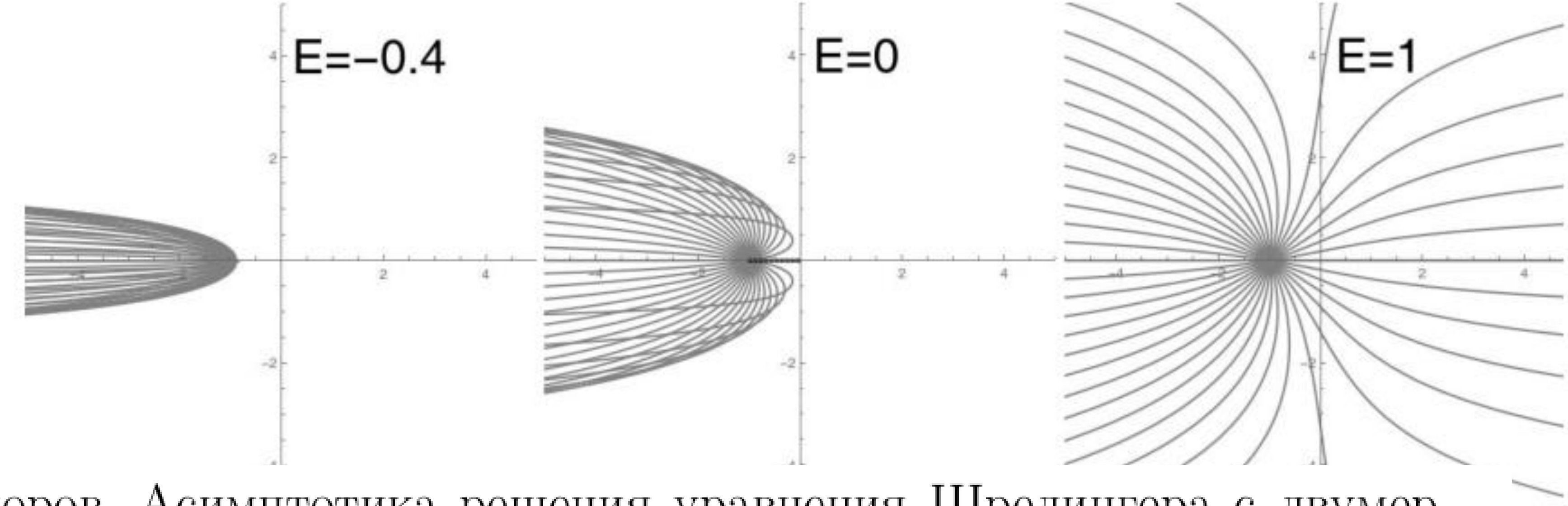




A simple example when the trajectories do not leave bounded areas: the Arnold singularity of Legendre type

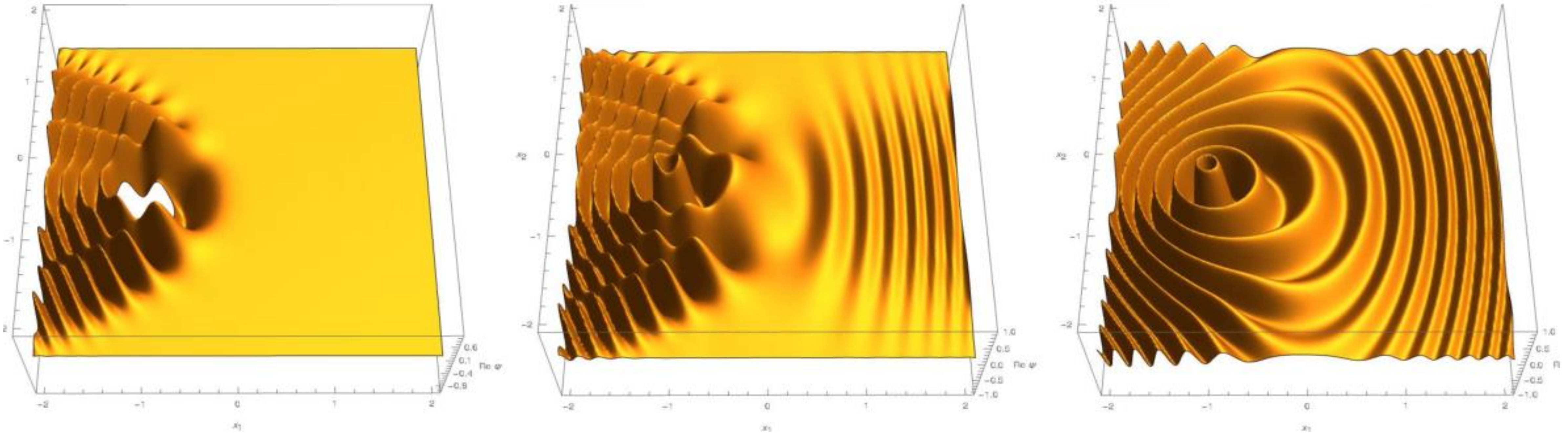
I. A. Bogaevsky, S. Yu. Dobrokhotov, A. A. Tolchennikov, Arnold Lagrangian singularity in the asymptotics of the solution of a model two-dimensional Helmholtz equation with a localized right-hand side, Theoret. and Math. Phys., 218:1 (2024), 19-40

$$\left(\frac{1}{2}(-h^2\triangle - x_1^2) - E\right)\psi = V\left(\frac{x-\xi}{h}\right), x = (x_1, x_2) \in \mathbb{R}^2$$



О. О. Федоров, Асимптотика решения уравнения Шредингера с двумерным квадратичным перевернутым потенциалом и локализованной правой частью, ТМФ, 226:3 (2026), 516-540

$E=-0.1$ $\quad$ $E=0$ $\quad$ $E=0.1$



$E=0$

$$x_1 \in [c_1, c_2]\,(c_1>0),\ x_2 \in \mathbb{R}$$

$$\psi(x)=\sqrt{\frac{h}{\ln\frac{1}{h}}}e^{-\frac{a^2}{2}}\frac{e^{i\pi/4}}{\sqrt{ax_1}}e^{\frac{i}{h}\frac{a^2+x_1^2}{2}}e^{\frac{ix_2^2}{2h\ln\frac{2ax_1}{h}}}e^{-\frac{\pi x_2^2}{4h\ln^2\frac{2ax_1}{h}}}\Gamma\left(\frac{1}{2}+\frac{ix_2^2}{2h\ln^2\frac{2ax_1}{h}}\right)+O\left(\frac{h^{\frac{1}{2}}}{(\ln\frac{1}{h})^{1-\delta}}\right),$$

Shallow water equation

Equations in dimensionless variables

$$h\frac{\partial \eta}{\partial t} + h\ \mathrm{div}((\eta + D(x))\mathbf{u}) = \mathrm{Re}\left[f_1\left(\frac{x-x^0}{h}\right)e^{i\frac{\omega}{h}t}\right],$$

h

$$h\frac{\partial \mathbf{u}}{\partial t} + h\langle \mathbf{u}, \nabla\rangle\mathbf{u} + g\nabla\eta = \mathrm{Re}\left[f_2\left(\frac{x-x^0}{h}\right)e^{i\frac{\omega}{h}t}\right].$$

Linearization of the system

$$h\frac{\partial N}{\partial t} + h\ \mathrm{div}(D(x)\mathbf{U}) = \mathrm{Re}\left[f_1\left(\frac{x-x^0}{h}\right)e^{i\frac{\omega}{h}t}\right],$$

$$h\frac{\partial \mathbf{U}}{\partial t} + h\, g\nabla N = \mathrm{Re}\left[f_2\left(\frac{x-x^0}{h}\right)e^{i\frac{\omega}{h}t}\right].$$

Relationship with the Helmholtz equation

We omit the vortical part of the solution, this gives

$$h^2\frac{\partial^2 N}{\partial t^2} - h^2\,\mathrm{div}(c^2(x)\nabla N) = \mathrm{Re}\left[e^{i\frac{\omega}{h}t}F\left(\frac{x-x^0}{h},x\right)\right],$$

$$c^2(x) = gD(x), \quad F(y,x) = i\omega f_1(y) - D(x)\,\mathrm{div}\,f_2(y) - h\langle\nabla D(x), f_2(y)\rangle$$

We put $N = \mathrm{Re}\left[e^{i\frac{\omega}{h}t}\Psi(x)\right]$ where Ψ is a solution to the equation

$$\langle\hat{p}, c^2(x)\hat{p}\rangle\Psi - \omega^2\Psi = F\left(\frac{x-x^0}{h},x\right), \quad \hat{p} = -ih\nabla, \quad c^2 = gD(x)$$

Far from the coast line one can put $\Psi(x) = \frac{1}{c(x)}\Phi(x)$

and obtain the Helmholtz equation

$$h^2\triangle\Phi + V(x,h)\Phi = -\frac{1}{c(x)}F\left(\frac{x-x^0}{h},x\right), \quad V(x,h) = \frac{\omega^2}{c^2(x)} - h^2\frac{\Delta c(x)}{c(x)}$$

The Hamilton systems, semiclassical asymptotics, standard and nonstandard caustics, the modified Maslov canonical operator

Hamiltonian $H_0 = c^2(x)p^2$

Trajectories $p = P(\psi,\tau), x = X(\psi,\tau)$ $\qquad H_0(X,P) = c^2(X)P^2 = \omega^2$

$$\dot{p} = -(c^2)_x p^2, \quad \dot{x} = 2c^2 p, \quad p|_{\tau=0} = \mathbf{n}(\psi)\frac{\omega}{c(x^0)}, \quad x|_{\tau=0} = x^0, \quad \mathbf{n}(\psi) = \begin{pmatrix}\cos\psi \\ \sin\psi\end{pmatrix}$$

First, we assume that $c(x) > 0$ and that in finite time the projections of trajectories $X(\psi,t)$ leave any bounded set in the domain $D(x) \geq 0$.

the Lagrangian manifold

$$\Lambda = \{p = P(\psi,\tau), x = X(\psi,\tau) \colon \psi \in S^1, \tau \in [0,\infty)\}$$

Relationship with the Finsler metric

Hamiltonian $\mathcal{H} = C(x)|p| \equiv \frac{\sqrt{H_0}}{\omega}, \quad C(x) = \frac{c(x)}{\omega}$

Trajectories $P(\psi,\tau) = \mathcal{P}(\psi,\tau_1), \quad X(\psi,\tau) = \mathcal{X}(\psi,\tau_1), \quad \tau_1 = 2\omega\tau$

$$\dot{p} = -2c|p|c_x|p| = -2\omega c_x|p|, \quad \dot{x} = 2c|p|c\frac{p}{|p|} = 2\omega c\frac{p}{|p|},$$

$$p|_{\tau=0} = \mathbf{n}(\psi)\frac{\omega}{c(x^0)}, \quad x|_{\tau=0} = x^0$$

$$\frac{dp}{d\tau_1} = -C_x|p|, \quad \frac{dx}{d\tau_1} = C\frac{p}{|p|},$$

$$p|_{\tau_1=0} = \mathbf{n}(\psi)\frac{1}{C(x^0)}, \quad x|_{\tau_1=0} = x^0, \quad \tau_1 = 2\omega\tau.$$

Finsler metric and Fermat's variational principle

$$\delta T[\gamma] = \delta\int_\gamma \frac{dl}{c(x)} = 0$$

The behavior near a shore

О. А. Олейник, Е. В. Радкевич, “Уравнения второго порядка с неотрицательной характеристической формой”, Итоги науки. Сер. Математика. Мат. анал. 1969, ВИНИТИ, М., 1971, 7–252.

T. Vukašinac, P. Zhevandrov, “Geometric asymptotics for a degenerate hyperbolic equation”, *Russ. J. Math. Phys.*, **9**:3 (2002), 371–381.

С. Ю. Доброхотов, В. Е. Назайкинский, “Униформизация уравнений с граничным вырождением бесселева типа и квазиклассические асимптотики”, *Матем. заметки*, **107**:5 (2020), 780–786.

В. Е. Назайкинский, “Об эллиптическом операторе, вырождающемся на границе области”, *Функц. анализ и его прил.*, **56**:4 (2022), 109–112.

В. Е. Назайкинский, “Геометрия фазового пространства для волнового уравнения, вырождающегося на границе области”, *Матем. заметки*, **92**:1 (2012), 153–156.

В. Е. Назайкинский, “Канонический оператор Маслова на лагранжевых многообразиях в фазовом пространстве, соответствующем вырождающемуся на границе волновому уравнению”, *Матем. заметки*, **96**:2 (2014), 261–276.

S. V. Bolotin, D. V. Treshev, “Another billiard problem”, *Russ. J. Math. Phys.*, **31**:1 (2024), 50–59.

Expanded phase space and non-standard coordinates in the vicinity of the coastline

The extended phase space $\mathbf{\Phi}$ obtained from the standard cotangent bundle $T^*\Omega$ by adding a plurality of "points at infinity"$\mathbf{\Phi}_\infty$ over $\partial\Omega$. Let $x_* \in \partial\Omega$ and Ω be near a point x_* is defined by the inequality $x_1 > f(x_2)$, where $f(x_2)$ is a smooth function. We introduce new coordinates (q, y, θ, ξ) on $T^*\Omega$ for x close to x_* and $p_1 = 0$ using the formulas

$$q = p_1^{-1}, \quad y = x_2, \quad \theta = p_1^2(x_1 - f(x_2)), \quad \xi = p_2 + f'(x_2)p_1$$

$$p_1 dx_1 + p_2 dx_2 = \theta dq + \xi dy + d(\theta q)$$

$$dp_1 \wedge dx_1 + dp_2 \wedge dx_2 = d\theta \wedge dq + d\xi \wedge dy$$

$$\mathcal{H}(x,p) = \mathcal{H}(q, y, \theta, \xi) = \sqrt{\theta}\gamma(f(y) + q^2\theta, y)\sqrt{1 + (f'(y) - q\xi)^2}$$

$$c(x) = \sqrt{x_1 - f(x_2)}\gamma(x).$$

$$J_{rs}(\psi,\tau) = \det\left(\frac{\partial(Q,Y)}{\partial(\psi,\tau)}\right), \quad J_{ss}(\psi,\tau) = \det\left(\frac{\partial(Q,\Xi)}{\partial(\psi,\tau)}\right)$$

The linear asymptotics near the coastline Γ (simple case)

$$[K^h_\Lambda A](x) = [\widetilde{K}_U A](2\sqrt{x_1 - f(x_2)}, x_2) + [\widetilde{K}_U A](-2\sqrt{x_1 - f(x_2)}, x_2) + O(h),$$

$$[\widetilde{K}^h_U A](\sigma, y) = e^{i\pi/4 - i\pi m/2} \left(\frac{\pi\tau_{odd}}{h\sigma}\right)^{\frac{1}{2}} e^{(i/h)\tau_{ev}} \mathbf{J}\left(\frac{\tau_{odd}}{h}\right) \frac{A(\alpha)}{|J_{\sigma y}|^{\frac{1}{2}}}\Bigg|_{\substack{\tau_{odd}=\tau_{odd}(\sigma,y),\\ \tau_{ev}=\tau_{ev}(\sigma,y)\\ \alpha=\alpha(\sigma,y)}}$$

$$\mathbf{J}(v) = J_0(v) + iJ_1(v),$$

$$J_{\sigma y}(\alpha) = \det \frac{\partial(\Sigma, Y)}{\partial \mu}(\alpha) \qquad \tau_{odd}(\sigma, y) = \frac{\tau(\alpha(\sigma, y)) - \tau(\alpha(-\sigma, y))}{2}$$

$$\Sigma(\alpha) = 2Q(\alpha)\sqrt{\Theta(\alpha)}, \qquad \tau_{ev}(\sigma, y) = \frac{\tau(\alpha(\sigma, y)) + \tau(\alpha(-\sigma, y))}{2}$$

$J_0(z)$, $J_1(z)$ are the Bessel functions

А. Ю. Аникин, С. Ю. Доброхотов, В. Е. Назайкинский, “Простые асимптотики обобщенного волнового уравнения с вырождающейся скоростью и их приложения в линейной задаче о набеге длинных волн на берег”, *Матем. заметки*, **104**:4 (2018), 483–504.

Asymptotics of the nonlinear system of shallow water equations in the vicinity of the coastline

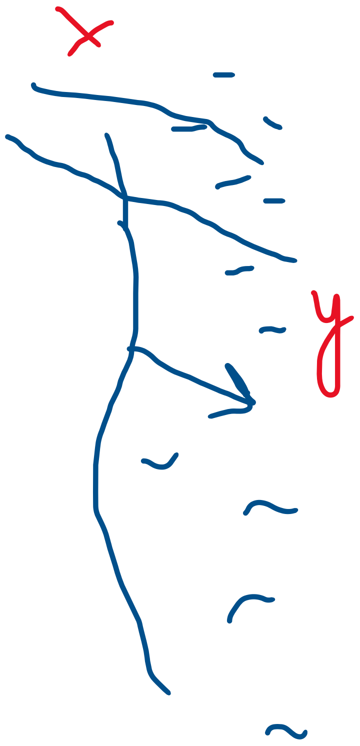
G. F. Carrier, H. P. Greenspan, “Water waves of finite amplitude on a sloping beach”, *J. Fluid Mech.*, **4** (1958), 97–109.

S. Yu. Dobrokhotov, D. S. Minenkov, V. E. Nazaikinskii, Asymptotic solutions of the Cauchy problem for the nonlinear shallow water equations in a basin with a gently sloping beach, Russ. J. Math. Phys., 29:1 (2022), 28-36.

С. Ю. Доброхотов, В. А. Калиниченко, Д. С. Миненков, В. Е. Назайкинский, Асимптотики длинных стоячих волн в одномерных бассейнах с пологими бере- гами: теория и эксперимент, Прикл. матем. мех., 87:2 (2023), 157-175.

S. Yu. Dobrokhotov, V. E. Nazaikinskii, I. A. Nosikov, A. A. Tolchennikov, Asymptotics of long waves generated by time-harmonic spatially localized sources in basins with gently sloping shores, Comput. Math. Math. Phys., 65:5 (2025), 966-981

Asymptotics of the nonlinear system of shallow water equations in the vicinity of the coastline



Let N(x,t),U(x,t) be the asymptotic solutions of a linearized system of shallow water equations. Introduce a cut-off function $\rho(y)$ equal to 1 in some small neighborhood of the curve Γ. Define the functions $\eta(x,t), \mathbf{u}(x,t)$ in a parametrically given form

$$x = y - N(y,t)\frac{\rho(y)\nabla_y D(y)}{|\nabla_y D(y)|^2}, \quad \eta = N(y,t), \quad \mathbf{u} = U(y,t)$$

Assume that the Jacobian $\det(\frac{\partial x}{\partial y}) \neq 0$ *in the vicinity of the coastline* Γ*, then the functions* $\eta(x,t), \boldsymbol{u}(x,t)$ *outside some neighborhood independent of* h *of* Γ *are asymptotic solutions of the original system of shallow water equations.*

CARRIER-GREENSPAN TRANSFORM

THE LINEAR WAVE EQUATION for $N(\tau, y), U(\tau, y)$:

$$N_\tau + \frac{\partial}{\partial y}(\gamma^2 y U) = 0, \quad U_\tau + g N_y = 0, \qquad g = 1,\ \gamma = 1$$

CONSIDER the SYSTEM

$$x = y - N(\tau, y) + \frac{1}{2} U^2(\tau, y), \quad t = \tau + U(\tau, y)$$

small

Let it defines one-to-one map from $\{y \geq 0, \tau \in \mathbb{R}\}$ to the value area of the right hand side

THEN

$$\eta(t, x) = N(\tau, y) - \frac{1}{2} U^2(\tau, y), \quad v(t, x) = U(\tau, y)$$

are the solution to the **ORIGINAL NONLINEAR SYSTEM** in a **PARAMETRIC FORM**

$$\eta_t + \frac{\partial}{\partial x}[v(\eta + \gamma x)] = 0,\ v_t + v v_x + g \eta_x = 0$$

S. Yu. Dobrokhotov, B. Tirozzi, Localized solutions of one-dimensional non-linear shallow-water equations with velocity $c = \sqrt{x}$, Russian Math. Surveys, 65:1 (2010), 177-179

S. Yu. Dobrokhotov, S. B. Medvedev, D. S. Minenkov, On Replacements Reducing One-Dimensional Systems of Shallow-Water Equations to the Wave Equation with Sound Speed $c^2 = x$, Math. Notes, 93:5 (2013), 704-714

Yu. A. Chirkunov, S. Yu. Dobrokhotov, S. B. Medvedev, D. S. Minenkov, Exact solutions of one-dimensional nonlinear shallow water equations over even and sloping bottoms, Theoret. and Math. Phys., 178:3 (2014), 278298

A. V. Aksenov, S. Yu. Dobrokhotov, K. P. Druzhkov, Exact Step-Like Solutions of One-Dimensional Shallow-Water Equations over a Sloping Bottom, Math. Notes, 104:6 (2018), 915-921

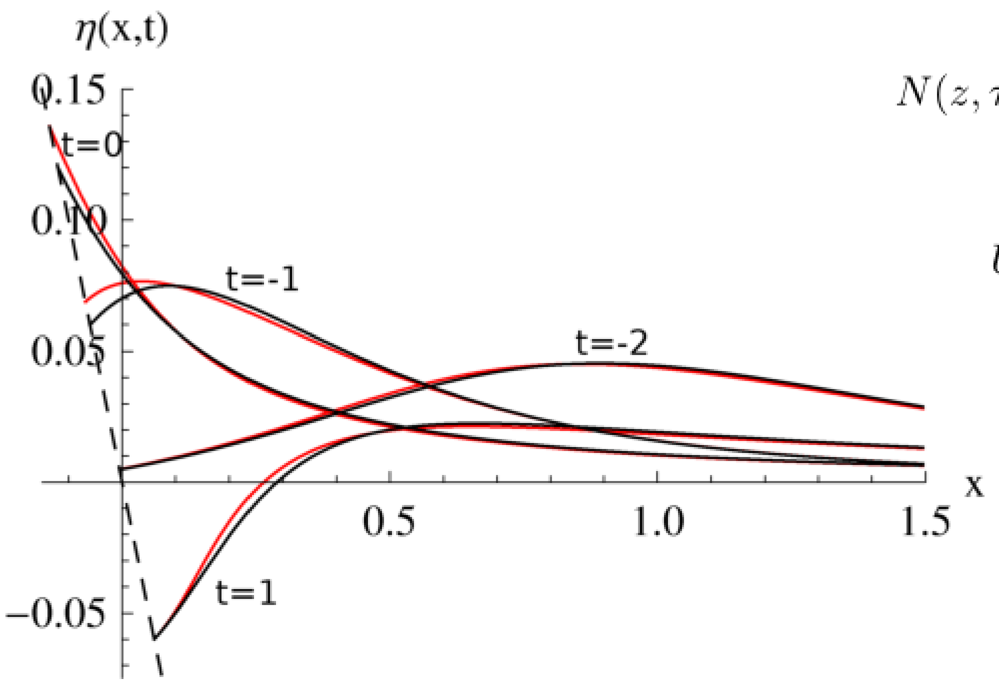
Reduced Carrier-Greenspan transformation (simplification):

$$D(y) = D(x) + \eta(x,t)\rho(x),\ \tau = t,\ N(y,\tau) = \eta(x,t),\ U(y,\tau) = u(x,t),$$

here $\rho(x)$ is a cut-off function, $\rho = 1$ near $x^0 : \quad D(x) = 0$ and $\rho = 0$ outside of the neighborhood of the point x^0.

The main property: **the boundary becomes fixed**

Example: exact solitary solution



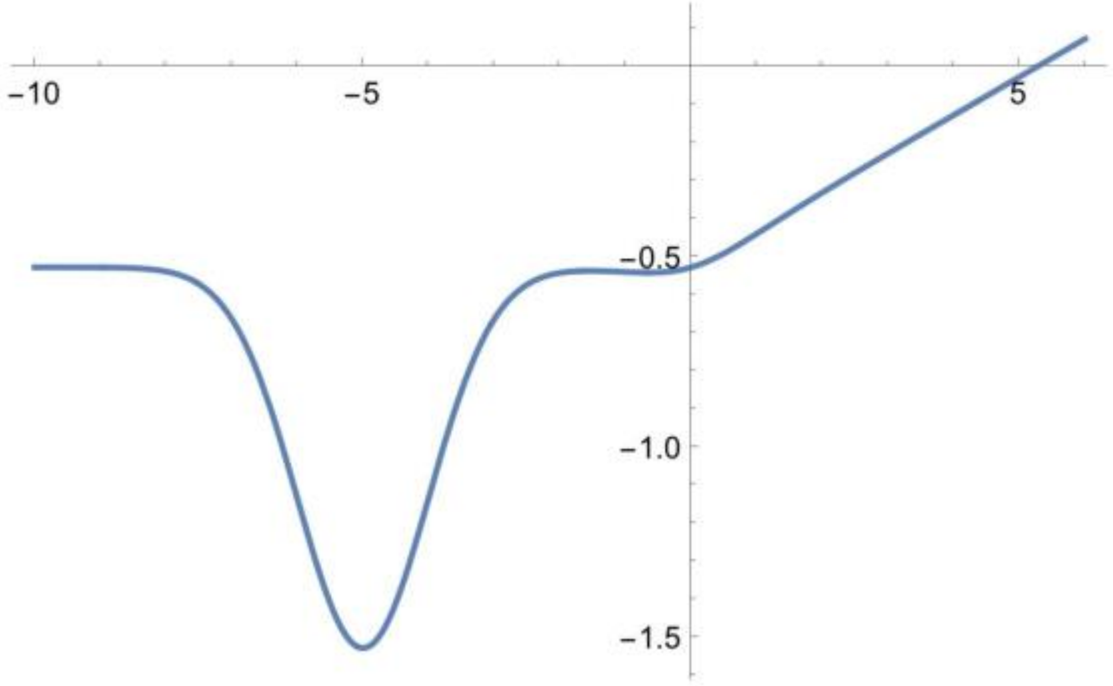
$$N(z,\tau) = \mathrm{Re}\,\frac{A(\tau + ib)}{(z - (\tau + ib)^2/4)^{3/2}},$$

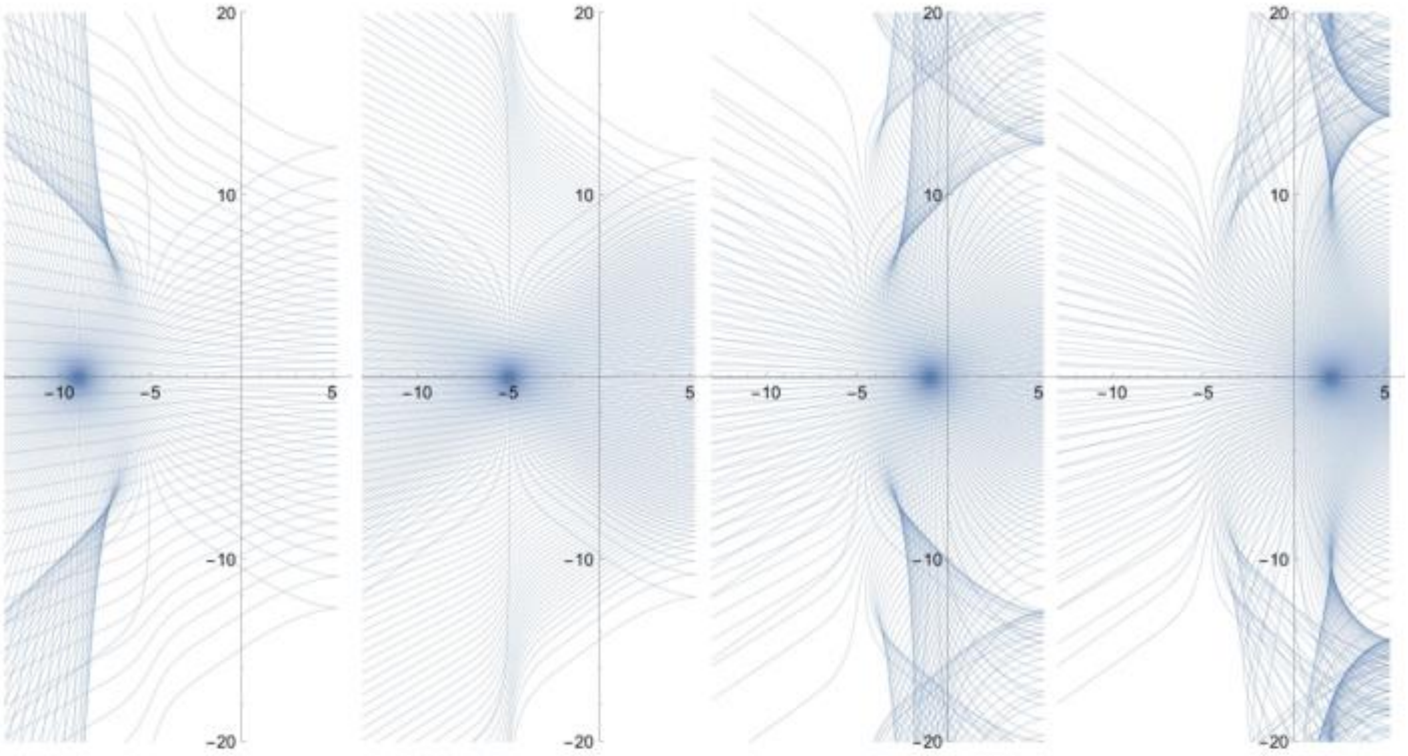
$$U(z,\tau) = 2\,\mathrm{Re}\,\frac{A}{(z - (\tau + ib)^2/4)^{3/2}},$$

Example

$$D(x) = -q(x_1(\mathrm{th}(x_1)+1)-a) - Ae^{-\frac{(x_1-x_1^0)^2}{2v}} + d$$

$$q = 0.05, a = 0.5, A = -1, v = 1, x_1^0 = -5, d = 0.5$$





$$x_2^0 \in \{-9, -5, -1, 2\}$$

Global and local properties of the Maslov canonical operator

Finsler metric and Fermat's variational principle

$$\delta T[\gamma] = \delta \int_\gamma \frac{dl}{c(x)} = 0$$

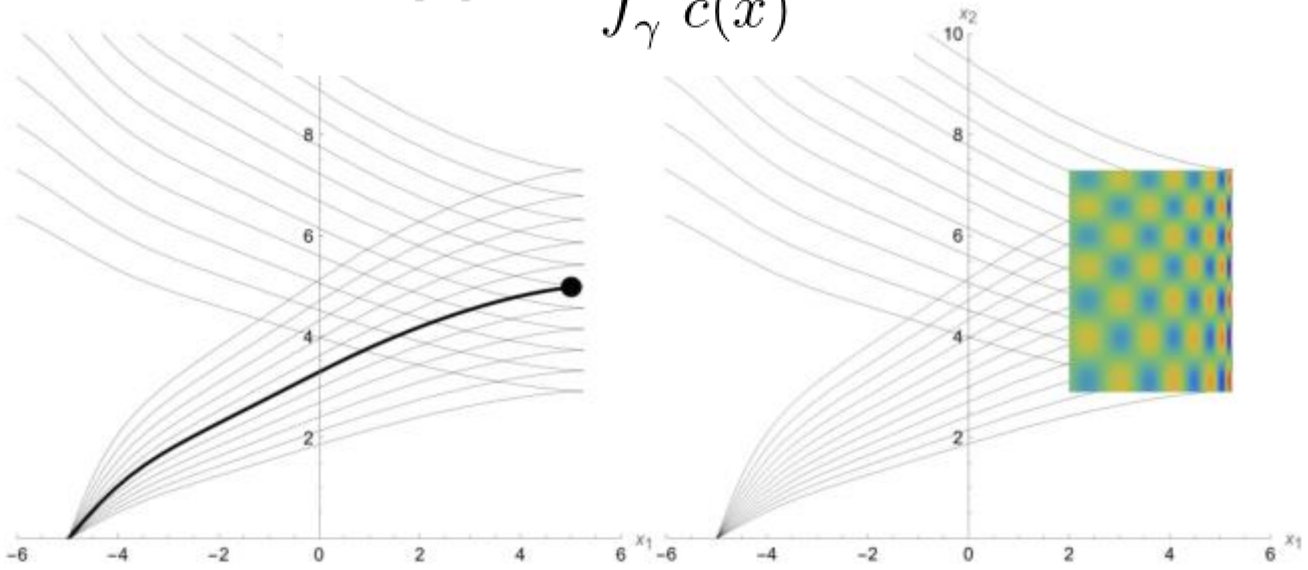
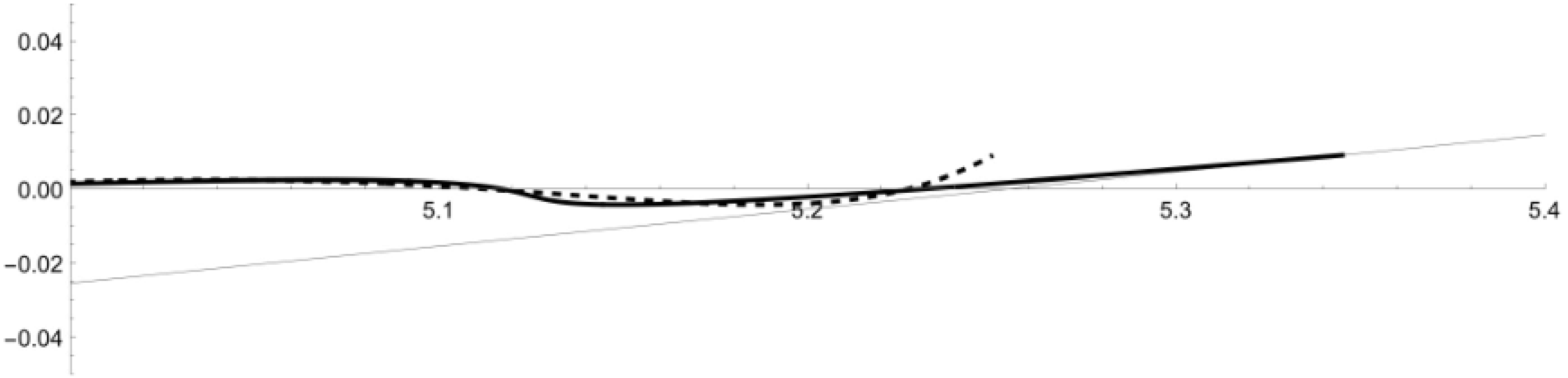


Рис. 5. Слева: жирная точка — $x^1 = (5, 5)$, жирная линия — луч из x^0 в x^1, найденный вариационным методом, тонкие линии — соседние лучи, справа: решение в окрестности точки x^1

С. Ю. Доброхотов, И. А. Носиков, А. А. Толченников, Принцип Мопертюи- Якоби и вариационный принцип Ферма в задаче о коротковолновой асимптотике решения уравнения Гельмгольца с локализованным источником, Ж. вычисл. матем. и матем. физ., 65 (2025)

И. А. Носиков, А. А. Толченников, М. В. Клименко, Краевая задача о расчете лучевых характеристик океанических волн, отраженных от береговой линии, Ж. вычисл. матем. и матем. физ., 64:3 (2024), 534-546.

From linear case to nonlinear case: run-up (splash)



Спасибо за внимание!

THANK YOU FOR YOUR ATTENTION!