On attractors of generalized iterated function systems from the viewpoint of semigroup dynamical systems

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A general iterated function system

A **general iterated function system** (GIFS) on a topological space X is a finite family $S = \{f_1, \dots, f_k\}$, where $f_i \colon X \to X$ is a continuous mapping, $i = 1, \dots, k$.

Let $\mathcal{K}(X)$ be the set of all nonempty compact subsets of the topological space X. The set $\mathcal{K}(X)$ can be endowed with the *Vietoris topology*, the subbase of which is formed by sets

$$V^+ = \{K \in \mathcal{K}(X) | K \subset V\}, \quad V^- = \{K \in \mathcal{K}(X) | K \cap V \neq \emptyset\},$$

where V runs through open subsets of X.

A GIFS $S = \{f_1, \dots, f_k\}$ given on X defines a mapping

$$F \colon \mathcal{K}(X) \to \mathcal{K}(X) \colon B \mapsto f_1(B) \cup \ldots \cup f_k(B) \ \forall B \in \mathcal{K}(X),$$

which is called the *Hutchinson operator*.

The strict and weak attractors of the GIFS

A nonempty compact subset $A \in \mathcal{K}(X)$ is called a **strict attractor** of the GIFS S if there exists an open subset $U \subset X$ such that $A \subset U$, $f_i(U) \subset U, \forall i = 1, \dots, k$, and, for an arbitrary set $B \in \mathcal{K}(X)$, $B \subset U$, the sequence $\{B_n = F^n(B)\}$ converges in the Vietoris topology to A.

A nonempty compact subset $A \in \mathcal{K}(X)$ is called a **weak attractor** of the GIFS S if there exists an open subset $U \subset X$ such that $f_i(U) \subset U, \forall i = 1, \dots, k$, and A is unique set in U such that $F(A) = A$.

In previous definitions the largest open set U satisfying these conditions is called the **basin of the attractor** A. If the basin U of an attractor A coincides with X, then the attractor A is called global.

Since the Hutchinson operator is continuous, the strict attractor is a weak attractor.

An attractor of semigroup dynamical system

Let G be a topological semigroup acting continuously on a topological space X. Then the pair (G, X) is called a semigroup dynamical system. We denote the action of $g \in G$ on an element $x \in X$ by $g.x$. The set $G.x = \{g.x \mid g \in G\}$ is called the orbit of the element $x \in X$.

Recall that a set A is called invariant if $G.A = \{g.a \mid g \in G,\ a \in A\} \subset A$. A proper nonempty closed invariant set $A \subset X$ is called an **attractor of the dynamical system** (G, X) if there exists an invariant neighborhood U of A such that $A \subset \overline{G.x}$ for all $x \in U \setminus A$. The neighborhood U is called the basin of the attractor A. If $U = X$, then the attractor A is called global.

A nonempty closed invariant subset $M \subset X$ is called **minimal** if $M = \overline{G.x}$ for all $x \in M$. If $X = \overline{G.x}$ for all $x \in X$, then the dynamical system (G, X) is called minimal.

The set G of all compositions of mappings from GIFS S defined on X forms a semigroup. Thus, a GIFS S defines a semigroup dynamical system (G, X).

Strict attractor theorem

Theorem 1. Let A be a strict attractor with basin U of a GIFS S defined on a Hausdorff topological space X, G be the semigroup generated by S. If $A = X$, then the dynamical system (G, X) is minimal. If $A \neq X$, then for the dynamical system (G, X), A is an attractor with basin U, where A is a minimal set, and there are no other minimal sets in U.

As is well known, if X is a metric space, then the topology on $\mathcal{K}(X)$ defined by the Hausdorff metric coincides with the Vietoris topology.

Corollary (Bagaev, 2024). Let G be a semigroup generated by contraction mappings $f_1, \ldots, f_k$ of a complete metric space X. Then:

1. there exists a unique global attractor A of the dynamical system (G, X);
2. A coincides with the global strict attractor of the IFS $S = \{f_1, \ldots, f_k\}$;
3. A is a minimal set of the dynamical system (G, X);
4. the dynamical system (G, X) has no other minimal sets distinct from A.

Weak attractor theorems

Theorem 2. Let S be GIFS on a Hausdorff topological space X, G be the semigroup generated by S, the set $A \subset X$ be a weak attractor of the GIFS S with basin U. Then A is a compact minimal set of the dynamical system (G, X) and there are no other compact minimal sets in U other than A.

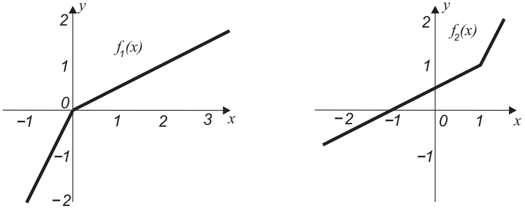
Theorem 3. Let S be GIFS on a Hausdorff topological space X, G be the semigroup generated by S. If A is a compact minimal set of the dynamical system (G, X) and there is an open set $U \subset X$ such that $A \subset U$ and there are no other compact G-invariant subsets in U other than A, then the set A is a weak attractor of the GIFS S with basin U.

Example 1

Let $f_1, f_2 \colon \mathbb{R} \to \mathbb{R}$ be defined by formulas:

$$f_1(x) = \begin{cases} \frac{x}{2}, & \text{if } x \geq 0 \\ 2x, & \text{if } x < 0 \end{cases}$$

$$f_2(x) = \begin{cases} 2x - 1, & \text{if } x \geq 1 \\ \frac{x+1}{2}, & \text{if } x < 1 \end{cases}$$



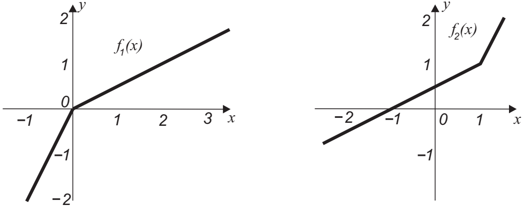
The set $[0, 1]$ is a weak attractor of GIFS $S = \{f_1, f_2\}$, but $[0, 1]$ is not a strict attractor. $[0, 1]$ is a minimal set and a global attractor of semigroup dynamical system $(G, \mathbb{R})$, where $G = < S >$.

Example 2

Let $S = \{f_1, f_2\}$, $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be defined by formulas:

$$f_1(x) = \begin{cases} \frac{x}{2}, & \text{if } x \geq 0 \\ x, & \text{if } x < 0 \end{cases}$$

$$f_2(x) = \begin{cases} x, & \text{if } x \geq 1 \\ \frac{x+1}{2}, & \text{if } x < 1 \end{cases}$$



The set $[0, 1]$ is a minimal set and a global attractor of semigroup dynamical system $(G, \mathbb{R})$, where $G =< S >$. Any segment containing $[0, 1]$ is a fixed point for F. So $[0, 1]$ is not a weak attractor or a strict attractor of GIFS S.

Example 3

Let $S = \{f\}$,

$f(x) = 2x$, $x \in \mathbb{R}^n$.

The set $A = \{0\}$ is a weak attractor of GIFS S, but is not a strict attractor.

A is not an attractor of semigroup dynamical system $(G, \mathbb{R}^n)$, where $G =< S >$.

Example 4

Consider IFS $S = \{f_1, f_2\}$ on $\mathbf{R}$, where $f_1(x) = \frac{x}{2}$, $f_2(x) = 1 \ \forall x \in \mathbf{R}$.

The strict attractor of the IFS S is a countable totally disconnected set

$$A = \{0, 1, \frac{1}{2^n}, \ n \in \mathbf{N}\}.$$

Really, for an arbitrary set $B \in \mathcal{K}(\mathbf{R})$, $B \subset X$, the sequence $\{B_n = F^n(B)\}$ converges to A.

Thus, A is a weak attractor of S and a global attractor of semigroup dynamical system $(G, \mathbb{R})$, where $G =< S >$.

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