Bifurcations through a non-semisimple hyperbolic 1:1 resonance, with examples in rigid body dynamics

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[2] A.Z. Ali, V.A. Kibkalo, E.A. Kudryavtseva, M.V. Onufrienko, Bifurcations in integrable Hamiltonian systems near corank-one singularities // Diff. Eq. **60**:10 (2024), 1311–1368.
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Hamiltonian system with $n = 2$ degrees of freedom ($\Omega = \sum_{j=1}^{n} dp_j \wedge dq_j$) depending on a parameter $\varepsilon \in \mathbb{R}$:

$$\frac{dq_i}{dt} = \frac{\partial H^\varepsilon}{\partial p_i}(q,p), \quad \frac{dp_i}{dt} = -\frac{\partial H^\varepsilon}{\partial q_i}(q,p), \quad i = 1,2. \tag{1}$$

Suppose F^ε is a first integral, i.e. $\{H^\varepsilon, F^\varepsilon\} = 0$ for each ε, where $\{f,g\} = \Omega(X_f, X_g)$, $dg = \omega(\ ,X_g)$.
Suppose that the origin is an equilibrium (a singular point of rank 0) at $\varepsilon = 0$:

$$H^0 = H_2^0 + H_3^0 + \dots, \quad F^0 = F_2^0 + F_3^0 + \dots.$$

Пусть $\{\pm\lambda_1, \dots, \pm\lambda_n\}$ — спектр гамильтонова оператора $\Omega^{-1}d^2H(x_0)$.

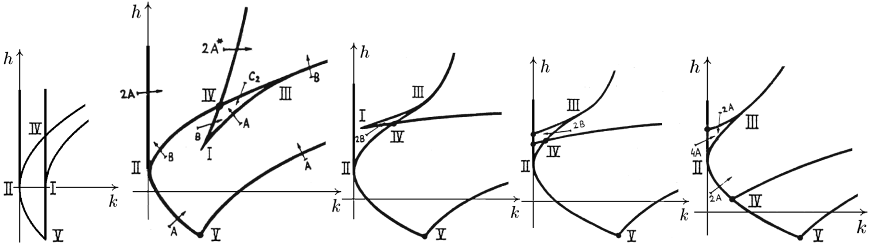
Theorem. (Int) [Birkhoff], [Bryuno 1988] If $\nexists\, \mathbf{k} \in \mathbb{Z}^n \setminus \{0\}$, $\langle \mathbf{k}, \vec{\lambda}\rangle = 0$, then $\exists$ local formal symplectic coordinates (P,Q), in which $H = h(B_1, \dots, B_n)$,

$$B_j = \begin{cases} \frac{1}{2}(P_j^2 + Q_j^2) & \text{(center)}, \\ P_jQ_j & \text{(saddle)}, \end{cases} \quad B_{j-1} + iB_j = (P_{j-1} + iP_j)(Q_{j-1} - iQ_j) \text{ (focus-focus)},$$

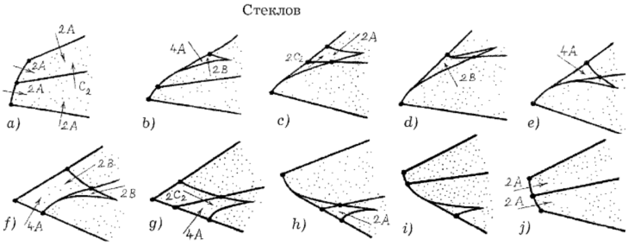
(Stab) if $d^2H^0(0)$ is positive definite, then x_0 is Lyapunov stable (elliptic); if $\exists \lambda_j \notin i\mathbb{R}$, then x_0 is not Lyapunov stable,
(Bifurc) $T_0\lambda_1 = 2\pi i$, $\frac{\lambda_j}{\lambda_1} \notin \mathbb{Z} \Rightarrow \exists$ a family of periodic solutions $N_1 \approx D^2(P_1, Q_1)$, $H|_{N_1} = h(P_1^2 + Q_1^2)$ (normal Lyapunov mode; $\frac{\lambda_1}{\lambda_2} = \frac{m}{n}$ [Duistermaat 1984]),
(StrStab) $\forall$ small perturbation $\widetilde{H}$ $\exists!$ equilibrium $\widetilde{x}_0 \in U$.

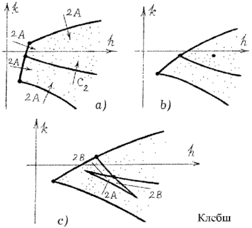
Definition of non-semisimple hyperbolic $1:1$ resonance at the equilibrium $(q,p)=(0,0)$. Suppose that the Hamiltonian operators $A = X_{H_2^0}$, $X_{F_2^0}$ are linearly independent, and A is non-diagonalizable (Jordan block structure), so it has a multiple spectrum $\operatorname{spec} A = \{\lambda, \lambda, -\lambda, -\lambda\}$, which we suppose to be real, $\lambda > 0$.

Examples: Kovalevskaya top at $g = 1$ (b → c, two anti-canonical involutions):



Steklov case (d → e), Clebsch case (b → c):





Properties: 1) $\exists$ linear changes of variables $(q,p) \to (\widetilde{q},\widetilde{p})$ and $(H,F) \to (H, \widetilde{F} = aH + bF)$ transforming H_2^0, F_2^0 to the form

$H_2^0 = \lambda \widetilde{S} + \widetilde{X}, \quad \widetilde{F}_2^0 = \widetilde{S},$ where $S = p_1 q_1 + p_2 q_2,\ X = p_1 q_2,\ Y = p_2 q_1,\ Z = p_1 q_1 - p_2 q_2$

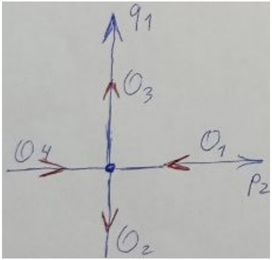
1') $\exists$ non-linear local changes $(q,p) \to (\widetilde{q},\widetilde{p}) = (\widetilde{q}^\varepsilon(q,p), \widetilde{p}^\varepsilon(q,p))$ and $(H,F) \to (H, \widetilde{F} = \widetilde{F}^\varepsilon(H,F))$ transforming H_2^0, F_2^0 to a normal form

$$H^\varepsilon = \lambda(\varepsilon)\widetilde{S} + \widetilde{X} + \varphi^\varepsilon(\widetilde{Y}, \widetilde{S}), \quad \widetilde{F}^\varepsilon = \widetilde{S},$$

2) equilibria (near the semi-simple 1:1 resonance) form a 1-parameter family $(\widetilde{q},\widetilde{p}) = (0,0)$ with parameter ε, so spec $A^\varepsilon = \{\pm\lambda_1^\varepsilon, \pm\lambda_2^\varepsilon\}$ where λ_1^ε and λ_2^ε collide at $\varepsilon = 0$,

3) locally $\exists$ two (families of commuting) anti-canonical involutions $\tau_\pm^\varepsilon : (\widetilde{q}_1, \widetilde{q}_2, \widetilde{p}_1, \widetilde{p}_2) \mapsto \pm(\widetilde{p}_2, \widetilde{p}_1, \widetilde{q}_2, \widetilde{q}_1)$, so the system is reversible w.r.t. them,

3') singular points on the integral manifold $L^{0,0} := \{H^0 = F^0 = 0\}$ (at which dH and dF are linearly dependent) form (locally) four rays $\{p_1 = q_2 = 0,\ p_2 q_1 = 0\} = 0 \cup \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_4$ where $\mathcal{O}_j$ are 4 1-dimensional local orbits of the local Hamiltonian action of $\mathbb{R}^2$ by the flows of X_{H^0}, X_{F^0}.



Each local involution $\tau_\pm$ permutes these local orbits: $\tau_+(\mathcal{O}_1) = \mathcal{O}_3$, $\tau_+(\mathcal{O}_2) = \mathcal{O}_4$.

From topological point of view, an integrable system is a singular Lagrangian foliation of the phase space by integral manifolds $\{H^\varepsilon = c_1, \ F^\varepsilon = c_2\}$. General problem: describe topology of this foliation and its bifurcations along the above family of equilibria.

Questions about the singularity with hyperbolic 1:1 resonance:

1) To which "standard form" can be locally reduced the pair of functions $H^\varepsilon, F^\varepsilon$ by some (non-linear, not necessarily symplectic) local coordinate changes?

2) How do eigenvalues λ_1^ε and λ_2^ε bifurcate along the family and collide at $\varepsilon = 0$?

3) What are local topological invariants of (the bifurcation of) the foliation? What are semi-global topological invariants (in an invariant neighborhood of the equilibrium) of this foliation?

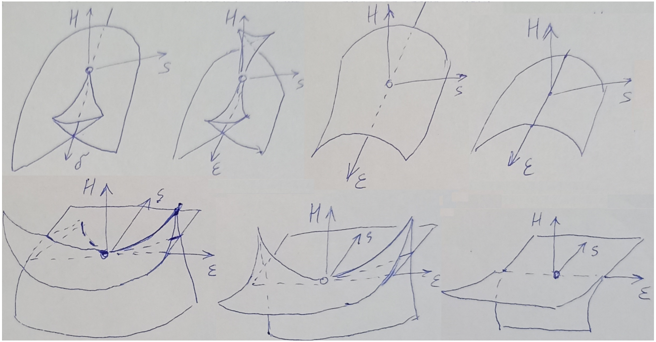
4) Is the given bifurcation (near the given equilibrium) structurally stable?

Theorem 1. Suppose that the integral manifold $L^{0,0} = \{H^0 = F^0 = 0\}$ is compact, contains a 0-dimensional orbit (equilibrium) with non-semisimple hyperbolic resonance, and no other 0-dimensional orbits.
Then the pair of functions $H^\varepsilon, F^\varepsilon$ can be locally reduced near this point, by some (non-linear, not necessarily symplectic) regular local coordinate changes, to one of the following standard forms: $(\widetilde{H}^\varepsilon, \widetilde{F}^\varepsilon) = (G^{\delta(\varepsilon)}, S)$ where

$$G^\delta = \begin{cases} X + Y^2 + \delta Y, & \text{(local singularity } \mathcal{KL}_{1:1} \text{ [KL 2024])} \\ X + Y^3 + YS + \delta Y^2 + \frac{\delta^2}{4} Y, & \text{(new local singularity } \mathcal{KL}^{\triangle}_{1:1}\text{)} \end{cases}$$

$S = p_1 q_1 + p_2 q_2,\ X = p_1 q_2,\ Y = p_2 q_1,\ Z = p_1 q_1 - p_2 q_2$, δ is a detuning parameter, $\delta(\varepsilon) = c\varepsilon^\ell$.
The pair $(c, \ell) \in \{(1, 2k-1), (1, 2k), (-1, 2k), (0, 0)\}$ is a smooth local invariant. It can be reduced to the case of $k = 1$ via topological equivalences.

- Bifurcations $\mathcal{KL}_{1:1}$ with $\delta(\varepsilon) = \varepsilon$ are locally structurally stable. Bifurcation $\mathcal{KL}^{\triangle}_{1:1}$ with $\delta(\varepsilon) = \varepsilon$ is semi-globally structurally stable (but not locally).
- Local bifurcation diagram is a local topological invariant of the above singularities. They are shown in Figure.



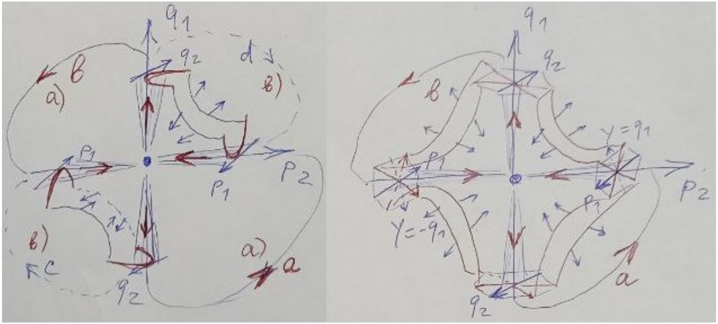
Spectrum $\operatorname{spec} A^{\varepsilon} = \{\pm\lambda_1^{\varepsilon}, \pm\lambda_2^{\varepsilon}\}$ along the family of equilibria has the following behaviour near the 1:1 resonance:

- For $\mathcal{KL}_{1:1}$:
  $(c,\ell) = (1,1)$, $\delta(\varepsilon) = \varepsilon$, $\lambda_{1,2} = \lambda(\varepsilon) \pm \sqrt{\varepsilon}$ (splitting),
  $(c,\ell) = (1,2)$, $\delta(\varepsilon) = \varepsilon^2$, $\lambda_{1,2} = \lambda(\varepsilon) \pm \varepsilon$ (passing real),
  $(c,\ell) = (-1,2)$, $\delta(\varepsilon) = -\varepsilon^2$, $\lambda_{1,2} = \lambda(\varepsilon) \pm i\varepsilon$ (passing complex),
  $(c,\ell) = (0,0)$, $\delta(\varepsilon) = 0$, $\lambda_1 = \lambda_2 = \lambda(\varepsilon)$ (merging real).
- For $\mathcal{KL}_{1:1}^{\triangle}$:
  $(c,\ell) = (1,1)$, $\delta(\varepsilon) = \varepsilon$, $\lambda_{1,2} = \lambda(\varepsilon) \pm \varepsilon/2$ (passing real),
  $(c,\ell) = (\pm 1,2)$, $\delta(\varepsilon) = \pm\varepsilon^2$, $\lambda_{1,2} = \lambda(\varepsilon) \pm \varepsilon^2/2$ (passing real),
  $(c,\ell) = (0,0)$, $\delta(\varepsilon) = 0$, $\lambda_1 = \lambda_2 = \lambda(\varepsilon)$ (merging real).

Theorem 2. For each local singularity, there correspond two semi-global singularities (in a small invariant neighborhood of the equilibrium) determined by the following condition: a) $\mathcal{O}_1 = \mathcal{O}_2$, b) $\mathcal{O}_1 = \mathcal{O}_3$.

The integral manifold $L^{0,0} = \{H^0 = F^0 = 0\}$ with its decomposition into orbits is a cell complex of the form $L^{0,0} = ((a \vee b) \cup_W D^2) \cup_{W^{-1}} D^2$, where the 2-cell is attached to the 1-skeleton in according to the word a) $W = aba^{-1}b^{-1}$, b) $W = a^2 b^{-2}$, where $a \approx b \approx S^1$.

Topology of $L^{0,0}$ is a semi-global topological invariant: a) $\pi_1(L^{0,0}) = \pi_1(\mathbb{T}^2)$, b) $\pi_1(L^{0,0}) = \pi_1(Kl^2)$.



Theorem 3. Suppose that

$$H^{\varepsilon} = \lambda S + X + aY^2 + c\varepsilon^{\ell} Y + \dots,$$

where $(c, \ell) \in \{(1, 2k-1), (1, 2k), (-1, 2k), (0, 0) \mid k \in \mathbb{N}\}$ as in Theorem 1.
If $a \neq 0$ then the bifurcation has local type $\mathcal{KL}_{1:1}$ with the corresponding pair (c, ℓ).
If $a = 0$ then the bifurcation has local type $\mathcal{KL}_{1:1}^{\triangle}$ with the corresponding pair (c, ℓ).

Thank you for your attention!