The theory of hidden oscillations and its applications

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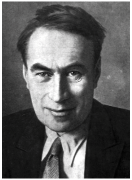
Theory: study of stability and oscillations



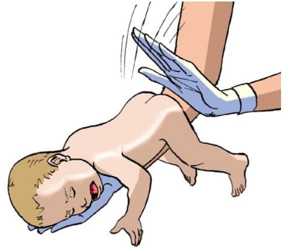
A.A. Andronov (1901-1952)
Academician (OTH) AS USSR

N.N. Krasovsky (1924-2012)
Academician (ОМПУ) RAS

G.A. Leonov (1947-2018)
Academician (ОЭММПУ) RAS

Cardiopulmonary system: self-excited & hidden attractors. Tumbler doll

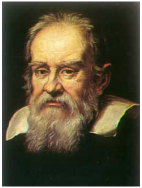
At the birth of a child, the start of the cyclic dynamics of breathing occurs without additional influences (by analogy with the self-excitation of oscillations from a small vicinity of an unstable equilibrium state): self-excited attractor.



In case of respiratory and cardiac arrest, indirect cardiac massage with hands or a defibrillator can help to launch cyclic breathing dynamics if the desired effect is correctly selected (by analogy with the choice of initial data outside a small neighborhood of a stable equilibrium state for attraction with a hidden attractor)

Tumbler doll: global stability — trivial global attractor
(no other local self-excited or hidden attractors).

Preface: mathematical modelling of dynamics



the Universe cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written... **It is written in the language of mathematics**, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering around in a dark labyrinth.
[**Galileo Galilei**, *The Assayer*, 1623]

The mathematical language for this lecture: ✓ dynamical system: linear vs nonlinear
✓ phase space & trajectories ✓ equilibrium points: stable vs unstable
✓ attractors & basins of attraction: global vs local, self-excited vs hidden

Problem: To reveal all attractors and basin of attractions in the phase space system.

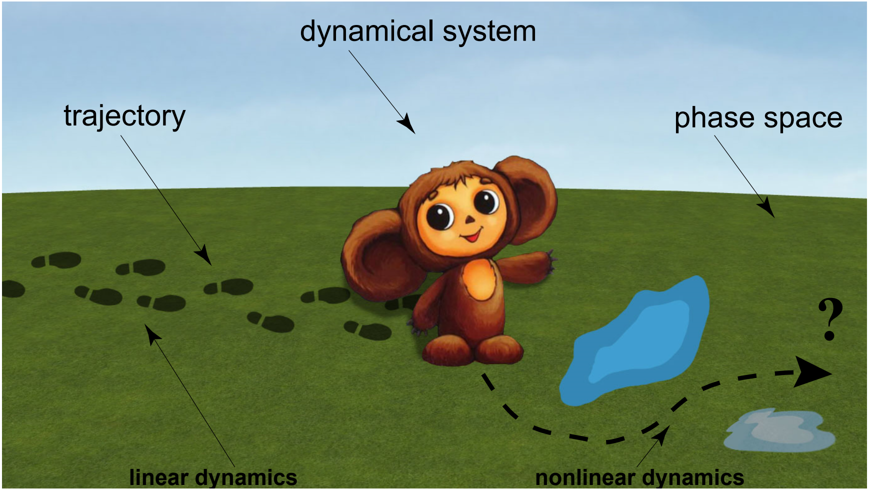
Problem: To check that the stationary set is the only attractor and is global attractor. I.e. there are no nontrivial oscillations and the system is globally stable.

Problem: To reveal the global stability boundary in the space of system's parameters.

New mathematical concepts and methods allow us to reveal common difficulties in a number of well-known theoretical and applied problems on stability and attractors, and also led to discovery of previously unknown attractors in well-studied systems.

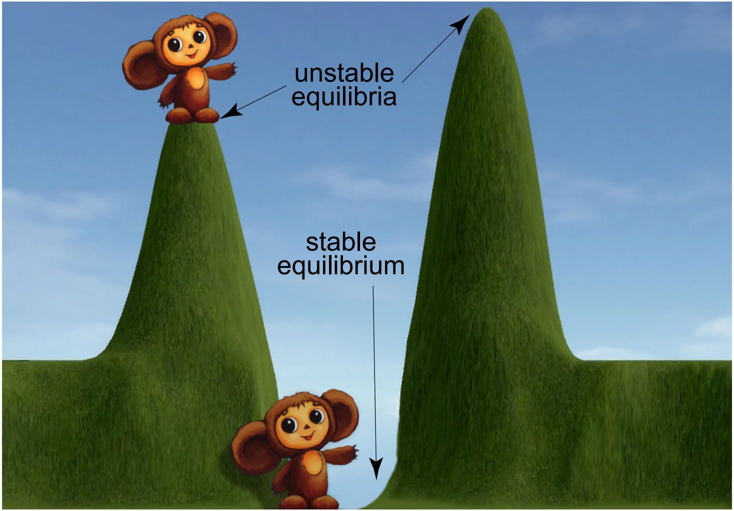
The lecture shows that widely used conjectures on stability and attractors may be wrong and discusses their impact on engineering technologies.

Illustration: Dynamical systems



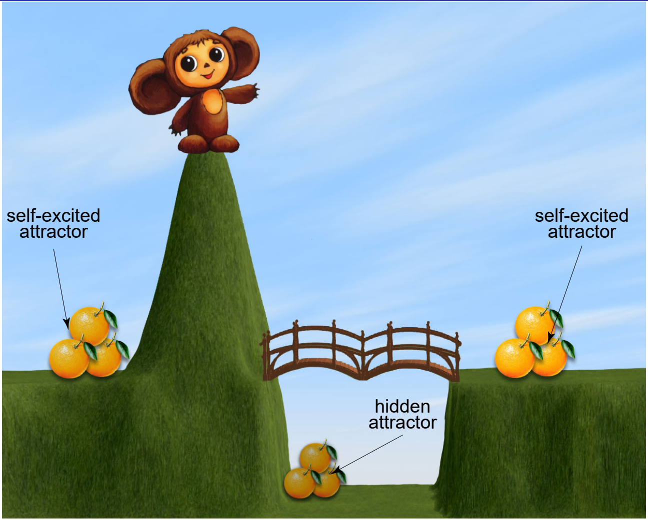
- dynamical system — mathematical law expressing evolution of a (long-term) process;
- phase space — a place, where a dynamical system evolves (realizes its dynamics);
- trajectory — a particular realization of dynamical system evolution;

Illustration: Equilibrium points



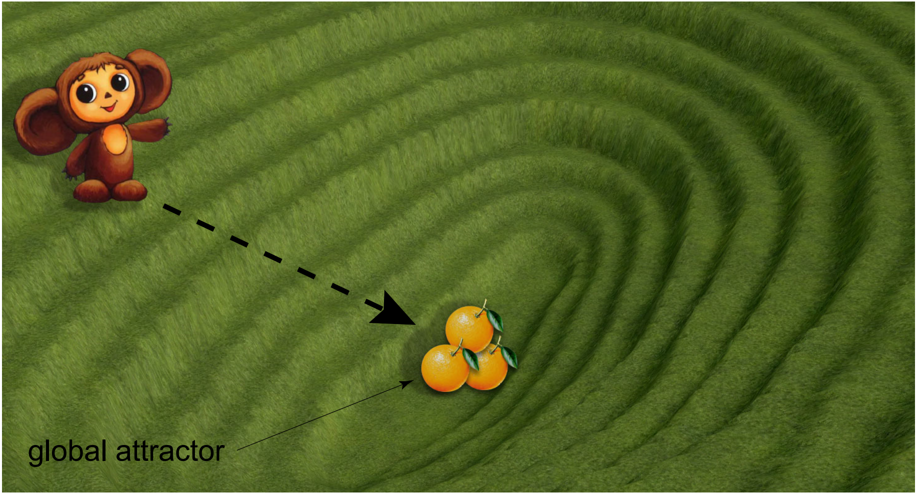
- equilibrium point — state of dynamical system, which does not change with time;
- stable equilibrium point — additionally attracts all the trajectories nearby;
- unstable equilibrium point — additionally repels some (all) trajectories nearby;

Illustration: Self-excited and Hidden Attractors



- attractor — a set of states of dynamical system attracting all trajectories nearby;
- self-excited attractor — is reachable from (a neighborhood of) an equilibrium point;
- hidden attractor — on the contrary, is non-reachable from any equilibrium point;

Illustration: Global attractor



- global attractor — a set attracting all trajectories of dynamical system;
- special case : globally attractive stable equilibrium point [global stability].

Poincare problem: analytical and numerical construction of dynamics

For simple dynamical models the trajectories, attractors and their basins of attraction can be found analytically ($x_{n+1} = 0.5x_n, x_n = (0.5)^n x_0 \to 0$ as $n \to \infty$).
How to reveal all attractors and basin of attractions in a general case?

It is clear that in the large majority of the cases that appear we cannot integrate differential equations by means of functions already known. We must begin the theory of any function and that is why the problem, which first occurs, is the following one:
to construct the curves defined by differential equations.
[Henri Poincaré, *On curves defined by differential equations*, 1881]

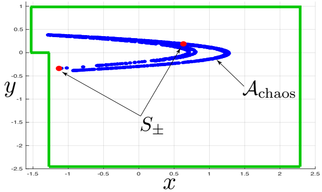
M. Hénon, 1976

$x_{k+1} = 1 + y_k - a x_k^2, \quad y_{k+1} = b x_k,$

Equilibria: $S_{\pm} = (x_{\pm},\, b\, x_{\pm}),$

$x_{\pm} = \frac{1}{2a}\left[(b-1) \pm \sqrt{(b-1)^2 + 4a}\right]$

Linear part: $x_k = 1 + b\, x_{k-2} \underset{k\to\infty}{\to} \infty$



Attractors may have very complex structure and can be described only numerically. In practice, it is often desirable to have unique attractor which attracts all trajectories - a global attractor, e.g., a stable equilibrium point (globally stable system).

Accuracy of simulation: parameters estimation & roundoff error

Roundoff errors in digital computing: $\sum_1^N \frac{1}{N} = 1?$
Fatal error: how Patriot overlooked a Scud, **SCIENCE**, 1992, p.1347: "*minute mathematical error ... allowed an Iraqi SCUD missile to slip through PATRIOT missile defenses a year ago and hit U.S. Army barracks in Dhahran, Saudi Arabia, killing 28 servicemen*".

The internal clock in PATRIOT kept time as an integer value in units of tenths of a second, and the computer's registers were only 24 bits long.: binary expansion of value 1/10: $1/10 = 1/2^4 + 1/2^5 + 1/2^8 + 1/2^9 + \ldots = 1/10 + 0.000000095$, velocity: 1676m/s, resulting position error was 0.5km. (**C-200** had analog architecture)

Parameters estimation: $x_{t+1} = ax_t, x_t = a^t x_0,\ x_t \to 0?\ 0.99^{365} = 0.03$ vs $1.01^{365} = 37.8$

Stability of zero solution of $a_n x^{(n)} + .. + a_2 x'' + a_1 x' + a_0 x = 0 \Leftrightarrow$
Stability of polynomial $a_n s^n + . + a_2 s^2 + a_1 s + a_0 = 0$: **Routh-Hurwitz** criterion (1876,95)
Robustness $a_i \in [l_i, u_i]$: **V. Kharitonov** criterium for interval polynomials (1978)

Stability of zero solution of $a_n x^{(n)} + .. + a_2 x'' + a_1 x' + a_0 x = f(x) \Leftrightarrow$
A. Lyapunov stability theorems (1892); **Perturbations** $k_1 x < f(x) < k_2 x$:
V. Popov absolute stability criteria (1959), **V. Yacubovich**, **R. Kalman** (KYP lemma)

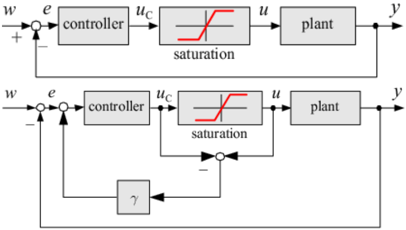
Contents

History: stability and oscillations in simulation of aircrafts

Accidents: crash of YF-22 (Lockheed/Boeing) 1992; crash of JAS-39 Gripen (SAAB) 1993.

Pilot-Induced Oscillation – sustained or uncontrollable oscillations resulting from efforts of the pilot to control with high precision and the actuator magnitude and rate saturation.

Windup: oscillations with increasing amplitude.



Actuator saturations may lead to controller windup, which dramatically changes the overall closed-loop system performance.

Antiwindup – a scheme to avoid windup in control systems with saturation.

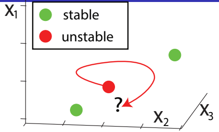
Lauvdal et al., *CDC, 1997*: **Since stability in simulations does not imply stability of the physical control system** *(an example is the crash of the YF22)* **stronger theoretical understanding is required.**

Contents

Simulation: analysis of stability and oscillations of dynamical models

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, .., x_n) \\ f_2(x_1, x_2, .., x_n) \\ \dots \\ f_n(x_1, x_2, .., x_n) \end{pmatrix} = \mathbf{F}(\mathbf{x})$$

$$\mathbf{x}_{t+1} = \mathbf{F}(\mathbf{x}_t)$$
$$\mathbf{x}^{eq} : \mathbf{F}(\mathbf{x}^{eq}) = \mathbf{0}$$
$$\mathbf{J}(\mathbf{x}^{eq}) = (\partial f_i / \partial x_j)$$

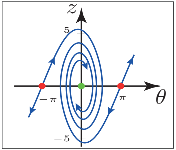


An oscillation can be visualized numerically if initial data from its vicinity lead to a long-term behavior that approaches the oscillation. From a computational point of view, such an oscillation (or a set osc.) is called an attractor, and its attracting set is called the basin of attraction. How to choose initial data in the basin of attraction?

Pendulum

$\dot{\theta} = z$

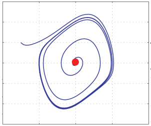
$\dot{z} = -\alpha z - \beta \sin\theta$



Van der Pol

$\dot{x} = y$

$\dot{y} = -x + \mu(1 - x^2)y$

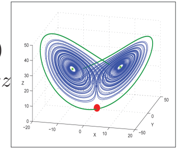


Lorenz

$\dot{x} = -\sigma(x - y)$

$\dot{y} = rx - y - xz$

$\dot{z} = -bz + xy$

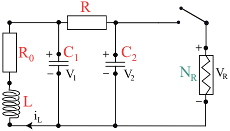


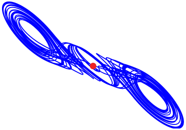
2009 (Kuznetsov, Leonov), 2009: An attractor is called a self-excited attractor if its basin of attraction intersects with small neighborhood of an equilibrium, otherwise it is called a hidden attractor. E.g. hidden attractors are attractors in the systems without equilibria, or with the only stable equilibrium (a case of multistability).

Attractors that do not attract trajectories from neighborhoods of equilibrium states are "hidden" from numerical and physical experiments.

Chua circuit and conjecture: self-excited and hidden attractors

$$\dot{x} = \alpha\big(y - x - f(x)\big),$$
$$\dot{y} = x - y + z,$$
$$\dot{z} = -(\beta y + \gamma z),$$



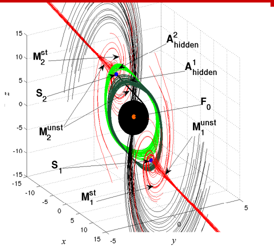


$$f(x) = m_1 x + \mathrm{sat}(x) = m_1 x + \tfrac{1}{2}(m_0 - m_1)(|x+1| - |x-1|)$$

1983–now: computations of hundreds Chua attractors by the standard computational procedure: a trajectory with initial data from vicinity of unstable zero equilibrium is attracted & visualizes the attractor. [*Bilotta&Pantano, A gallery of Chua attractors, 2008*]

L.Chua, 1990 (**conjecture**): If the zero equilibrium is stable ⇒ no attractors.

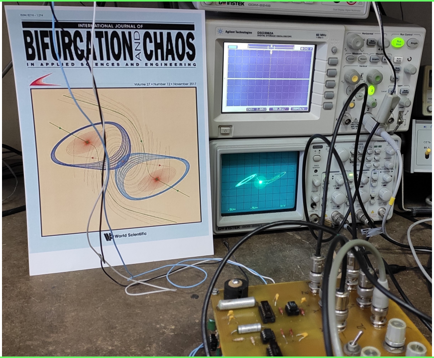
Two symmetric hidden chaotic attractors ($\mathcal{A}^{1,2}_{\text{hidden}}$ - green domains) in the classical Chua system: trajectories (red) from unstable manifolds $\mathrm{M}^{\text{unst}}_{1,2}$ of two saddle points $S_{1,2}$ are either attracted to the locally stable zero equilibrium F_0, or tend to infinity; trajectories (black) from stable manifolds $\mathrm{M}^{\text{st}}_{0,\pm}$ tend to F_0 or $S_{1,2}$



✓ Leonov G.A.,Kuznetsov N.V., Vagaitsev V.I, Localization of hidden Chua's attractors, Physics Letters A, 375(23), 2011, 2230-2233

Hidden Chua attractors and multistability: 5 coexisting attractors

"Chaotic" generalization of Hilbert's 16th problem: on **the number and disposition of attractors** in multi-dimensional chaotic dynamical systems; e.g., depending on the degree of polynomials in the rhs of the model.



What is the maximum number of coexisting attractors that can be found in the Chua system? How many of the coexisting attractors can be hidden?

2 trivial attractors (stable equilibria)
unstable equilibrium (2d stable manifold, 1d unstable manifold attracted by 2 trivial attractors);
3 hidden Chua attractors: periodic limit cycle and two symmetric chaotic Chua attractors

✓ Kuznetsov N., Mokaev T., Ponomarenko V., Seleznev E., Stankevich N., Chua L., Hidden attractors in Chua circuit: mathematical theory meets physical experiments, **Nonlin Dynamics**, 111(6), **2023**, 5859–5887
✓ N.V. Stankevich, N.V. Kuznetsov, G.A. Leonov, L. Chua, Scenario of the birth of hidden attractors in the Chua circuit, **Int. J. of Bifurcation and Chaos**, 27(12), **2017**, art. num. 1730038

16th Hilbert problem (second part) 1900:
number and mutual disposition of limit cycles for

$$\dot{x} = P_n(x,y) = a_1x^2 + b_1xy + c_1y^2 + \alpha_1 x + \beta_1 y + ...$$
$$\dot{y} = Q_n(x,y) = a_2x^2 + b_2xy + c_2y^2 + \alpha_2 x + \beta_2 y + ...$$

- ✓ N.N. Bautin 1949-1952: 3 nested small-amplitude limit cycles (LCs)
- ✓ I.G. Petrovskii, E.M. Landis 1955–1959: only 3 LCs
- ✓ L. Chen & M. Wang, S. Shi 1979-80: 1 large & 3 nested small LCs
- ✓ Y. Ilyashenko 1995: finiteness of the number of limit cycles

Visualization problem: nested limit cycles are hidden attractors!

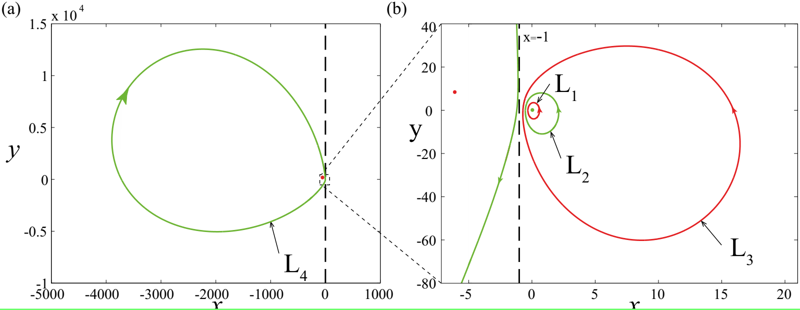
V. Arnold (2005): To estimate the number of LCs of square vector fields on plane, academician A.N. Kolmogorov had distributed several hundreds of such fields (with randomly chosen coefficients of quadratic expressions) among a few hundreds of students of Mech.& Math. Faculty of Moscow Univ. as a mathematical practice. Each student had to find the number of LCs of his/her field. The result of this experiment was absolutely unexpected: not a single field had a LC! It is known that a limit cycle persists under a small change of field coefficients. Therefore, the systems with one, two, three (and even, as has become known later, four) limit cycles form an open set in the space of coefficients.

Hidden attractors in Hilbert-Kolmogorov visualization problem

D.Hilbert (1900): number & mutual disposition of limit cycles

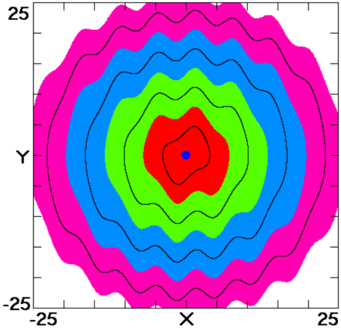
$$\dot{x} = P_n(x, y) = a_1x^2 + b_1xy + c_1y^2 + \alpha_1x + \beta_1y + \dots$$
$$\dot{y} = Q_n(x, y) = a_2x^2 + b_2xy + c_2y^2 + \alpha_2x + \beta_2y + \dots$$

In the right subfig there are 3 nested limit cycles around stable zero equilibrium.
Red – unstable limit cycles, green – stable limit cycles and equilibrium:
stable equilibrium coexists with stable limit cycle – a hidden atractor L_2.



N.V. Kuznetsov, O.A. Kuznetsova, G.A. Leonov, Visualization of four normal size limit cycles in two-dimensional polynomial quadratic system, Differential equations and Dynamical systems, 21(1-2), 2013, 29-34 (doi:10.1007/s12591-012-0118-6)

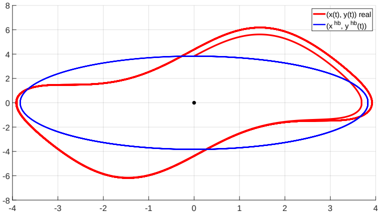
How to reveal all attractors in the phase space: infinite number of hidden attractors. Numerical search can't help to reveal infinite number of hidden attractors in unbounded domain of the phase space. Thus, analytical-numerical methods are needed.



The first four of infinitely many limit cycles and their basins of attraction. The blue dot is an unstable equilibrium.

$\ddot{x} + x - b\dot{x}\cos x = 0$

Hint: The harmonic balance method provides an infinite number of periodic solutions $x^{\mathrm{hb}}(t) \approx \beta_c \sin t$ where $J_1(\beta_c) = 0$ (Bessel function).



Parameters: $\beta_c = 3.83,\ b = 2.$

Approximation $\cos x \approx 1 - \frac{x^2}{2!} + ..$ gives the 16th Hilbert problem: is still unsolved.

Contents

Local stability and the birth (bifurcation) of oscillations

$\dot{x} = f(x)$ $\quad f : \mathbb{R}^n \to \mathbb{R}^n$, $f \in C^1$, $f(0) = 0$, $J(x) = Df(x)$ - Jacobian matrix

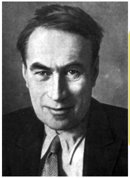
A.M. Lyapunov [1892] : Theory of stability

Def. Equilibrium $x = 0$ is said to be **Lyapunov stable**, if, $\forall \epsilon > 0$, $\exists \delta > 0$ such that, if $\|x(0)\| < \delta$, then $\forall t \geq 0$ we have $\|x(t, x_0)\| < \epsilon$.

Def. Equilibrium x_e is said to be **asymptotically stable** if it is Lyapunov stable and $\exists \delta > 0$ such that, if $\|x(0)\| < \delta$, then $\lim_{t \to \infty} \|x(t, x_0)\| = 0$.

Thm. Consider a function $V : \mathbb{R}^n \to \mathbb{R}$, $V \in C^1$: $V(x) = 0$ iff $x = 0$, $V(x) > 0$ iff $|x| < h$, $x \neq 0$, $\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) \leq 0$, $|x| < h$.
Then $x_e \equiv 0$ is locally stable (for asymptotic stability: $\dot{V}(x) = 0$ iff $x = 0$).

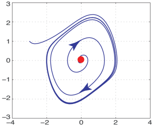
Thm. If the Jacobian matrix $J(0)$ has all eigenvalues with negative real parts: $\lambda_{1,..,n}(0) < 0$, then exist the Lyapunov function and $x = 0$ is asymptotically stable.



A.A. Andronov [1937] : Theory of oscillations

The birth of self (excited) oscillations (1929): the *generation* and *maintenance* of a periodic motion by a source of power that lacks any corresponding periodicity. [limit cycle in Van der Pol model: $\dot{x} = y, \dot{y} = -x + (1 - x^2)y$]

A.A. Andronov [1937]: Theory of bifurcations



Van der Pol oscillator

Local and global stability: boundaries in the space of parameters

A.Andronov, N.Bautin (1937-1939): Andnonov-Hopf local bifurcation, Bautin safe & dangerous boundaries of local stability

A.Andronov, G.Mayer (1944): Boundaries of linear and global stability coincide in the central problem of direct regulation theory $\ddot{x}+B\dot{x}+Ax=y-\frac{1}{2}\mathrm{sign}\dot{x}$, $\dot{y}=-x$. All trajectories tend to the rest segment. I.Vyshnegradsky studied linear model. In the summary of Andronov's results for election to the Academy of Science, 1946.

M.Aizerman (1949), **R.Kalman** (1957), **L.Markus-H.Yamabe** (1960): conjectures on global stability by the first approximation. If the Jacobian $J(x)$ has all eigenvalues with negative real parts $\forall x \in \mathbb{R}^n : \lambda_{1,..,n}(x) < -\delta < 0$, then the system is globally stable.

Cima et al.[1997]: $\dot{x}_1 = -x_1 + x_3(x_1+x_2x_3)^2$, $\dot{x}_2 = -x_2 - (x_1+x_2x_3)^2$, $\dot{x}_3 = -x_3$, $\lambda_{1,2,3}(x) = -1$, solution $(x_1(t), x_2(t), x_3(t)) = (18\exp(t), -12\exp(2t), -\exp(-t))$.

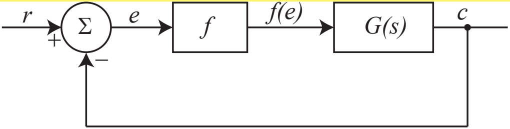
Boundary of global stability in the space of parameters: *explicit parts* defined by local bifurcations, *hidden parts* defined by nonlocal bifurcation (in the conjectures on global stability it is expected that the boundary is trivial).

Outer estimation of the global stability boundary: linearisation around equilibria and local bifurcations analysis. Inner estimations: sufficient criteria of global stability

Kuznetsov N.V., Theory of hidden oscillations and stability of control systems, Journal of Computer and Systems Sciences International RAS, 59(5), 2020, 647–668.

Kalman conjecture on absolute stability of control systems

if $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{b}k\mathbf{c}^*\mathbf{z}$, is asympt. stable $\forall\, k \in (k_1, k_2)$: $\mathbf{z}(t, \mathbf{z}_0) \to 0\ \forall \mathbf{z}_0$, then
is $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\varphi(\sigma)$, $\sigma = \mathbf{c}^*\mathbf{x}$, $\varphi(0) = 0$, $k_1 < \varphi(\sigma)/\sigma$ and $\varphi'(\sigma) < k_2$: $\mathbf{x}(t, \mathbf{x}_0) \to 0\ \forall \mathbf{x}_0$?



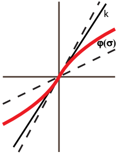


Fig.1. Nonlinear control system. $G(s)$ is a linear transfer function, $f(e)$ is a single-valued, continuous, and differentiable

In 1957 R.E. Kalman wrote [1]: "If $f(e)$ in Fig. 1 is replaced by constants K corresponding to all possible values of $f'(e)$, and it is found that the closed-loop system is stable for all such K, then it is intuitively clear that the system must be monostable; i.e., all transient solutions will converge to a unique, stable critical point."

DFM: there are no periodic solutions in the case of Kalman's conditions

Engineering intuition: the statement is true for models in $\mathbb{R}^{1,2,3}$!

In $\mathbb{R}^4$ it is not true: a nontrivial attractor can coexist with stable zero equilibrium!

In 2013 he wrote about examples with hidden attractors: "I was far too young and lacking technical mathematical knowledge to go more deeply into the matter."

Global stability and the birth of oscillations

A.Lurie, V.Postnikov (1944): $\dot{x}=Ax+b\varphi(c^*x), \varphi(0)=0$, matrix A, vectors b,c
Lyapunov function for the global analysis: $V(x,\theta_e) = x^T Hx + \int_0^{c^*x} \varphi(\sigma)d\sigma$.

E.A. Barbashin, N.N. Krasovsky [1952]: **Theory of global stability**

Thm. Let $V(x) : \mathbb{R}^n \to \mathbb{R}$, $V \in C^1$:
$V(x) > 0 \ \forall x \neq 0$ and $V(0) = 0$,
$\dot{V}(x) < 0 \quad \forall x \neq 0$, $V(x) \to +\infty$ as $||x|| \to +\infty$.
Then any solution tends to the equilibrium $x \equiv 0$.

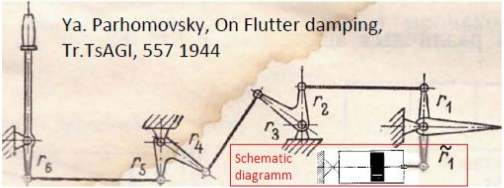
J.P. LaSalle [1960]: stability in large and generalization for multiple equilibria.

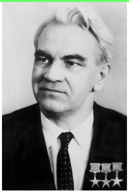
A.Gelig, G. Leonov (discontinuous systems); G.Leonov (cylindrical phase space)
Cylindrical phase space: $V(z,\sigma) = z^T Hz + \int_0^{\sigma} \sin(s)ds \not\to +\infty$ as $||(z,\sigma)|| \to +\infty$

The boundaries of global stability are determined by the birth of oscillations due to local bifurcations in the vicinity of the stationary set (explicit boundary of the global stability) or nonlocal bifurcations (hidden boundary of the global stability).
If an oscillation is born via such a nonlocal bifurcation of the loss of global stability, then it is a hidden oscillation, since the basin of its attraction is separated from the locally attractive stationary set. The revealing of hidden oscillations and obtaining initial data for their computation are challenging problems.

Hidden attractors in aircraft control and Keldysh model

M. Keldysh, On dampers with a nonlinear characteristic, Tr. TsAGI 557 (1944) 26-37

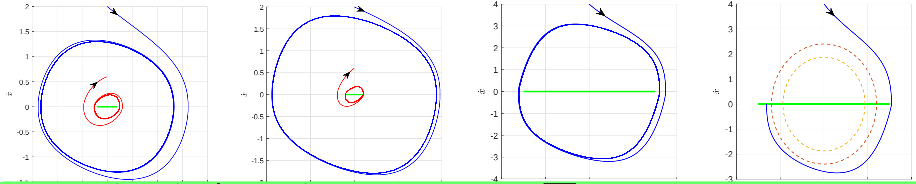




Flutter suppression by hydraulic damper: resistance $f(\dot{x}) = \lambda\dot{x} + (\Phi + \kappa\dot{x}^2)\mathrm{sign}(\dot{x})$

Keldysh by DFM: $J\ddot{x} + kx = h\dot{x} - f(\dot{x}), \mu = -h + \lambda, \varphi(\dot{x}) = f(\dot{x}) - \lambda\dot{x}$ (1d of freedom)
$h < \lambda + 2.08\sqrt{\Phi\kappa} \Rightarrow$ all trajectories converge to the equilibria segment $[-\Phi/k, \Phi/k]$
$h > \lambda + 2.08\sqrt{\Phi\kappa} \Rightarrow$ there are two limit cycles $a_\pm(\mu)\cos(\omega t)$, $a_\pm(\mu) = \ldots$;

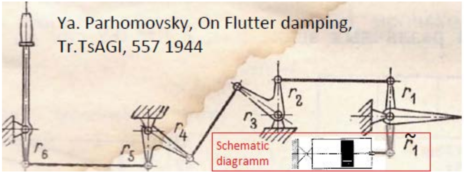
Leonov G.A., Kuznetsov N.V., On the Keldysh problem of flutter suppression, **Doklady Physics**, 2018

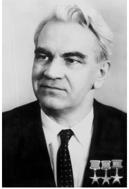


$V = \frac{1}{2}(J\dot{x}^2 + kx^2)$, $\dot{V} = -\mu\dot{x}^2 - \dot{x}\varphi(\dot{x}) \Rightarrow h < \lambda + 2\sqrt{\Phi\kappa}$

Hidden attractors in aircraft control and Keldysh model

M. Keldysh (TsAGI,1944): Flutter suppression by hydraulic damper





$$c_{11}\ddot{x}+c_{12}\ddot{y}+b_{11}\dot{x}+b_{12}\dot{y}+a_{11}x+a_{12}y=0, \quad \mu=b_{22}+\lambda$$

$$c_{21}\ddot{x}+c_{22}\ddot{y}+b_{21}\dot{x}+(b_{22}+h)\dot{y}+a_{21}x+a_{22}y=(h-\lambda)\dot{y}-(\Phi+\kappa\dot{y}^2)\mathrm{sign}(\dot{y})$$

He had only DFM (developed by 1944) and wrote: " We do not give a rigorous math proof, we construct a number of conclusions on intuitive considerations ..."

Nowadays we can apply rigorous analytical and reliable numerical methods:
Keldysh by DFM: stability $\lambda > 1.33$; hidden attractor $\lambda \in [1.33, 1.34]$, stability $\lambda > 1.65$

Theorem (2018). If $\varphi(\sigma)\sigma \geq \delta_e\sigma^2$ for $\delta_e \geq 0$; $\varphi'(\sigma) > -\delta$, $\sigma \neq 0$, for $\delta \geq 0$; $\tau \geq 0$, θ: $\mathrm{Re}(\omega^2+\theta i\omega+\tau)W(i\omega)-\delta|W(i\omega)|^2\omega^2+\tau\delta_e|W(i\omega)|\geq 0, \forall\omega\in\mathbb{R}$,
then all solutions tend to the stationary set.

Keldysh model of flutter suppression

M. Keldysh (TsAGI,1944): Flutter suppression by hydraulic damper

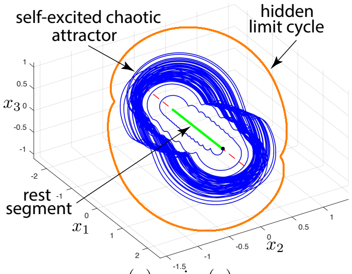
$$c^*(A - pI)b = \frac{p(p+0.01)}{((p+0.1)^2+0.9^2)((p+0.1)^2+1.1^2)}.$$

Sector of linear stability: $(0, +\infty)$.

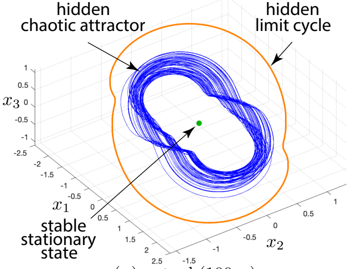
Counterexamples to the Kalman conjecture

Lauvdal T., Murray R., Fossen T., *IEEE CDC, 1997*: *Since stability in simulations does not imply stability of the physical control system (an example is the crash of YF-22 (Lockheed/Boeing) in 1992) stronger theoretical understanding is required.*

E. Kudryashova, N. Kuznetsov, O. Kuznetsova, G. Leonov, R. Mokaev, **Harmonic Balance Method and Stability of Discontinuous Systems**. Dynamics and Control of Advanced Structures and Machines. Eds: Matveenko V., Krommer M., Belyaev A., Irschik H., Springer 2019

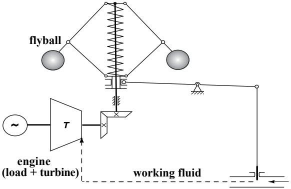


$\varphi(\sigma) = \mathrm{sign}(\sigma)$



$\varphi(\sigma) = \tanh(100\ \sigma)$

Classical engineering models: Watt regulator

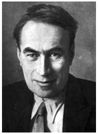


Watt's flyingball regulator (1788): maintaining the constant speed.
Vyshnegradsky (1877): math model
$\ddot{x} + B\dot{x} + Ax = y - \frac{1}{2}\text{sign}\dot{x}, \ \dot{y} = -x$
working regime: locally stable equilibrium

Local analysis (without $\text{sign}(\cdot)$): $AB > 1 \Rightarrow$ locally stable equilibrium state

Léauté H. (1885), Zhukovsky N.E. (1909): Vyshnegradsky 'conjecture' (local stability of equilibrium state implies global stability) is not justified.

Approx. global analysis: harmonic balance method (Krylov, Bogolyubov, 1937)



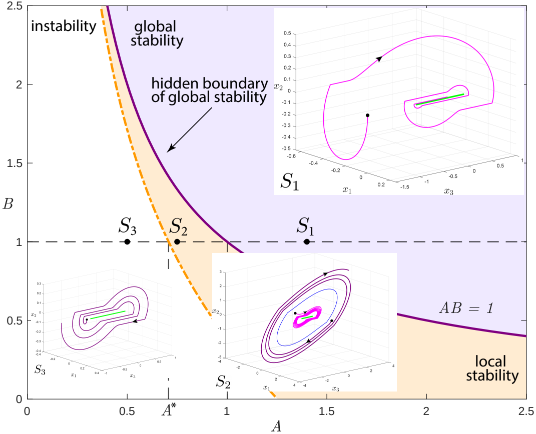
Andronov, A., (Vitt A.,) Khaikin, S. The theory of oscillations, 1937
- explanation of physical oscillating regimes as Poincare's limit cycles
- rigorous treatment of discontinuous dynamical models
- did not cover multidimensional systems and chaos, numerical methods

Precise global analysis: point-mapping method (Andronov, Mayer, 1944).
Stability of 'linearized' system implies global stability: no hidden oscillations!

Watt regulator: boundary of global stability

$\ddot{x}+B\dot{x}+Ax=y-f(\dot{x})$, $\dot{y}=-x$, $f(\dot{x})=\frac{1}{2}\text{sign}\dot{x}$ 'Linearized': $\dddot{x}+B\ddot{x}+A\dot{x}=-x$
Sector of linear stability: $(0,+\infty)$ Kalman conjecture holds for smooth nonlinearity.
Trivial boundary of global stability for the class of $f(\cdot)$ in the Kalman conjecture.

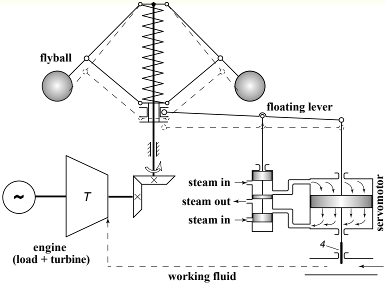
Lyapunov method: the stationary set is globally stable if $AB>1$.
HBM: no periodic solutions if $AB>1$;
Tsypkin&LPRS methods: there's a region in the parameter space where unstable periodic solutions exist (when $AB<1$).
Numerical modeling: there's a region in the parameter space where the stationary set is locally stable (when $AB<1$). $\Rightarrow$ The global stability boundary is **hidden**.

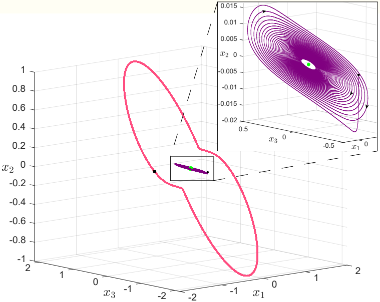


N.V. Kuznetsov et al., Andronov-Vyshnegradsky problem on Watt governor and Kalman conjecture on global stability, IFAC-PapersOnLine 56(2), 2023, 4460-4465

Classical engineering problem: Watt regulator with servomotor

Math model: $\ddot{x} + B\dot{x} + Ax = y - \frac{1}{2}\text{sign}(\dot{x}),\ \dot{y} = -z,\ \alpha\dot{z} + z = x.$





Linear analysis: $A > 0, B > 0, AB > 1, 0 \leq \alpha \Rightarrow$ locally stable equilibrium state

Consider values of parameters $A = 1.3, B = 1.2, \alpha = 1.1.$

Approx. global analysis (Harmonic balance method): no periodic oscillations.

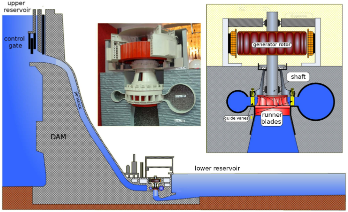
Precise global analysis (Point-mapping method): difficult to apply (model is 4D).

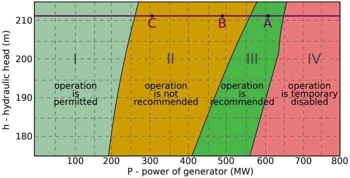
Sufficient conditions of global stability: $A > 0, B > 0, AB > 1, 0 \leq \alpha < \frac{AB-1}{B}$.
G. Leonov (1971): global Lyapunov functions (discontinous) for differential inclusions

Stability of 'linearized' system does not implies global stability!

Control system for hydraulic units of the Sayano-Sheshenskaya HPP

Stability (regulation) and oscillations in a closed system: $\dot{x} = F(x)$
A.I. Vyshnegradsky (1877), A.A. Andronov (1944): **"regulator+turbine"**
$\ddot{y} + \alpha\dot{y} + \beta y = z - \mathrm{sign}(\dot{y}), \quad \dot{z} = -y$
F. Tricomi (1933), A.A. Yanko-Trinitsky (1958): **"electric machine+load"**
$\ddot{\theta} + \alpha\theta + \beta\sin\theta = \gamma$





Oscillations in a closed nonlinear model of a hydraulic unit: $\dot{x} = F(x)$
"regulator + turbine + generator + load" (10th order system)
When modeling hydraulic units of the Sayano-Shushenskaya HPP, areas corresponding to undesirable oscillatory modes were found, consistent with full-scale tests at the Sayano-Shushenskaya HPP in 1988 (CNSCS: Kuznetsov et. al., 2019)

Analysis of external disturbances $f(t,x)$: $\dot{x} = F(x) + f(t,x)$
V.S. Seleznev, A.V. Liseykin, A.A. Bryksin, P.V. Gromyko (2015): seismology
R.F. Ganiev (2013): nonlinear oscillations in a water pipeline and resonances

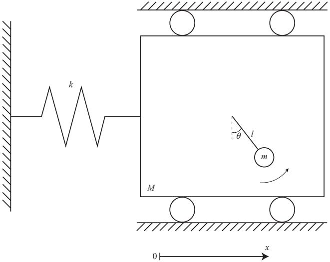
Hidden attractors in Sommerfeld effect

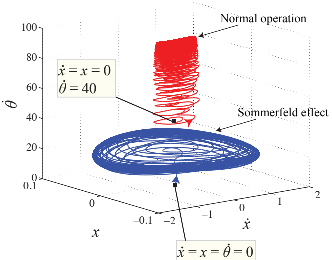
Hidden attractors in systems without equilibria

D.S.Bernstein et al. (199x); A.Fradkov, O.Tomchina, D.Tomchin (2011) Controlled passage through resonance in mechanical systems. J. of Sound and Vibration, 330(6), 1065-1073.

A. Sommerfeld effect (1902): represents the inability of a system to be spun up by a torque-limited rotor to a desired rotational velocity due to its resonant interaction with another part of the system.

$$(M+m)\ddot{x}+k_1\dot{x}+ml(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta)+kx=0, \quad J\ddot{\theta}+k_\theta\dot{\theta}+ml\ddot{x}\cos\theta=u$$





Kiseleva M.A., Kuznetsov N.V., Leonov G.A., Hidden attractors in electromechanical systems with and without equilibria, **IFAC-PapersOnLine**, 49(14), **2016**, 51-55
G.A. Leonov, N.V. Kuznetsov, E.P. Solovyeva, Mathematical modeling of vibrations in turbogenerator sets of Sayano-Shushenskaya hydroelectric power station, **Doklady Physics**, 61(2), **2016**, 55-60

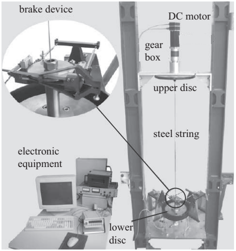
Hidden attractors in drilling models

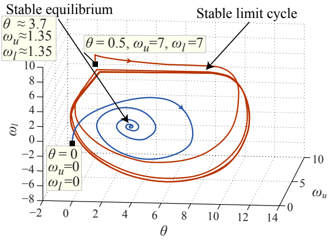
De Bruin et al., 2009 (Eindhoven University of Technology):

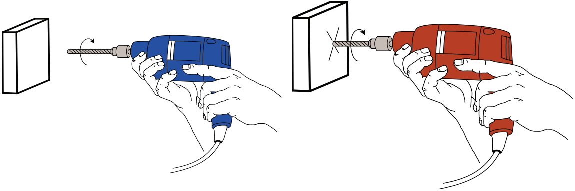
$$J_u\ddot{\theta}_u + k_\theta\left(\theta_u - \theta_l\right) + b\left(\dot{\theta}_u - \dot{\theta}_l\right) + T_{fu}\left(\dot{\theta}_u\right) - k_m v = 0,$$

$$J_l\ddot{\theta}_l - k_\theta\left(\theta_u - \theta_l\right) - b\left(\dot{\theta}_u - \dot{\theta}_l\right) + T_{fl}\left(\dot{\theta}_l\right) = 0.$$

$\theta = \theta_u - \theta_l$ are angular displacements of the upper and lower discs
$J_{u,l}$ are constant inertia torques,
b is rotational friction, k_θ is the torsional spring stiffness,
DC motor: v – constant input voltage, k_m – motor constant,
$T_{fu,fl}(\dot{\theta}_{u,l})$ are friction torques acting on the upper and lower discs.







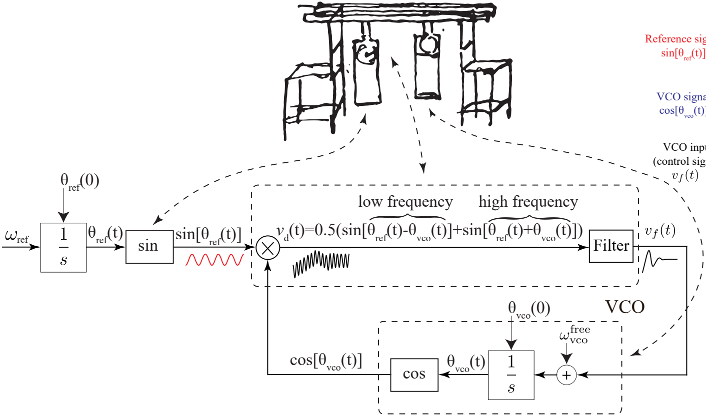
Computer modeling.
Hidden oscillations in space $(\theta = \theta_u - \theta_l, \dot{\theta}_u, \dot{\theta}_l)$:
stable limit cycle coexists with stable equilibrium state.

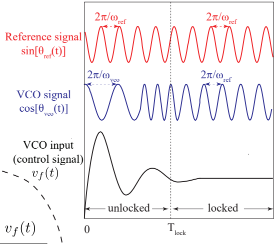
Leonov G.A., Kuznetsov N.V., Kiseleva M.A. et al., Hidden oscillations in mathematical model of drilling system actuated by induction motor with a wound rotor, **Nonlinear Dynamics**, 77(1-2), 2014, 277-288

Contents

Phase-Locked Loop operation principle

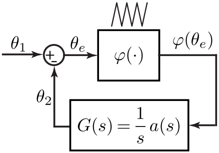
Mutual synchronization of oscillators phases: synchronization of two weakly coupled pendulum clocks, Christiaan Huygens, “Horologium Oscillatorium”, 1673





Master-slave synchronization of oscillators phases:
Phase-locked loop synchronization, Edward Appleton, 1923
(Nobel Prize winner (1947) and pioneer in radiophysics)

Phase-Locked Loop mathematical model



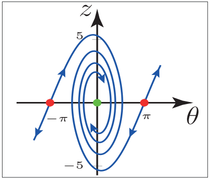
$$\varphi(\theta_e + 2\pi) = \varphi(\theta_e),\ \forall \theta_e \in \mathbb{R},\ G(s) = \frac{1}{s}a(s) = \frac{1}{s}K_{\text{vco}}F(s)$$

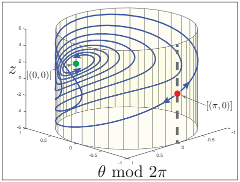
$$\dot{z} = Pz - q\varphi(r^T z), \quad z(t) = \big(u_1(t), \quad u_2(t)\big)^T \in \mathbb{R}^2,$$

Cylindrical phase space: $V(z,\sigma) = z^T H z + \int_0^\sigma \sin(s)ds \not\to +\infty$ as $||(z,\sigma)|| \to +\infty$
G.Leonov, Y.Bakaev (Barbashin-Krasovsky theorem for cylindrical space)

Pendulum

$$\dot{\theta} = z$$
$$\dot{z} = -\alpha z - \beta \sin\theta$$



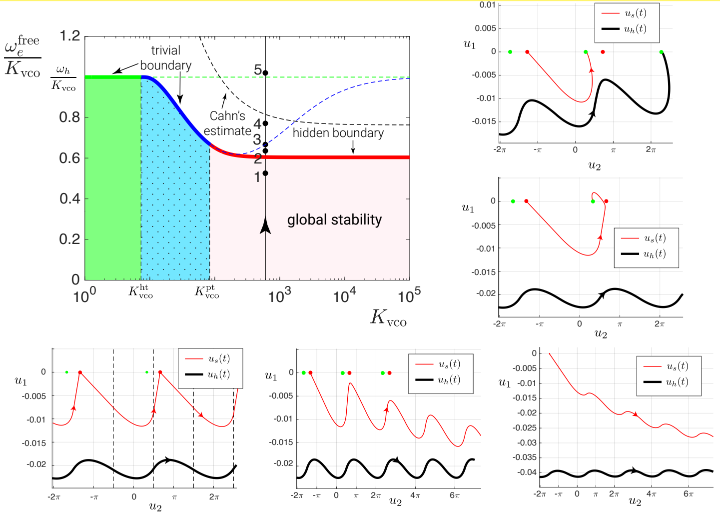


Although a PLL model can be written as a closed-loop negative feedback system, general theory of PLLs and its usage is somewhat undertreated in the control literature.

D. Abramovitch, "Phase-locked loops: A control centric tutorial," in American Control Conf. Proc., vol. 1. IEEE, 2002, 1-15.

Global stability problem: trivial and hidden boundary

Kapranov's conjecture (1956): the loss of global stability is determined by the birth of self-excited oscillations (i.e. the global stability boundary is trivial).



Kuznetsov N.V., Lobachev M.Y., Mokaev T.N., Hidden boundary of global stability in a counterexample to the Kapranov conjecture on the pull-in range, Doklady Mathematics, 108(4), 2023, 300–308

Robust Regression with a Huber-like Loss and Polyak Momentum (Heavy–Ball) Method

Dataset. The CALIFORNIA HOUSING dataset ($\sim 2\,000$ training samples) relates socio-economic indicators to the median house value. E.g., pick single feature (*median income*) x_i and target y_i (median house value).

Model and loss. A scalar weight w predicts $\hat{y}_i = wx_i$.

Following Lessard et. al. (2016), loss is introduced via the *Huber-like* error, with threshold $\delta = 1$:

$$L(w) = \frac{1}{N}\sum_{i=1}^{N} \begin{cases} \frac{1}{2}(y_i - wx_i)^2, & |y_i - wx_i| \le \delta, \\ \delta\big(|y_i - wx_i| - \frac{1}{2}\delta\big), & \text{otherwise.} \end{cases}$$

Polyak, 1964: for a differentiable loss $L : \mathbb{R}^d \to \mathbb{R}$ and constants $\gamma > 0$, $\mu \in [0, 1]$:

$$w_{k+1} = w_k - \gamma \nabla L(w_k) + \mu\big(w_k - w_{k-1}\big), \qquad k \ge 1,$$

with $w_{0,1}$ initialized arbitrarily. Here, take $\gamma = 0.18$, $\mu = 0.15$.

Hidden dynamics. Starting at $w_{0,1} = 1.5$ the iterates settle into the unique minimizer $w^* \approx 1.41$ (locally stable equilibrium). In contrast, distant initialization $w_{0,1} = -20$ converges to the **hidden 3-cycle** $\{-8.93, 3.78, 8.21\}$ (locally attractive periodic orbit).

✓ B.T. Polyak. Some methods of speeding up the convergence of iteration methods. USSR Computational Mathematics and Mathematical Physics, 1964.
✓ L. Lessard, B. Recht, A. Packard. Analysis and Design of Optimization Algorithms via Integral Quadratic Constraints, 2016.
✓ N.V. Kuznetsov, T.N. Mokaev, R.N. Mokaev. Hidden Oscillations in the Dynamical Models of the First–Order Optimization Algorithms, 2025 (in progress).

Conclusion

The considered examples are the motivation to apply nonlinear rigorous method for the study of dynamical system, where it is possible; where it is impossible – to use simulation, but take in account possible problems with attractors localization.

Boundary of global stability in the space of parameters: trivial parts defined by local bifurcations, hidden parts defined by nonlocal bifurcation (in the conjectures on global stability it is expected that the boundary is trivial).

Inner estimations: sufficient criteria of global stability. There special criteria for the cylindrical phase space. Outer estimation of the global stability boundary: linearisation around equilibria and local bifurcations analysis.

Boundary of global attractivity (monoatractivity: unique local attractor is a global attractor). In Euclidean phase space the global attractor contains an equilibrium (thus various local bifurcations methods can be used). However hidden (local) attractors have basins which are not connected with equilibria. In cylindrical phase space there are global attractors without equilibria.

The approach based on the existing methods of bifurcation theory and the numerical packages is effective in studying possible scenarios for the birth of attractors and for the analysis of already discovered attractors. However, when solving problems of multistability or global stability, where it is necessary to find all nontrivial attractors or to establish their absence, the key problem of this approach is inability to guarantee the absence of other attractors in the phase space than those found.

Leading Scientific school (Center of Excellence)

Свидетельство

Выдано

коллективу ведущей научной школы

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Математика и механика

Руководитель
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Control of multistability

Alexander N. Pisarchik [a, b], Ulrike Feudel [c]

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Physics Reports

Volume 637, 3 June 2016, Pages 1-50

Hidden attractors in dynamical systems

Dawid Dudkowski [a], Sajad Jafari [b], Tomasz Kapitaniak [a], Nikolay V. Kuznetsov [c, d], Gennady A. Leonov [c], Awadhesh Prasad [e]


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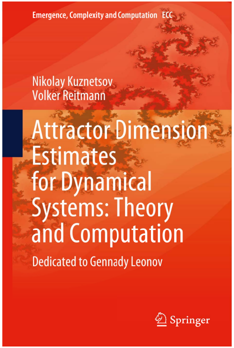

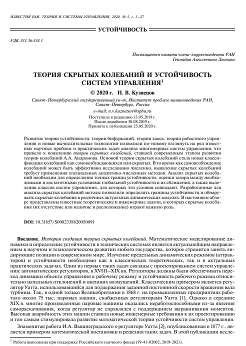

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ТЕОРИЯ СКРЫТЫХ КОЛЕБАНИЙ И УСТОЙЧИВОСТЬ СИСТЕМ УПРАВЛЕНИЯ[1]

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Развитие теории устойчивости, теории бифуркаций, теории хаоса, теории робастного управления и новые вычислительные технологии позволили по-новому взглянуть на ряд известных научных проблем и практических задач анализа многомерных систем управления, что привело к появлению *теории скрытых колебаний*, ставшей современным этапом развития теории колебаний А.А. Андронова. Основой теории скрытых колебаний стала новая классификация колебаний как самовозбуждающихся или скрытых. В то время как самовозбуждение колебаний может быть эффективно исследовано численно, выявление скрытых колебаний требует применения специальных аналитико-численных методов. Анализ скрытых колебаний необходим для определения точных границ устойчивости, оценки зазора между необходимыми и достаточными условиями глобальной устойчивости и их сближения, а также выделения классов систем управления, для которых эти условия совпадают. Разработанные для анализа скрытых колебаний методы позволили определить границы устойчивости и обнаружить скрытые колебания в различных актуальных динамических моделях. В настоящем обзоре представлены известные теоретические и инженерные задачи, в которых скрытые колебания (их отсутствие или наличие и расположение) играют важную роль.

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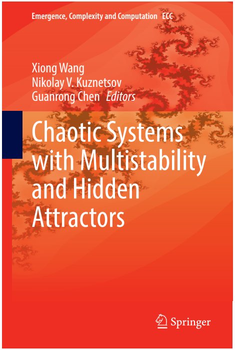
Введение. *История создания теории скрытых колебаний.* Математическое моделирование динамики и определение устойчивости в технических системах является актуальнейшим направлением в научном и технологическом развитии любого государства, которое стремится занять лидирующие позиции в современном мире. Изучение предельных динамических режимов (аттракторов) и устойчивости необходимо как в классических теоретических, так и в актуальных практических задачах. Одни из первых таких задач связаны с проектированием систем управления: автоматических регуляторов, в XVIII–XIX вв. Регуляторы должны были обеспечивать переход динамики объекта управления к рабочему режиму и устойчивость рабочего режима относительно начальных отклонений и внешних возмущений. Классическим примером является регулятор Уатта, использовавшийся для поддержания заданной постоянной скорости вращения вала турбины. Так, в одной только Великобритании в 1868 г. на промышленных предприятиях работало около 75 тыс. паровых машин, снабженных регуляторами Уатта [1]. Однако в середине XIX в. многие произведенные паровые машины оказались неработоспособными из-за явления самораскачивания, когда регулятор не справлялся с поддержанием выравнивания моментов. Высокая аварийность этих машин ставила новые инженерные требования к их проектированию и тем самым стимулировала развитие математической теории устойчивости систем управления.

Знаменитая работа И.А. Вышнеградского о регуляторе Уатта [2], опубликованная в 1877 г., является примером математической постановки и решения таких задач. В этой публикации иссле-

[1] Работа выполнена при поддержке Российского научного фонда (19-41-02002, 2019-2021).



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