Gaudin models and Laplace operator on a flag manifold

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Based on the joint work with D. Bykov:

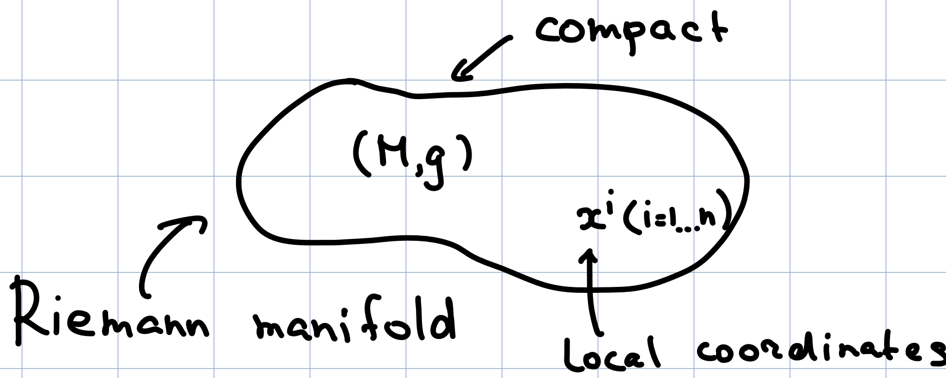
arXiv: 2508.20889 [hep-th]

arXiv: 2404.15900 [hep-th]

Plan:

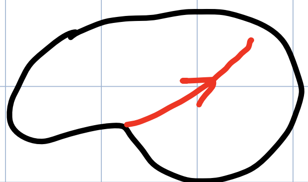
- 1D sigma models
- Flag manifolds and their geometry
- Spin chains:
  1) quantum level
  2) „classical" level
- Correspondence between $SU(n)$ spin chains and 1D sigma models
- Example: $\mathcal{F}(3)$ (If time permits...)

1D sigma models



$$\boxed{S = \frac{1}{2}\int dt \; g_{ij}(x)\,\dot{x}^i\dot{x}^j \; \text{„} + \int dt \; A_i(x)\,\dot{x}^i \text{"}}$$ ← 1D sigma model „with magnetic field"

CM = geodesics on M



$$\nabla_{\dot\gamma}\dot\gamma = 0$$

$$\boxed{\frac{d^2}{dt^2}x^i + \Gamma^i_{jk}\,\dot{x}^j\dot{x}^k = 0}$$

geodesic equation

QM = spectrum of the Laplace-Beltrami operator

$$H\psi = E\psi \quad , \psi \in \mathcal{L}^2(M)$$

$$H = -\Delta = -\boxed{\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{g}\,\frac{\partial}{\partial x^j}\right)}$$

Laplace - Beltrami operator

These are two very tricky problems even for homogeneous spaces...

Flag manifolds

Complete flag manifold: $\mathcal{F}(N) = \dfrac{U(N)}{U(1)^{\times N}}$

symplectic, homogeneous space of $U(N)$

[(co)adjoint orbit of $SU(N)$]

Parametrization: $g \in U(N)$: $\quad g = (u_1, u_2 \dots u_N)$, u_i - column of g

$$\boxed{\bar{u}_i \cdot u_j = \sum_{\alpha=1}^{N} (u_i^\alpha)^* u_j^\alpha}$$

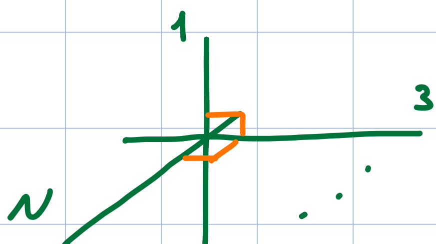
$$g \in \mathcal{F}(N): \quad g \sim gh, \text{ where } h = \mathrm{Diag}(e^{i\varphi_1} \dots e^{i\varphi_N})$$

$$\Downarrow$$

$$u_i \sim u_i e^{i\varphi_i}, \text{ i.e. } u_i \in \mathbb{CP}^{N-1} \text{ and } \underbrace{|u_i| = 1}$$

just a matter of convenience

Geometric picture: a point of $\mathcal{F}(N)$ = N mutually orthogonal lines in $\mathbb{C}^N$



Geometry of $\mathcal{F}(N)$ I

(Left) invariant metric on $\mathcal{F}(N)$ [Arvanitoyeorgos '93]:

$$ds^2 = \sum_{i<j} \frac{1}{\alpha_{ij}} |\bar{u}_i \cdot du_j|^2$$

So, the action of 1D model:

$$S = \frac{1}{2}\int dt \sum_{i<j} \frac{1}{\alpha_{ij}} |\bar{u}_i \cdot \dot{u}_j|^2 , \quad \bar{u}_i \cdot u_j = \delta_{ij}$$

$$u_i \sim u_i \, e^{i\varphi_i}$$

Lagrangian embedding: $\mathcal{F}(N) \ni (u_1 \ldots u_N) \xhookrightarrow{\mathcal{E}} \{u_i\}_{i=1}^{N} \in (\mathbb{CP}^{N-1})^{\times N}$

$$\omega_P = \sum \omega^i_{FS} ,$$

$$\omega^i_{FS} \equiv i \, d\bar{u}_i \wedge du_i \leftarrow \text{Standard Fubini-Studi form}$$

$$\omega_P|_{\mathcal{E}(\mathcal{F}(N))} = 0$$

$$\dim \mathcal{F}(N) = \tfrac{1}{2} \dim (\mathbb{CP}^{N-1})^{\times N}$$

Geometry of $\mathcal{F}(N)$ II

Even more general observation:

$\sharp\ (\mathbb{CP}^{N-1})^{\times N} \ni Z = (z_1 \dots z_N) \overset{\mathcal{M}}{\rightsquigarrow} Z = UH$, $\quad U^{\dagger}U = \mathbb{1}$, i.e. U is unitary

$\det z \neq 0$ $\qquad$ $H^{\dagger} = H$ $\qquad$ $(\|H\| < \text{const} = C)$

Polar decomposition

$$\boxed{U = z\,(z^{\dagger}z)^{-1/2}\,,\quad H = (z^{\dagger}z)^{1/2}}$$

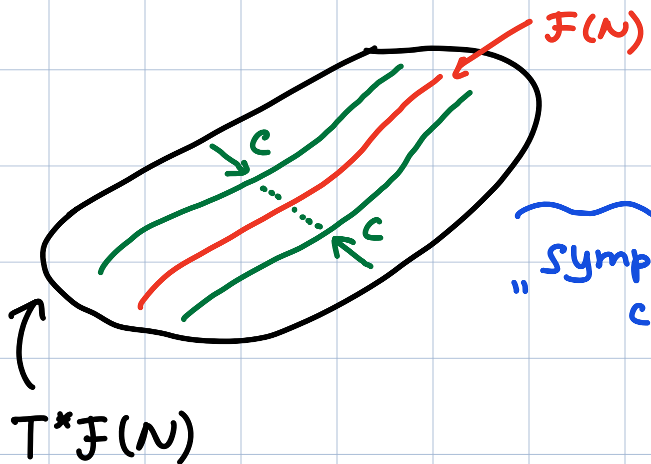
$\exists\ \mathfrak{X} = \{\det z \neq 0\}$

$$\begin{array}{ccc} \omega_P|_{\mathfrak{X}} & & \omega^{ct} \\ \downarrow & & \downarrow \\ \mathfrak{X} & \overset{\mathcal{M}}{\rightsquigarrow} & T^{*}\mathcal{F}(N) \end{array} : \quad \mathcal{M}^{*}\,\omega^{ct}\big|_{\mathcal{M}(\mathfrak{X})} = \omega_P\big|_{\mathfrak{X}}$$

$z \longmapsto (U,\ K = H^2)$

$\boxed{\mathfrak{X} \text{ is symplectomorphic to an open subset of } T^{*}\mathcal{F}(N) \text{ which contains } \mathcal{F}(N)}$

U is a point of $\mathcal{F}(N)$, H is a „momentum"

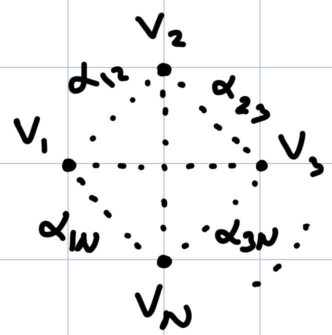


„symplectic compactification"

$$\underbrace{\overset{1}{\mathbb{CP}^{N-1}} \times \overset{2}{\mathbb{CP}^{N-1}} \times \dots \times \overset{N}{\mathbb{CP}^{N-1}}}$$

actually, it's without z: $\det z = 0$

Spin chains (or models? :)



$\exists\ V_i$ is a irrep of $SU(N)$ $\left(\text{Our case: } V_i = \underbrace{\square\square\square\square\cdots\square}_{p}\right)$

$\exists\ S_i^a$ is $\underset{\substack{\uparrow \\ \text{generator of } su(N)}}{\tau^a}$ in V_i $(a=1\dots \dim SU(N)) = \mathbb{1}\otimes\mathbb{1}\otimes\dots\otimes \underset{\substack{\uparrow \\ i}}{S^a}\otimes\dots\otimes\mathbb{1}$

$$\boxed{H = \sum_{i<j}\alpha_{ij}\sum_a S_i^a S_j^a} \leftarrow \text{Hamiltonian} \qquad \left(\mathrm{Tr}(\tau^a\tau^b) = \delta^{ab}\right)$$

$$\text{Hilbert space} = V_1\otimes V_2\otimes\dots\otimes V_N = \underbrace{\square\square\square\square\cdots\square}_{p}{}^{\otimes N}$$

If $\alpha_{ij} = \dfrac{t_i - t_j}{m_i - m_j}$ then the model is a Gaudin model

$$\Downarrow$$

- $H_i \equiv \sum_{j\neq i}\dfrac{1}{m_i - m_j}\sum_a S_i^a S_j^a, \quad i=1\dots N \ : \ [H_i, H_j] = 0$
- There exists Bethe ansatz method, which provides a way to compute the spectrums of H_i's! (and of H)

"Classical" spin chain

$\Rightarrow \quad (\mathbb{CP}^{N-1})^{\times N} \ni Z = (z_1 \dots z_N) : \ \bar{z}_i \cdot z_i = p \leftarrow$ they are NOT necessarily orthogonal!

$$S = \int dt \left(i \sum_j \bar{z}_j \cdot \dot{z}_j - \sum_{i<j} \alpha_{ij} |\bar{z}_i \cdot z_j|^2 \right)$$

Its quantization: $z_j^\alpha \rightsquigarrow a_j^\alpha$ $\qquad [a_i^\alpha, (a_j^\beta)^+] = \delta^{\alpha\beta}\delta_{ij} \quad$ $i,j = 1..N$, $\alpha,\beta = 1...N$

$\bar{z}_j^\alpha \rightsquigarrow (a_j^\alpha)^+$

$\bar{z}_i \cdot z_i = p \rightsquigarrow (a_i^\alpha)^+ a_i^\alpha = p \leftarrow$ condition on the Fock space

The $SU(N)$ action: τ^a is in fund. irrep $\rightsquigarrow$ $\boxed{S_i^a \equiv (a_i^\alpha)^+ \tau^a_{\alpha\beta} a_i^\beta}$

So, $(a_i^\alpha)^+|0\rangle \leftarrow$ is a fundamental representation

$$\mathrm{Hilb}(p) = \Big\{ \prod_{i=1}^N \underbrace{(a_i^{\alpha_1^i})^+ \dots (a_i^{\alpha_p^i})^+}_{\text{Symmetric in indices}} |0\rangle \Big\} = \Big(\underbrace{\boxed{}\boxed{}\boxed{}\cdots\boxed{}}_{p} \Big)^{\otimes N}$$

Schwinger-Wigner quantization

Note: $H = \sum_{i<j} \alpha_{ij} |\bar{z}_i \cdot z_j|^2 = \sum_{i<j} \alpha_{ij} \sum_{i<j} S_i^a S_j^a$

Correspondence

Spin chain: $$S = \int_0^\tau dt \left(i \sum_j \bar{z}_j \cdot \dot{z}_j - \sum_{i<j} \alpha_{ij} |\bar{z}_i \cdot z_j|^2 \right), \quad \bar{z}_i \cdot z_i = p$$

for general $Z = (z_1 \dots z_N)$ use polar decomposition,
$$Z = UH$$
where $U = (u_1 \dots u_N)$

after integration over $K \equiv H^2$

1D sigma model on $\mathcal{F}(N)$: $$S = \int_0^\tau dt \sum_{i<j} \frac{1}{\alpha_{ij}} |\bar{u}_i \cdot \dot{u}_j|^2, \quad \text{where} \quad \bar{u}_i \cdot u_j = \delta_{ij}$$

Remark: Z is non-degenerate $\Rightarrow \|K\| < \text{const}(p)$

So: Solutions to the spin chain $\xleftrightarrow{1 \text{ to } 1}$ geodesics on $\mathcal{F}(N)$ with restricted momenta

disappears when $p \to \infty$

Spin chains vs Flag manifolds

Let us define $\mathcal{J}(A)_{ij} = \begin{cases} \alpha_{ij} A_{ij} & i<j \\ \alpha_{ji} A_{ij} & i>j \\ 0 & i=j \end{cases}$

E.o.m.:

$$i \frac{d}{dt} \mathcal{L} = [\mathcal{L}, \mathcal{J}(\mathcal{L})],$$

where $\mathcal{L} \equiv Z^{+} Z$

E.o.m.:

$$\frac{d}{dt} L = [L, \mathcal{J}(L)],$$

where $L_{ij} = \begin{cases} \frac{1}{\alpha_{ij}} \bar{u}_i \cdot \dot{u}_j & i<j \\ \frac{1}{\alpha_{ji}} \bar{u}_i \cdot \dot{u}_j & i>j \\ 0 & i=j \end{cases}$

the same Euler equations

Integrability: Mischenko & Fomenko '78 :

if $\alpha_{ij} = \dfrac{t_i - t_j}{m_i - m_j}$

Quantum level

Key: consider twisted partition function and path integral representation

Twisted boundary conditions:

$\forall t$: $\quad Z(t+\tau) = g Z(t)$ for $g \in SU(N)$

$$\Downarrow$$

$\mathcal{U}(t+\tau) = g\, \mathcal{U}(t)$

$$\boxed{\lim_{p\to\infty} \mathrm{Tr}_{\mathrm{Hilb}(p)}\left(g e^{-\tau H}\right) = \mathrm{Tr}_{\mathcal{L}^2(\mathcal{F}(N))}\left(g e^{-\tau H_p}\right), \quad \text{where } H_p = -\Delta} \quad (*)$$

$$H = \sum_{i<j} \alpha_{ij} \sum_a S_i^a S_j^a \quad \longleftrightarrow \quad ds^2 = \sum_{i<j} \frac{1}{\alpha_{ij}} |\bar{u}_i \cdot du_j|^2$$

Quantum properties:

From ⊛:

- $\mathrm{Hilb}(p) \subset \mathrm{Hilb}(p+1) \subset \ldots \subset \text{"}\mathrm{Hilb}(\infty)\text{"} = \mathcal{L}^2(\mathcal{F}(\mathcal{N}))$

In the representation theory sense

- $\mathrm{Spec}_p(H) \subset \mathrm{Spec}_{p+1}(H) \subset \ldots \subset \text{"}\mathrm{Spec}_\infty(H)\text{"} = \mathrm{Spec}(-\Delta)$

The spectrum of the spin chain Hamiltonian H

$\boxed{\text{For } \alpha_{ij} = \dfrac{t_i - t_j}{m_i - m_j} \text{ one could find } \mathrm{Spec}(-\Delta) \text{ using Bethe ansatz method}}$

Gaudin-Mischenko-Fomenko metric on $\mathcal{F}(\mathcal{N})$

Example: $\mathcal{F}(3)$

Integrability

Metric on $\mathcal{F}(3)$:

$$ds^2 = \frac{1}{\alpha_{12}} |\bar{u}_1 \cdot du_2|^2 + \frac{1}{\alpha_{13}} |\bar{u}_1 \cdot du_3|^2 + \frac{1}{\alpha_{23}} |\bar{u}_2 \cdot du_3|^2,$$

$$\bar{u}_i \cdot u_j = \delta_{ij}$$

$$\boxed{\begin{array}{l}\text{Main observation:} \\ \forall\ \alpha_{12}, \alpha_{13}, \alpha_{23}\ \exists\ t_i\text{'s and } m_i\text{'s such that } \alpha_{ij} = \dfrac{t_i - t_j}{m_i - m_j}\end{array}}$$

$$\Downarrow$$

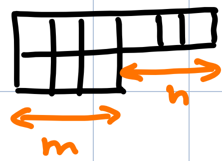
- Geodesic equation is integrable for all invariant metrics on $\mathcal{F}(3)$

(Paternain & Spatzier '94 (Thimm '81))

- The associated $SU(3)$ spin chain is always Gaudin

Quantum SU(3) spin chain

$$\mathrm{Hilb}(p) = \Big(\underbrace{\text{}}_{p}\Big)^{\otimes 3} = \bigoplus_{(n,m)}$$



$(a_i^\alpha)^+ , a_i^\alpha \qquad i,\alpha = 1,2,3$

Note:

← trivial representation of $SU(3)$

$$\downarrow$$

$$|\psi\rangle = Q(a_1^+, a_2^+, a_3^+)|0\rangle$$

↑ homogeneous polynomial of poly-degree (p,p,p)

- $\mathrm{Hilb}(p) \hookrightarrow \mathrm{Hilb}(p+1)$:

  $|\psi\rangle \longrightarrow \langle a_1^+ \, a_2^+ \, a_3^+ \rangle |\psi\rangle$, where $\langle a_1^+ \, a_2^+ \, a_3^+ \rangle \equiv \varepsilon_{\alpha\beta\gamma} (a_1^\alpha)^+ (a_2^\beta)^+ (a_3^\gamma)^+$

- $\mathrm{Spec}_p(H) \subset \mathrm{Spec}_{p+1}(H)$:

  $[H, \langle a_1^+ \, a_2^+ \, a_3^+ \rangle] = 0$

  $\left(H \sim \sum_{i<j} \alpha_{ij} (a_i^\alpha)^+ a_j^\alpha (a_j^\alpha)^+ a_i^\alpha \right)$

- $\mathrm{Hilb}(p) \hookrightarrow \mathcal{L}^2(\mathcal{F}(3))$

  $$f(x,y,z) \equiv \frac{Q(x,y,z)}{\langle x,y,z\rangle^p}$$

  ← well-defined function on $(\mathbb{CP}^2)^{\times 3}$ $(x,y,z \in \mathbb{CP}^2)$

  $\cup$ $\mathcal{F}(3)$

  The map $|\psi\rangle = Q|0\rangle \mapsto f|_{\mathcal{F}(3)}$ is injective

Hilbert space of the spin chain

$\mathfrak{H}\mathrm{ilb}(p) \setminus \mathcal{H}\mathrm{ilb}(p-1) \ni$ irreps with (n,m) : $\boxed{n+2m=3p}$

↑ multiplicity $= \min(n,m)+1$

$m \backslash n$	0	1	2	3	4	5	6	7
0	1			1			1	
1		2			2			2
2			3			3		
3	1			4			4	
4		2			5			5
5			3			6		
6	1			4			7	

$p=0$ $\quad p=1 \quad p=2$

↑ multiplicities

Table 1. Decomposition of $\mathcal{L}^2(\mathcal{F}(3))$ into irreps

Hamiltonian of the SU(3) spin chain

$$H = \frac{\alpha_{12}+\alpha_{23}+\alpha_{13}}{2}\, C_2 - \left(\alpha_{23}^2 H_1 + \alpha_{13}^2 H_2 + \alpha_{12}^2 H_3\right)$$

quadratic Casimir operator acting in $\big(\underbrace{\square\square\cdots\square}_{p}\big)^{\otimes 3}$

$$\boxed{H_1 = \frac{S_1^a S_2^a}{\alpha_{23}-\alpha_{13}} + \frac{S_1^a S_3^a}{\alpha_{23}-\alpha_{12}} \qquad H_2 = \frac{S_1^a S_2^a}{\alpha_{13}-\alpha_{23}} + \frac{S_2^a S_3^a}{\alpha_{13}-\alpha_{12}} \qquad H_3 = \frac{S_1^a S_3^a}{\alpha_{12}-\alpha_{23}} + \frac{S_2^a S_3^a}{\alpha_{12}-\alpha_{13}}}$$

Heighest weight of n,m: $n\omega_1 + m\omega_2 = 3p\omega_1 - \ell\alpha_1 - \ell'\alpha_2$

fundamental weights

simple roots

For (n,m) such that $n+2m=3p \Rightarrow \ell'=0,\ \ell=0,1\dots p$

Bethe ansatz equations [Feigin, Frenkel, Reshetikhin '94]

$$p\left(\frac{1}{t_a-\alpha_{23}}+\frac{1}{t_a-\alpha_{13}}+\frac{1}{t_a-\alpha_{12}}\right) - 2\sum_{\substack{b=1\\ b\neq a}}^{\ell}\frac{1}{t_a-t_b} = 0, \quad a=1\dots\ell$$

Bethe root

$$\mu_i = p\sum_{a=1}^{\ell}\left(\frac{1}{t_a-\alpha_{23}}+\frac{1}{t_a-\alpha_{13}}+\frac{1}{t_a-\alpha_{12}}\right) \leftarrow \text{eigenvalue of } H_i$$

Polynomial equations on eigenvalues

Equivalent reformulation: Fuchs differential equation

(Stieltjes 1885, Mukhin et al 2007)

$\exists \quad P(x) = \prod_{a=1}^{e} (t_a - x)$:

$$\boxed{P'' - pP'\left(\frac{1}{x-\alpha_{23}} + \frac{1}{x-\alpha_{13}} + \frac{1}{x-\alpha_{12}}\right) - P\left(\frac{\mu_1}{x-\alpha_{23}} + \frac{\mu_2}{x-\alpha_{13}} + \frac{\mu_3}{x-\alpha_{12}}\right) = 0}$$

We are interested in polynomial solutions with distinct roots of the given degree e

$\oplus \quad \mu_1 + \mu_2 + \mu_3 = 0$

Polynomial eq. on an eigenvalue ε of H:

$$\det\left(M_e - \varepsilon \mathbb{1}_{e+1}\right) = 0 \qquad (**)$$

We know it

Spectrum of the Laplace-Beltrami operator on $\mathcal{F}(3)$

Prop.:

Eigenvalues of $-\Delta$ are solutions to $(**)$ for $p \in \mathbb{N}_0$ and $\ell \in \{0, 1, \dots, p\}$, each with multiplicity $(3p-2\ell+1)(\ell+1)(3p-\ell+2)$ for $\ell \neq p$ and $(p+1)^3$ for $\ell = p$

Conclusion

- We have found the spectrum of Δ on $\mathcal{F}(3)$

Outlook

- ~~SUSY!~~ (arXiv: 2412.21024 [hep-th] with D. Bykov & V. Krivorol)
- ~~Magnetic case~~ (arXiv: 2503.08646 with D. Bykov)
- SO and Sp groups (nothing nice, see :( )
- Non-compact groups? (arXiv: 2602.22984 D. Bykov & V. Krivorol)
- Infinite-dimensional Lie groups and 2d models