On closure of invariant manifolds of saddle fixed points of Morse-Smale systems.

E. Gurevich, I. Saraev

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Preliminary definitions

Let M^n be a closed smooth manifold of dimension $n \geq 1$, $d : M^n \times M^n \to [0, +\infty)$ be metric and f^t be a smooth flow i.e. an action $(t, x) \mapsto f^t(x)$ of the group $(\mathbb{R}, +)$ on M^n smoothly depending on t.

- A point $p \in M^n$ is equilibrium of f^t if $f^t(p) = p$ for all $t \in \mathbb{R}$;
- the equilibrium p of the flow f^t is hyperbolic if all eigenvalues of $D_p\left(\frac{\partial f^t(x)}{\partial t}|_{t=0}\right)$ have non-zero real parts;
- the stable (unstable) invariant manifold of p is the set

$$W^s_p = \{x \in M^n | \lim_{t \to +\infty} d(f^t(x), p) = 0\}$$

$$(W^u_p = \{x \in M^n | \lim_{t \to -\infty} d(f^{-t}(x), p) = 0\})$$

- a point $x \in M^n$ is called wandering if there exists an open neighborhood U of x and a time value t_0 s.t. $f^t(U) \cap U = \varnothing$ for all $t \geq t_0$.

Gradient-like flows

A smooth flow f^t on M^n is called gradient-like if

1. it has finitely many equilibria and other points are wandering;
2. invariant manifolds of equilibria intersect each other transversely.

Let $\Psi(M^n)$ be a class of gradient-like flows on M^n, suppose that $f^t \in \Psi(M^n)$ and consider its equilibrium p. Since p is hyperbolic then all eigenvalues of the differential $D_p\left(\frac{\partial f^t(x)}{\partial t}\big|_{t=0}\right)$ have non-zero real parts. Let i_p be the number of eigenvalues with negative real parts. It follows from Hartman-Grobman theorem that i_p is equal to dimension $\dim W^u_p$ of the unstable manifold W^u_p of p. The number i_p is called a Morse index of p.

Embedding of invariant manifolds

In general if p is a hyperbolic equilibrium of a flow f^t then W^u_p is the image of an injective smooth immersion $\mathrm{im}:\mathbb{R}^{i_p}\to M^n$. For equilibria of gradient-like flows this fact can be refined as follows

Theorem 1. (Smale, 1967)

For any $f^t\in\Psi(M^n)$:

1. $M^n=\bigcup\limits_{p\in\Omega_{f^t}} W^s_p=\bigcup\limits_{p\in\Omega_f} W^u_p$;
2. for any point $p\in\Omega_{f^t}$ the set W^u_p is a smooth submanifold of M^n homeomorphic to $\mathbb{R}^{i_p}$;
3. for any point $p\in\Omega_{f^t}$ the following equality holds $\mathrm{cl}W^u_p\setminus W^u_p=\bigcup\limits_{\substack{q\in\Omega_{f^t}\\ W^s_q\cap W^u_p\neq\varnothing}} W^u_q$, where Ω_{f^t} denotes the set of equilibria of the flow f^t.

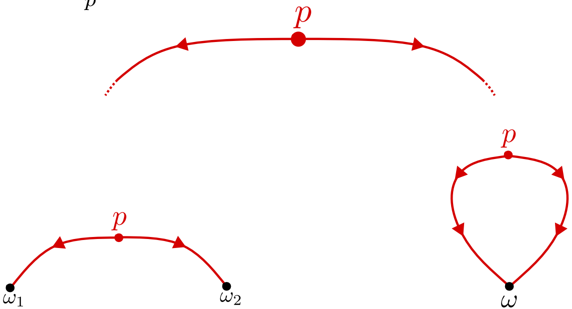
Closures of invariant manifolds without heteroclinic trajectories

Let p be a saddle of a flow $f^t \in \Psi(M^n)$. Suppose that W^u_p does not intersect stable manifolds of saddle equilibria different to p. Then $W^u_p \setminus \{p\} \subset \bigcup\limits_{\omega \in \Omega_{f^t}} W^s_\omega$.

- If $i_p = 1$ then, $W^u_p \setminus \{p\}$ is a pair of intervals. By Theorem 1 there exist no more than two sink points ω_1, ω_2 of f^t such that $\mathrm{cl} W^u_p = W^u_p \cup \{\omega_1, \omega_2\}$. Hence $\mathrm{cl} W^u_p$ is either a segment or a circle (when $\omega_1 = \omega_2$).
- If $i_p > 1$ then $W^u_p \setminus \{p\}$ is connected hence there is only one sink equilibrium of f^t such that $W^u_p \setminus \{p\} \subset W^s_\omega$. It follows from Theorem 1 that $\mathrm{cl} W^u_p = W^u_p \cup \{\omega\}$. Then the set $\mathrm{cl} W^u_p$ is homeomorphic to i_p-dimensional sphere.

Closures of invariant manifolds without heteroclinic trajectories

$\dim W_p^u = 1$



$\mathrm{cl} W_p^u = W_p^u \cup \{\omega_1, \omega_2\}$

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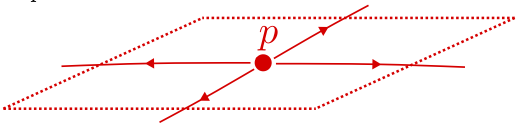
Closures of invariant manifolds without heteroclinic trajectories

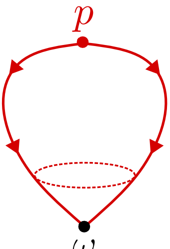
Let p be a saddle of a flow $f^t \in \Psi(M^n)$. Suppose that W^u_p does not intersect stable manifolds of other saddle equilibria. Then $W^u_p \setminus \{p\} \subset \bigcup\limits_{\omega \in \Omega_{f^t}} W^s_\omega$.

- If $i_p = 1$ then, $W^u_p \setminus \{p\}$ is a pair of intervals. By Theorem 1 there exist no more than two sink points ω_1, ω_2 of f^t such that $\mathrm{cl} W^u_p = W^u_p \cup \{\omega_1, \omega_2\}$. Hence $\mathrm{cl} W^u_p$ is either a segment or a circle (when $\omega_1 = \omega_2$).
- If $i_p \geq 2$ then $W^u_p \setminus \{p\}$ is connected hence there is only one sink equilibrium of f^t such that $W^u_p \setminus \{p\} \subset W^s_\omega$. It follows from Theorem 1 that $\mathrm{cl} W^u_p = W^u_p \cup \{\omega\}$. Then the set $\mathrm{cl} W^u_p$ is homeomorphic to i_p-dimensional sphere.

Closures of invariant manifolds without heteroclinic trajectories

$\dim W_p^u > 1$



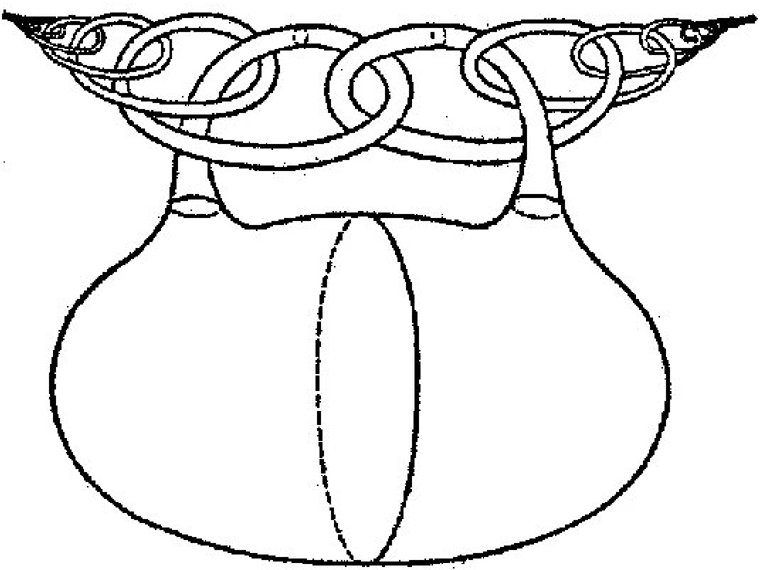


$$\mathrm{cl} W_p^u = W_p^u \cup \{\omega\}$$

Embedding of $\mathrm{cl} W_p^u$ in M^n

We recall that the set $S \subset M^n$ is locally flat if for any $x \in S$ there exist an open neighborhood $U_x \subset M^n$ and a homeomorphism $h : U_x \to \mathbb{R}^n$ such that $h(U_x \cap S) = \mathbb{R}^k$. If S contains a point x in which the condition fails then S is wild at x. By definition the set S is a topological submanifold of M^n iff S is locally flat in M^n.

There are wild 2-spheres in $\mathbb{R}^3$



Embedding of $\mathrm{cl}W_p^u$ in M^n

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Proposition 1.

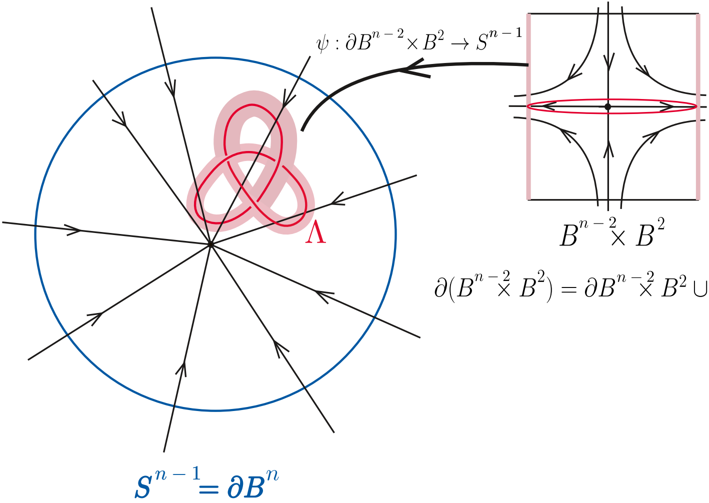
Let $f^t \in \Psi(M^n)$ and p be its saddle equilibrium such that W_p^u does not intersect stable manifolds of other saddles. The following statements hold for embedding of $\mathrm{cl}W_p^u$ in M^n:

1. for $n \in \{2, 3\}$ and for any i_p the set $\mathrm{cl}W_p^u$ is locally flat in M^n;
2. for $n \geq 4$ if $i_p \neq n-2$ then $\mathrm{cl}W_p^u$ is a locally flat sphere in M^n.

Construction of a flow with wild closures of $(n-2)$-dimensional unstable manifolds

Let $S_\omega \subset W^s_\omega \setminus \{\omega\}$ be a cross-section for the flow f^t. Then S_ω is homeomorphic to $\mathbb{S}^{n-1}$ and the intersection $\Lambda^u_p = W^u_p \cap S_\omega$ is transversal. Hence Λ^u_p is the cross-section for $f^t|_{W^u_p \setminus \{p\}}$ which implies that Λ^u_p is homeomorphic to $\mathbb{S}^{i_p - 1}$. The sphere $\Lambda^u_p \subset S_\omega$ is unknotted if there exists an embedding $h : S_\omega \to \mathbb{R}^n$ such that $h(S_\omega) = \mathbb{S}^{n-1}$ and $h(\Lambda^u_p) = \mathbb{S}^{i_p - 1}$. Otherwise Λ^u_p is knotted in S_ω.

Construction



$$\partial(B^{n-2} \times B^2) = \partial B^{n-2} \times B^2 \cup B^{n-2} \times \partial B^2$$

Wild closures of $(n-2)$-dimensional unstable manifolds

Let f^t be a flow obtained by the construction above and M^n be an ambient manifold. It has one sink, one source (such flows are called polar) and two equilibria p, q with $\dim W_p^u = n-2$ and $\dim W_q^u = 2$.

Theorem 4. (V. Medvedev and E. Zhuzhoma, 2013)

The manifold M^n is simply connected ($\pi_1(M^n) = 1$). Since Λ is knotted then $\mathrm{cl} W_p^u$ is a wild $(n-2)$-dimensional sphere.

Proposition 2. (E. Gurevich, I.)

For any disjoint k-union Λ of smooth $(n-3)$-dimensional spheres in $\mathbb{S}^{n-1}$ there exists a polar flow f^t with k saddles $p_1, \dots, p_k$ such that there exists a homeomorphism $h : S_\omega \to \mathbb{S}^{n-1}$ satisfying $h\left(\bigcup\limits_p \Lambda_p^u\right) = \Lambda$. Moreover, $\mathrm{cl} W_p^u$ is locally flat iff Λ_p^u is unknotted.

For $n = 4$ the ambient manifold M^4 is homeomorphic either to connected sum $\#_k(\mathbb{S}^2 \times \mathbb{S}^2)$ or to connected sum $\#_k(\mathbb{S}^2 \widetilde{\times} \mathbb{S}^2)$.

Topological classification

Two flows f^t, f'^t on manifolds M^n, M'^n are topologically equivalent if there exists a homeomorphism $H : M^n \to M'^n$ mapping trajectories of the flow f^t to trajectories of the flow f'^t preserving orientation on them with respect to increasing time. Let $G(M^n)$ be a class of gradient-like flows without heteroclinic intersections on M^n. Due to Proposition 2 topological classification of flows in the class $G(M^n)$ is not algorithmically solvable for $n \geq 4$.

Polar flows on 4-manifolds

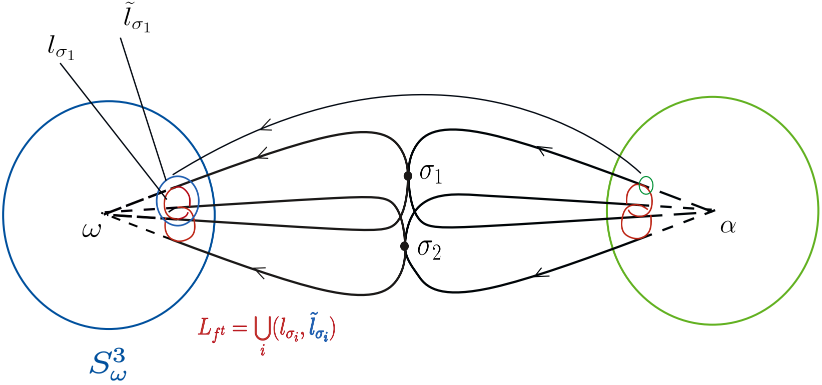


Figure: Kirby diagram of a polar flow f^t

Necessary and sufficient conditions of topological equivalence

The set L_{f^t} is a Kirby diagram of f^t.

Theorem 5. (E. Gurevich, I.)

Two polar flows without heteroclinic intersections on M^4 are topologically equivalent iff there exists a homeomorphism $h : S_\omega \to S_{\omega'}$ such that $h(L_{f^t}) = L_{f'^t}$ and $h(\widetilde{l}_{\sigma_i}) = \widetilde{l}_{\sigma'_i}$ for any pair l_{σ_i}, $l_{\sigma'_i} = h(l_{\sigma_i})$.

The link $l_\sigma \cup \widetilde{l}_\sigma$ determines uniquely up to isotopy by l_σ and the number $n = \mathrm{lk}(l_\sigma, \widetilde{l}_\sigma)$.

Classification and realization

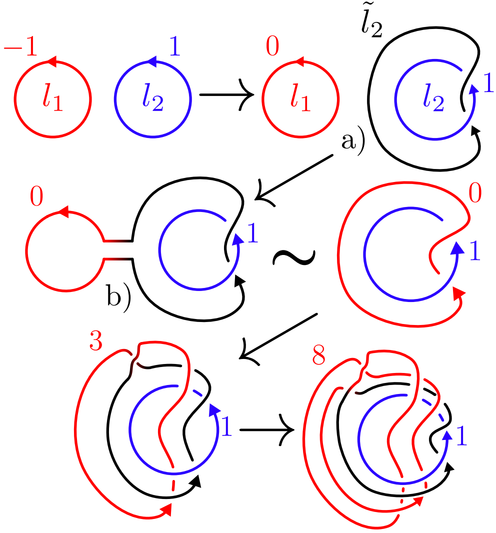
Theorem 6. (E. Gurevich, I.)

For any Kirby diagram $L \subset S^3$ of a simply connected smooth manifold M^4 which consists of framed knots there exists a polar flow on M^4 such that its Kirby diagram L_{f^t} is equivalent up to homeomorphism to L.

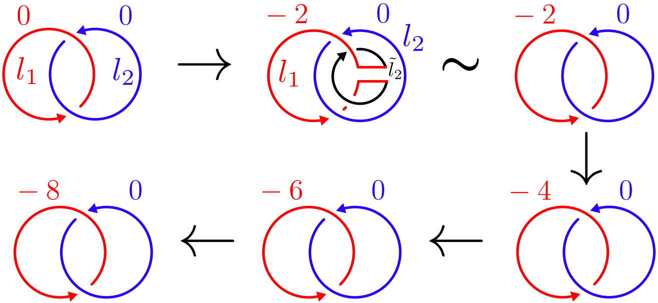
Theorem 7. (R. Kirby, E. de Sa)

Two Kirby diagrams L, $L' \subset S^3$ consisting of framed links represent the same manifold iff they can be obtained from each other by moves of two types.

Kirby moves of type II



Kirby moves of type II



Infinitely many non-equivalent polar flows on M^4

Coroallry 1. (V. Medvedev, E. Zhuzhoma 2013; E. Gurevich, I. 2024)

1. Any two polar flows f^t, f'^t with one saddle on M^4 are topologically equivalent, M^4 is homeomorphic with $\mathbb{CP}^2$;
2. for any simply connected M^4 not homeomorphic to $\mathbb{S}^4$ and for any number $k \geq 2$ there exist infinitely many polar flows on M^4 with k saddles of Morse index 2 and with no saddles of other indexes.

Bifurcations

Hypothesis

Let f^t, f'^t be polar flows on M^4 such that Kirby diagram $L_{f'^t}$ is equivalent to Kirby diagram L obtained from L_{f^t} with Kirby move of type II. Then there exists a smooth arc γ in the space of smooth flows such that:

1. γ connects f^t, f'^t;
2. γ passes through polar flows except for one bifurcation point;
3. the flow ϕ^t corresponding to the bifurcation point has a pair of saddles p, q for which the intersection $W_p^u \cap W_q^s$ is quasi-transversal.