Гиростат Жуковского-Вольтерра как система ван Диена-Руйсенаарса, алгебры Склянина и интегрируемые цепочки

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Plan of the talk:

- Integrable systems: Lax equations and r-matrices
- Calogero-Moser – Euler top correspondence
- Zhukovsky-Volterra gyrostat and reflection equation
- Painlevé VI
- Ruijsenaars - van Diejen model and BC_1 Sklyanin algebra
- Elliptic Toda chain and elliptic Ruijsenaars-Toda chain from XYZ model
- Elliptic Toda chain with boundaries

Lax equations and classical r-matrix

Let L and M are matrices (matrix-valued functions on the phase space), and equations of motion are represented in the form of Lax equation:

$$\dot{L} = [L, M]\,, \quad \dot{L} = \{H, L\} = \sum_{i,j} E_{ij}\{H, L_{ij}\}\,, \qquad L, M \in \mathrm{Mat}(N, \mathbb{C})$$

Then $H_k = \mathrm{tr}(L^k)$ are integrals of motion:

$$\dot{L^k} = [L^k, M]\,, \quad \frac{d}{dt}\,\mathrm{tr}(L^k) = \mathrm{tr}(L^k M - M L^k) = 0\,.$$

Introduce notation $\{L_1\,, L_2\} = \sum\limits_{ijkl} E_{ij} \otimes E_{kl}\,\{L_{ij}, L_{kl}\} \in \mathrm{Mat}_N^{\otimes 2}$. E_{ij} – standard matrix basis. If there exists such $r_{12} \in \mathrm{Mat}_N^{\otimes 2}$ (classical r-matrix) that

$$\{L_1\,, L_2\} = [L_1, r_{12}] - [L_2, r_{21}]\,, \quad L_1 = L \otimes 1\,, \quad L_2 = 1 \otimes L$$

$$r_{12} = \sum_{ijkl} r_{ij,kl}\, E_{ij} \otimes E_{kl}\,, \qquad r_{21} = \sum_{ijkl} r_{ij,kl}\, E_{kl} \otimes E_{ij}$$

then the conservation laws are in involution $\{H_i, H_j\} = 0$.

The same with spectral parameter

Let $L(z)$ and $M(z)$ are matrices (matrix-valued functions on the phase space), z – auxiliary parameter and equations of motion are represented in the form

$$\dot{L}(z) = [L(z), M(z)]\,, \quad \forall z \quad \dot{L}(z) = \{H, L(z)\} = \sum_{ij} E_{ij}\{H, L_{ij}(z)\}$$

Then $H_k(z) = \mathrm{tr}(L^k(z))$ are generating functions of integrals of motion:

$$\mathrm{tr}(L^k(z)) = \sum_m (z - z_0)^m H_{k,m}\,, \qquad \frac{d}{dt} H_{k,m} = 0 \ \forall k, m$$

Introduce notation $\{L_1(z)\,, L_2(w)\} = \sum_{ijkl} E_{ij} \otimes E_{kl} \, \{L_{ij}(z), L_{kl}(w)\}$. If there exists such $r_{12} \in \mathrm{Mat}^{\otimes 2}$ (classical r-matrix) that

$$\{L_1(z)\,, L_2(w)\} = [L_1(z), r_{12}(z, w)] - [L_2(w), r_{21}(w, z)]\,,$$

where $L_1(z) = L(z) \otimes 1$ and $L_2(w) = 1 \otimes L(w)$. Then $\{H_{i,m}, H_{j,n}\} = 0$.

The Calogero-Moser model:

$$H_2 = \sum_{i=1}^{N} \frac{p_i^2}{2} - \nu^2 \sum_{i<j}^{N} \wp(q_i - q_j)\,, \qquad \wp(x) \to \frac{1}{\sin^2(x)} \to \frac{1}{x^2}$$

where ν – coupling constant, $\wp(q)$ – Weierstrass $\wp$-function. Its equations of motion are written in the Lax form with the Lax matrix

$$L(z) = \begin{pmatrix} p_1 & \nu\phi(z, q_1 - q_2) & \nu\phi(z, q_1 - q_3) & \dots & \nu\phi(z, q_1 - q_N) \\ \nu\phi(z, q_2 - q_1) & p_2 & \nu\phi(z, q_2 - q_3) & \dots & \nu\phi(z, q_2 - q_N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \nu\phi(z, q_N - q_1) & \nu\phi(z, q_N - q_2) & \nu\phi(z, q_N - q_3) & \dots & p_N \end{pmatrix}$$

where z – spectral parameter (Krichever 1980). It is a local coordinate on (elliptic) curve. Elliptic Kronecker function is used here:

$$\phi(z, q) = \frac{\vartheta'(0)\vartheta(z+q)}{\vartheta(z)\vartheta(q)}\,.$$

Quadratic classical r-matrix structure:

$$\{L_1(z)\,,L_2(w)\} = [L_1(z)L_2(w), r_{12}(z,w)].$$

Main properties are the same as for the linear r-matrix structure.
The functions $\mathrm{tr}(L^k(z))$ are generating functions for conservation laws. Locally, for

$$\mathrm{tr}(L^k(z)) = \sum_m (z-z_0)^m H_{k,m}\,, \qquad \frac{d}{dt}H_{k,m} = 0\ \forall k,m$$

we have

$$\{H_{i,m}, H_{j,n}\} = 0.$$

The Yang-Baxter equation for r-matrix is the same.
Additional properties:
1. $\det L(z)$ is a generating function for **center of underlying algebra**.
2. Construction of chain. Having a set of $L^i(z)$ with the quadratic brackets

$$\{L^i_1(z)\,,L^j_2(w)\} = \delta^{ij}[L^i_1(z)L^i_2(w), r_{12}(z,w)].$$

one gets the same equation for the monodromy matrix $T(z) = L^1(z)L^2(z)...L^n(z)$:

$$\{T_1(z)\,,T_2(w)\} = [T_1(z)T_2(w), r_{12}(z,w)].$$

Complexified Euler top in $\mathbb{C}^3$ $(S_1,\ S_2,\ S_3)$. It is defined by the Hamiltonian

$$H^{\rm top}=\frac{1}{2}\sum_{\alpha=1}^{3}S_\alpha^2\wp(\omega_\alpha)$$

where the half-periods ω_j

$$\omega_1=\frac{\tau}{2}\,,\quad \omega_2=\frac{1+\tau}{2}\,,\quad \omega_3=\frac{1}{2}$$

are numerated according to numeration of the Pauli matrices. The Poisson-Lie structure on $\mathrm{sl}^*(2,\mathbb{C})$ (complexification of $\mathrm{su}^*(2)$)

$$\{S_\alpha,S_\beta\}=-\sqrt{-1}\varepsilon_{\alpha\beta\gamma}S_\gamma\,.$$

The variables S_α are naturally arranged into the traceless 2×2 matrix $S=\sum_{\alpha=1}^{3}\sigma_\alpha S_\alpha$ in the Pauli matrices basis. Equations of motion $\dot S=\{H_0^{\rm top},S\}$ then takes the form

$$\dot S=[S,J(S)]\qquad J(S)=\sum_{\alpha=1}^{3}S_\alpha J_\alpha\sigma_\alpha\,,\quad J_\alpha=-\frac{1}{2}\wp(\omega_\alpha)\,.$$

The vector (S_1,S_2,S_3) is a complexification of the angular momentum vector of a rigid body (the Euler top). The constants J_α are components of the inverse inertia tensor in principle axes.

All J_α depend on a single parameter – elliptic moduli τ. In order to have a set of free parameters J_α one may multiply $H_0^{\rm top}$ by an arbitrary constant and also shift it by an expression proportional to the Casimir function $C_2 = \frac{1}{2}\sum_{\alpha=1}^{3} S_\alpha^2$. Equations of motion for the Hamiltonian $H^{\rm top}$ admit the Lax representation:

$$L^{\rm top}(z,S) = \sum_{\alpha=1}^{3} S_\alpha \varphi_\alpha(z)\sigma_\alpha\,, \qquad M^{\rm top}(z,S) = \frac{1}{2}\sum_{\alpha=1}^{3} S_\alpha f_\alpha(z)\sigma_\alpha\,.$$

Also, we have

$$\frac{1}{4}\,{\rm tr}\left(L^{\rm top}(z)^2\right) = \frac{1}{2}\sum_{\alpha=1}^{3} S_\alpha^2\varphi_\alpha(z)^2 = H_0^{\rm top} + C_2\wp(z)\,,$$

$$H^{\rm top} = \frac{1}{2}\,{\rm tr}(SJ(S))\,, \qquad C_2 = \frac{1}{2}\,{\rm tr}(S^2)\,.$$

Classical r-matrix structure:

$$\{L_1(z), L_2(w)\} = \frac{1}{c}\,[L_1(z) + L_2(w), r_{12}(z-w)]$$

$$r_{12}(z-w) = \frac{1}{2}\,E_1(z-w)\sigma_0\otimes\sigma_0 + \frac{1}{2}\sum_{k=1}^{3}\varphi_k(z-w)\sigma_k\otimes\sigma_k\,,$$

Description through quadratic algebra:

$$\{\mathcal{L}_1(z), \mathcal{L}_2(w)\} = \frac{1}{c}\,[\mathcal{L}_1(z)\mathcal{L}_2(w), r_{12}(z-w)]$$

The Lax matrix

$$\mathcal{L}(z, \mathbb{S}) = \sigma_0 \mathbb{S}_0 + \sum_{k=1}^{3} \sigma_k \varphi_k(z) \mathbb{S}_k \,,$$

and the elliptic r-matrix

$$r_{12}(z-w) = \frac{1}{2}\,E_1(z-w)\sigma_0 \otimes \sigma_0 + \frac{1}{2}\sum_{k=1}^{3} \varphi_k(z-w)\sigma_k \otimes \sigma_k \,,$$

where $E_1(z) = \partial_z \log \vartheta(z)$ and

$$\varphi_1(z) = \frac{\theta_1'(0)\theta_2(z)}{\theta_1(z)\theta_2(0)}\,, \qquad \varphi_2(z) = \frac{\theta_1'(0)\theta_3(z)}{\theta_1(z)\theta_3(0)}\,, \qquad \varphi_3(z) = \frac{\theta_1'(0)\theta_4(z)}{\theta_1(z)\theta_4(0)}\,.$$

lead to the classical Sklyanin algebra:

$$c\{\mathbb{S}_i, \mathbb{S}_j\} = -\imath\varepsilon_{ijk}\mathbb{S}_0\mathbb{S}_k\,,$$

$$c\{\mathbb{S}_0, \mathbb{S}_i\} = -\imath\varepsilon_{ijk}\mathbb{S}_j\mathbb{S}_k\big(\wp(\omega_j) - \wp(\omega_k)\big)\,.$$

where the half-periods ω_j

$$\omega_1 = \frac{\tau}{2}\,, \quad \omega_2 = \frac{1+\tau}{2}\,, \quad \omega_3 = \frac{1}{2}$$

are numerated according to numeration of the Pauli matrices.

The classical Sklyanin algebra has two Casimir functions (on 4-dimensional phase space S_a $a = 0, 1, 2, 3$)

$$\mathbf{C}_1 = (\mathbb{S}_1)^2 + (\mathbb{S}_2)^2 + (\mathbb{S}_3)^2\,, \qquad \mathbf{C}_2 = (\mathbb{S}_0)^2 + \sum_{k=1}^{3}(\mathbb{S}_k)^2\wp(\omega_k)$$

appearing from determinant – center of quadratic r-matrix brackets

$$\det\mathcal{L}(z, \mathbb{S}) = \mathbf{C}_2 - \wp(z)\mathbf{C}_1\,.$$

The Hamiltonian is given by $S_0 = \frac{1}{2}\operatorname{tr}\mathcal{L}(z)$.

Quadratic r-matrix structure in many-body systems: Ruijsenaars-Schneider model.

$$H^{\rm RS} = c\sum_{j=1}^{N} \prod_{k:k\neq j}^{N} \frac{\vartheta(q_j - q_k - \eta)}{\vartheta(q_j - q_k)}\, e^{p_j/c}\,,$$

where p_i and q_i, $i = 1, ..., N$ are canonically conjugated momenta and positions of particles on a complex plane, $c \in \mathbb{C}$ and $\eta \in \mathbb{C}$ are constants and $\vartheta(w)$ is the first Jacobi theta-function. In the non-relativistic limit $c \to \infty$ (together with redefining $\eta = \nu/c$) one gets the elliptic Calogero-Moser model:

$$H^{\rm CM} = \sum_{i=1}^{N} \frac{p_i^2}{2} - \nu^2 \sum_{i>j}^{N} \wp(q_i - q_j)\,,$$

where $\wp(w)$ is the Weierstrass $\wp$-function.
The r-matrix structure for the Calogero-Moser model ("non-relativistic") is linear:

$$\{L_1^{\rm CM}(z), L_2^{\rm CM}(w)\} = [L_1^{\rm CM}(z), r_{12}^{\rm CM}(z,w)] - [L_2^{\rm CM}(w), r_{21}^{\rm CM}(w,z)]$$

and it is quadratic for the Ruijsenaars-Schneider model ("relativistic"):

$$c\,\{L_1^{\rm RS}(z), L_2^{\rm RS}(w)\} = L_1^{\rm RS}(z)L_2^{\rm RS}(w)r_{12}^{-}(z,w) - r_{12}^{+}(z,w)L_1^{\rm RS}(z)L_2^{\rm RS}(w) +$$

$$+ L_1^{\rm RS}(z)s_{12}^{+}(z,w)L_2^{\rm RS}(w) - L_2^{\rm RS}(w)s_{12}^{-}(z,w)L_1^{\rm RS}(z)\,.$$

Simplest examples for 1 degree of freedom models:

1. Many-body non-relativistic – 2-body Calogero-Moser model (in the center of mass frame):

$$H^{\mathrm{CM}} = \frac{p^2}{2} - \nu^2 \wp(2q)\,,$$

2. Many-body relativistic – 2-body Ruijsenaars-Schneider model (in the center of mass frame):

$$H^{\mathrm{RS}} = \frac{1}{2}\Big(\frac{\vartheta(2\mathbf{q}-\eta)}{\vartheta(2\mathbf{q})}\, e^{\mathbf{p}/2c} + \frac{\vartheta(2\mathbf{q}+\eta)}{\vartheta(2\mathbf{q})}\, e^{-\mathbf{p}/2c}\Big)\,,$$

3. Euler top, non-relativistic – Euler top with Poisson-Lie brackets:

$$H^{\mathrm{top}} = \frac{1}{2}\sum_{\alpha=1}^{3} S_\alpha^2 \wp(\omega_\alpha),$$

4. Euler top, relativistic – Euler top with quadratic Sklyanin brackets:

$$H^{\mathrm{top}} = S_0.$$

Gauge equivalence. Introduce the matrix (by Baxter from IRF-Vertex correspondence):

$$g(z,\mathbf{q}_a)=\frac{1}{\theta_1(2\mathbf{q}_a|\tau)}\left(\begin{array}{cc}\theta_3(z-2\mathbf{q}_a|2\tau) & -\theta_3(z+2\mathbf{q}_a|2\tau)\\ -\theta_2(z-2\mathbf{q}_a|2\tau) & \theta_2(z+2\mathbf{q}_a|2\tau)\end{array}\right).$$

The gauge equivalence

$$L^{\rm top}(z)=g(z)L^{\rm CM}(z)g^{-1}(z)$$

holds true identically in z and provides explicit change of variables $S_\alpha=S_\alpha(p,q,\nu)$:

$$\left\{\begin{array}{l}
S_1(p,q,\nu)=p\dfrac{\theta_{01}(0)}{\vartheta'(0)}\dfrac{\theta_{01}(2q)}{\vartheta(2q)}+\dfrac{\nu}{2}\dfrac{\theta_{01}^2(0)}{\theta_{00}(0)\theta_{10}(0)}\dfrac{\theta_{00}(2q)\theta_{10}(2q)}{\vartheta^2(2q)}\,,\\[3ex]
S_2(p,q,\nu)=p\dfrac{\sqrt{-1}\theta_{00}(0)}{\vartheta'(0)}\dfrac{\theta_{00}(2q)}{\vartheta(2q)}+\dfrac{\nu}{2}\dfrac{\sqrt{-1}\theta_{00}^2(0)}{\theta_{10}(0)\theta_{01}(0)}\dfrac{\theta_{10}(2q)\theta_{01}(2q)}{\vartheta^2(2q)}\,,\\[3ex]
S_3(p,q,\nu)=p\dfrac{\theta_{10}(0)}{\vartheta'(0)}\dfrac{\theta_{10}(2q)}{\vartheta(2q)}+\dfrac{\nu}{2}\dfrac{\theta_{10}^2(0)}{\theta_{00}(0)\theta_{01}(0)}\dfrac{\theta_{00}(2q)\theta_{01}(2q)}{\vartheta^2(2q)}\,.
\end{array}\right.$$

The map $(p,q) \mapsto (S_1, S_2, S_3)$ **is a Poisson map**, i.e. the Poisson brackets between the functions $S_\alpha(p,q,\nu)$ evaluated through the canonical brackets $\{p,q\} = 1$ provide the linear Poisson-Lie structure.

Gauge equivalence provides relation between the Casimir function C_2 (length of angular momentum) and the coupling constant:

$$C_2 = \nu^2/2$$

Short form for the change of variables:

$$S_\alpha(p,q,\nu) = c_\alpha(\tau)\Big(p - \frac{\nu}{2}\,\partial_q\Big)\varphi_\alpha(q)\,,$$

$$c_1(\tau) = \Big(\frac{\theta_{01}(0)}{\vartheta'(0)}\Big)^2, \quad c_2(\tau) = \sqrt{-1}\Big(\frac{\theta_{00}(0)}{\vartheta'(0)}\Big)^2, \quad c_3(\tau) = \Big(\frac{\theta_{10}(0)}{\vartheta'(0)}\Big)^2$$

and

$$\varphi_1(z) = \frac{\theta_1'(0)\theta_4(z)}{\theta_1(z)\theta_4(0)}\,, \qquad \varphi_2(z) = \frac{\theta_1'(0)\theta_3(z)}{\theta_1(z)\theta_3(0)}\,, \qquad \varphi_3(z) = \frac{\theta_1'(0)\theta_2(z)}{\theta_1(z)\theta_2(0)}\,.$$

At the level of relativistic models the gauge equivalence means:

$$L^{\rm top}(z) = g(z) L^{\rm RS}(z) g^{-1}(z)\,.$$

Change of variables:

$$\mathbb{S}_0 = \frac{1}{2}\Big(\frac{\vartheta(2\mathbf{q}-\eta)}{\vartheta(2\mathbf{q})}\, e^{\mathbf{p}/2c} + \frac{\vartheta(2\mathbf{q}+\eta)}{\vartheta(2\mathbf{q})}\, e^{-\mathbf{p}/2c}\Big)\,,$$

$$\mathbb{S}_1 = \frac{1}{2}\frac{\theta_4(0)}{\vartheta'(0)}\Big(\frac{\theta_4(2\mathbf{q}-\eta)}{\vartheta(2\mathbf{q})}\, e^{\mathbf{p}/2c} - \frac{\theta_4(2\mathbf{q}+\eta)}{\vartheta(2\mathbf{q})}\, e^{-\mathbf{p}/2c}\Big)\,,$$

$$\mathbb{S}_2 = \frac{\imath}{2}\frac{\theta_3(0)}{\vartheta'(0)}\Big(\frac{\theta_3(2\mathbf{q}-\eta)}{\vartheta(2\mathbf{q})}\, e^{\mathbf{p}/2c} - \frac{\theta_3(2\mathbf{q}+\eta)}{\vartheta(2\mathbf{q})}\, e^{-\mathbf{p}/2c}\Big)\,,$$

$$\mathbb{S}_3 = \frac{1}{2}\frac{\theta_2(0)}{\vartheta'(0)}\Big(\frac{\theta_2(2\mathbf{q}-\eta)}{\vartheta(2\mathbf{q})}\, e^{\mathbf{p}/2c} - \frac{\theta_2(2\mathbf{q}+\eta)}{\vartheta(2\mathbf{q})}\, e^{-\mathbf{p}/2c}\Big)\,.$$

These are the generators of the classical Sklyanin algebra. The change of variables provides the Poisson map.

Summary:

Quadratic r-matrix structure:	2-body RS model	$\overset{\text{gauge equivalence}}{\longleftrightarrow}$	rel. top
	$\downarrow$ non-rel. limit		$\downarrow$ non-rel. limit
Linear r-matrix structure:	2-body Calogero model	$\overset{\text{gauge equivalence}}{\longleftrightarrow}$	non-rel top

Generalization to BC_n type models.

The elliptic Calogero-Moser model of gl_n type:

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2} - g^2 \sum_{i>j}^{n} \wp(q_i - q_j)\,,$$

where $\wp(z)$ is the Weierstrass $\wp$-function, $g \in \mathbb{C}$ is a coupling constant.

BC_n type Calogero-Inozemtsev system. It is described by the Hamiltonian:

$$H = \frac{1}{2}\sum_{k=1}^{n} p_k^2 - g^2 \sum_{i<j}^{n} \Big(\wp(q_i - q_j) + \wp(q_i + q_j)\Big) - \frac{1}{2}\sum_{a=0}^{3}\sum_{k=1}^{n} \nu_a^2 \wp(q_k + \omega_a)\,, \qquad (1)$$

where ω_γ are half-periods , and the **five arbitrary constants** are $g, \nu_0, \nu_1, \nu_2, \nu_3 \in \mathbb{C}$. One degree of freedom case – 4 constants:

$$H^{\mathrm{Inoz}} = \frac{p^2}{2} - \sum_{a=0}^{3} \nu_a^2 \wp(q + \omega_a)\,.$$

In particular case when all $\nu_a = \nu_b$ it reduces to $H^{\mathrm{CM}} = p^2/2 - \nu^2 \wp(2q)\,.$

Zhukovsky-Volterra gyrostat:

$$\dot{\vec{S}} = \vec{S} \times J(\vec{S}) + \vec{S} \times \vec{\lambda}\,,$$

where $\vec{S} = (S_1, S_2, S_3)$ is an angular momentum vector of rotation of rigid body in 3-dimensional space (in our consideration the 3d space is complexified to $\mathbb{C}^3$), $J(\vec{S}) = (J_1S_1, J_2S_2, J_3S_3)$ with some constants J_1, J_2, J_3 (these are the components of the inverse inertia tensor written in principle axes) and $\vec{\lambda}$ is a constant vector.
The Lax matrix for the Zhukovsky-Volterra gyrostat has the form:

$$L^{\mathrm{ZhV}}(z) = \sum_{\alpha=1}^{3} \Big(S_\alpha \varphi_\alpha(z) - \frac{\lambda_\alpha}{\varphi_\alpha(z)} \Big) \sigma_{4-\alpha}\,,$$

where σ_k are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and below we also use notation $\sigma_0 = \sigma_4$ for the identity 2×2 matrix.

The Poisson structure is given by the Poisson-Lie brackets on sl_2^*:

$$\{S_\alpha, S_\beta\} = \imath\varepsilon_{\alpha\beta\gamma}S_\gamma\,, \quad \alpha,\beta,\gamma \in \{1,2,3\}\,.$$

From the Lax matric we conclude that

$$\frac{1}{4}\,\mathrm{tr}\left(\left(L^{\mathrm{ZhV}}(z)\right)^2\right) = C\wp(z) + H^{\mathrm{ZhV}}\,,$$

where

$$C = \frac{1}{2}\left(S_1^2 + S_2^2 + S_3^2\right)$$

is the Casimir function of the Poisson-Lie brackets and H^{ZhV} is the Hamiltonian:

$$H^{\mathrm{ZhV}} = \frac{1}{2}\sum_{\alpha=1}^{3} S_\alpha^2\wp(\omega_\alpha) + \sum_{\alpha=1}^{3} S_\alpha\lambda_\alpha\,.$$

Introduce the set of matrices:

$$S=\sum_{\alpha=1}^{3} S_\alpha \sigma_{4-\alpha}\,, \qquad \hat{\wp}(S)=\sum_{\alpha=1}^{3} S_\alpha \wp(\omega_\alpha)\sigma_{4-\alpha}\,, \qquad \lambda=\sum_{\alpha=1}^{3} \lambda_\alpha \sigma_{4-\alpha}\,.$$

Then the equations of motion generated by the Hamiltonian $H^{\rm ZhV}$ and the Poisson-Lie brackets take the form

$$\dot{S}=[S,\hat{\wp}(S)]+[S,\lambda]\,.$$

These equations are **represented in the Lax form** $\dot{L}^{\rm ZhV}(z)=[L^{\rm ZhV}(z),M^{\rm ZhV}(z)]$ with M-matrix

$$M^{\rm ZhV}(z)=\frac{1}{2}\sum_{\alpha=1}^{3} S_\alpha\,\frac{\varphi_1(z)\varphi_2(z)\varphi_3(z)}{\varphi_\alpha(z)}\,\sigma_{4-\alpha}\,.$$

Let $r_{12}(z\pm w)$ be the classical elliptic r-matrix:

$$r_{12}(z\pm w)=\frac{1}{2}\sum_{\alpha=1}^{3}\varphi_\alpha(z\pm w)\sigma_{4-\alpha}\otimes\sigma_{4-\alpha}\,.$$

The Lax matrix of the Zhukovsky-Volterra model satisfies the classical linear reflection equation:

$$\{L_1^{\rm ZhV}(z),L_2^{\rm ZhV}(w)\}=\frac{1}{2}\,[L_1^{\rm ZhV}(z)+L_2^{\rm ZhV}(w),r_{12}(z-w)]-\frac{1}{2}\,[L_1^{\rm ZhV}(z)-L_2^{\rm ZhV}(w),r_{12}(z+w)]$$

identically in z,w and provides the linear Poisson-Lie brackets.

$$g(z)=\begin{pmatrix}\theta_3(z-2q|2\tau) & -\theta_3(z+2q|2\tau)\\ \\ -\theta_2(z-2q|2\tau) & \theta_2(z+2q|2\tau)\end{pmatrix}.$$

The gauge transformation of the Lax matrix of BC_1 Calogero-Inozemtsev system with this gauge transformation yields the Lax matrix of the Zhukovsky-Volterra (non-relativistic) gyrostat

$$g(z)L^{\mathrm{Inoz}}(z)g^{-1}(z)=L^{\mathrm{ZhV}}(z)$$

and provides the Poisson map given by the change of variables:

$$S_\alpha(p,q)=c_\alpha\Big(\frac{p}{2}\,\varphi_\alpha(2q)+\breve{\nu}_0\varphi_\beta(2q)\varphi_\gamma(2q)+\varphi_\alpha(2q)\sum_{k=1}^{3}\breve{\nu}_k\varphi_k(2q)\Big)$$

for distinct $\alpha,\beta,\gamma\in\{1,2,3\}$ and

$$\begin{pmatrix}\breve{\nu}_0\\ \breve{\nu}_1\\ \breve{\nu}_2\\ \breve{\nu}_3\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1&1&1\\ 1&1&-1&-1\\ 1&-1&1&-1\\ 1&-1&-1&1\end{pmatrix}\begin{pmatrix}\nu_0\\ \nu_1\\ \nu_2\\ \nu_3\end{pmatrix}.$$

and

$$\lambda_\alpha = \frac{\breve{\nu}_\alpha}{c_\alpha}\,, \qquad \alpha = 1, 2, 3\,,$$

where c_α are the constants

$$c_1(\tau) = -\Big(\frac{\vartheta_2(0)}{\vartheta'(0)}\Big)^2, \quad c_2(\tau) = -\imath\Big(\frac{\vartheta_3(0)}{\vartheta'(0)}\Big)^2, \quad c_3(\tau) = -\Big(\frac{\vartheta_4(0)}{\vartheta'(0)}\Big)^2.$$

The Casimir function $S_1^2 + S_2^2 + S_3^2$ of the Poisson-Lie brackets is mapped to $\breve{\nu}_0^2$.

Three of four linear combinations of ν_0, ν_1, ν_2, ν_3 are arranged into gyrostatic momenta vector $\vec{\lambda}$. And the last one is the Casimir function – the length of angular momentum vector.

Painlevé VI.

Three forms of the Painlevé VI equation.

1. *Rational form* is the original one. It was found by Gambier and proved to be the last and the most general in the list of the second order ordinary differential equations satisfying the Painlevé property:

$$\frac{d^2X}{dT^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-T}\right)\left(\frac{dX}{dT}\right)^2 - \left(\frac{1}{T} + \frac{1}{T-1} + \frac{1}{X-T}\right)\frac{dX}{dT} +$$

$$+ \frac{X(X-1)(X-T)}{T^2(T-1)^2}\left(\alpha + \beta\frac{T}{X^2} + \gamma\frac{T-1}{(X-1)^2} + \delta\frac{T(T-1)}{(X-T)^2}\right).$$

It contains four free complex constants $\alpha\,,\beta\,,\gamma\,,\delta$.

2. Elliptic form was found by P. Painlevé:

$$\frac{d^2u}{d\tau^2} = \sum_{a=0}^{3} \nu_a^2 \wp'(u+\omega_a)\,, \quad \{\omega_a\,,\ k=0,\ldots,3\} = \{0\,,\frac{1}{2}\,,\frac{1+\tau}{2}\,,\frac{\tau}{2}\}\,.$$

The equation is defined on the elliptic curve Σ_τ with half-periods ω_a and moduli τ which is the time variable here. The relation between rational and elliptic forms is given by the following change of variables:

$$(u\,,\tau) \to \left(X(u,\tau) = \frac{\wp(u)-e_1}{e_2-e_1}\,, \quad T(\tau) = \frac{e_3-e_1}{e_2-e_1} \right)\,,\ e_\alpha \equiv \wp(\omega_\alpha)\,,$$

$$(\nu_0^2\,,\nu_1^2\,,\nu_2^2\,,\nu_3^2) = -4\pi^2(\alpha\,,-\beta\,,\gamma\,,-\delta+\tfrac{1}{2})\,.$$

In such a form the equation is obviously Hamiltonian. The mechanical (non-autonomous) model is given by the Hamiltonian

$$H^{PVI} = \frac{1}{2}v^2 - \sum_{a=0}^{3} \nu_a^2 \wp(u+\omega_a)$$

and the canonical Poisson bracket

$$\{v,u\} = 1\,.$$

The mechanics is non-autonomous since two of the half-periods and the Weierstrass $\wp$-function depend on τ.

3. Non-autonomous version of the Zhukovsky-Volterra gyrostat was introduced and proved to be equivalent to the Painlevé VI:

$$\partial_\tau S = [S, J(S)] + [S, \nu'] \,, \quad \text{or} \quad \partial_\tau \vec{S} = \vec{S} \times J(\vec{S}) + \vec{S} \times \vec{\nu'} \,,$$

where S is sl_2^*-valued dynamical variable. In the basis of Pauli matrices

$$S = \sum_{\alpha=1}^{3} S_\alpha \sigma_\alpha \,, \quad J(S) = \sum_{\alpha=1}^{3} J_\alpha(\tau) S_\alpha \sigma_\alpha \,, \quad J_\alpha(\tau) = \wp(\omega_\alpha) = e_\alpha \,, \, \alpha = 1\,,2\,,3\,,$$

$$\nu' = \sum_{\alpha=1}^{3} \nu'_\alpha \sigma_\alpha \,, \quad \nu'_a = -\tilde{\nu}_a \exp(-2\pi i\, \omega_a \partial_\tau \omega_a) \left(\frac{\vartheta'(0)}{\vartheta(\omega_a)} \right)^2 \,, \; a = 0\,,1\,,2\,,3\,,$$

where $\tilde{\nu}_a$ are four free complex constants. The autonomous version (when the time variable is not related to the moduli τ) of the equation is known as the Zhukovsky-Volterra gyrostat. Vector $(\nu'_1\,, \nu'_2\,, \nu'_3)$ plays role of the gyrostatic momentum while $J_\alpha(\tau)$ are the inverse components of the inertia tensor in the principal axes of inertia. The equation is Hamiltonian with the Hamiltonian function

$$H^{ZVG} = \frac{1}{2} \sum_{\alpha=1}^{3} J_\alpha S_\alpha^2 + S_\alpha \nu'_\alpha$$

and the Poisson-Lie brackets on sl_2^* are $\{S_\alpha, S_\beta\} = \epsilon_{\alpha\beta\gamma} S_\gamma$.

The relation between equations is given by the following **change of variables**:

$$S_1 = -v\frac{\theta_2(0)}{\vartheta'(0)}\frac{\theta_2(2u)}{\vartheta(2u)} - \frac{\kappa}{2}\frac{\theta_2(0)}{\vartheta'(0)}\frac{\theta_2'(2u)}{\vartheta(2u)} +$$
$$\tilde{\nu}_0\frac{\theta_2^2(0)}{\theta_3(0)\theta_4(0)}\frac{\theta_3(2u)\theta_4(2u)}{\vartheta^2(2u)} + \tilde{\nu}_1\frac{\theta_2^2(2u)}{\vartheta^2(2u)} + \tilde{\nu}_2\frac{\theta_2(0)}{\theta_4(0)}\frac{\theta_2(2u)\theta_4(2u)}{\vartheta^2(2u)} + \tilde{\nu}_3\frac{\theta_2(0)}{\theta_3(0)}\frac{\theta_2(2u)\theta_3(2u)}{\vartheta^2(2u)}\ ,$$

$$iS_2 = v\frac{\theta_3(0)}{\vartheta'(0)}\frac{\theta_3(2u)}{\vartheta(2u)} + \frac{\kappa}{2}\frac{\theta_3(0)}{\vartheta'(0)}\frac{\theta_3'(2u)}{\vartheta(2u)} -$$
$$\tilde{\nu}_0\frac{\theta_3^2(0)}{\theta_2(0)\theta_4(0)}\frac{\theta_2(2u)\theta_4(2u)}{\vartheta^2(2u)} - \tilde{\nu}_1\frac{\theta_3(0)}{\theta_2(0)}\frac{\theta_3(2u)\theta_2(2u)}{\vartheta^2(2u)} - \tilde{\nu}_2\frac{\theta_3(0)}{\theta_4(0)}\frac{\theta_3(2u)\theta_4(2u)}{\vartheta^2(2u)} - \tilde{\nu}_3\frac{\theta_3^2(2u)}{\vartheta^2(2u)}\ ,$$

$$S_3 = -v\frac{\theta_4(0)}{\vartheta'(0)}\frac{\theta_4(2u)}{\vartheta(2u)} - \frac{\kappa}{2}\frac{\theta_4(0)}{\vartheta'(0)}\frac{\theta_4'(2u)}{\vartheta(2u)} +$$
$$\tilde{\nu}_0\frac{\theta_4^2(0)}{\theta_2(0)\theta_3(0)}\frac{\theta_2(2u)\theta_3(2u)}{\vartheta^2(2u)} + \tilde{\nu}_1\frac{\theta_4(0)}{\theta_2(0)}\frac{\theta_2(2u)\theta_4(2u)}{\vartheta^2(2u)} + \tilde{\nu}_2\frac{\theta_4^2(2u)}{\vartheta^2(2u)} + \tilde{\nu}_3\frac{\theta_4(0)}{\theta_3(0)}\frac{\theta_4(2u)\theta_3(2u)}{\vartheta^2(2u)}\ .$$

and the following identification of constants:

$$\tilde{\nu}_0 = \frac{1}{2}\left(\nu_0+\nu_1+\nu_2+\nu_3\right)\ ,$$
$$\tilde{\nu}_1 = \frac{1}{2}\left(\nu_0+\nu_1-\nu_2-\nu_3\right)\ ,$$
$$\tilde{\nu}_2 = \frac{1}{2}\left(\nu_0-\nu_1+\nu_2-\nu_3\right)\ ,$$
$$\tilde{\nu}_3 = \frac{1}{2}\left(\nu_0-\nu_1-\nu_2+\nu_3\right)\ .$$

Notice that while three constants $(\tilde{\nu}_1\,, \tilde{\nu}_2\,, \tilde{\nu}_3)$ enter the equation explicitly the last one ν_0' appears to be related to the value of the Casimir function of the brackets:

$$\frac{1}{2}\sum_{\alpha=1}^{3} S_\alpha^2 = \tilde{\nu}_0^2\,,$$

i.e. three linear combinations of the four Painlevé VI constants are arranged into the gyrostatic momentum vector while the last one linear combination is the length of the angular momentum. The gauge transformation has the following form in this case

$$L^{\text{Zh-V}}(z) = g(z)L^{\text{CM-Inoz}}(z)g^{-1}(z) - \kappa\partial_z g(z)g^{-1}(z)\,.$$

since in contrast to autonomous mechanics here we deal with the monodromy preserving equations

$$\partial_\tau L(z) - \partial_z M(z) = [L(z), M(z)].$$

Description of Zhukovsky-Volterra gyrostat through quadratic algebra.

Quadratic Poisson structure: BC_1 Sklyanin algebra. Introduce the following quadratic Poisson algebra:

$$c\{S_\alpha, S_\beta\} = -i\varepsilon_{\alpha\beta\gamma} S_0 S_\gamma \,,$$

$$c\{S_0, S_\alpha\} = -i\varepsilon_{\alpha\beta\gamma} S_\beta S_\gamma (\wp_\beta - \wp_\gamma) + i\varepsilon_{\alpha\beta\gamma}(S_\beta \lambda_\gamma - \lambda_\beta S_\gamma) \,.$$

The constant c is not necessary, and one can choose $c = 1$, but we keep it since it will be identified with the "light speed" c entering $e^{\pm p/2c}$.

The Hamiltonian is

$$H^{\text{Zh-V}} = S_0 \,.$$

In the case $\lambda_1 = \lambda_2 = \lambda_3 = 0$ the BC_1 Sklyanin algebra define the custom classical Sklyanin algebra. The **Casimir functions are as follows:**

$$C_1 = S_1^2 + S_2^2 + S_3^2 \,,$$

$$C_2 = S_0^2 + \sum_{k=1}^{3} S_k^2 \wp_k + 2 S_k \lambda_k \,, \quad \wp_k = \wp(\omega_k) \,.$$

Introduce the Lax matrix (the same as in the non-relativistic case)

$$L^{\mathrm{ZhV}}(z) = \sigma_0 S_0 + \sum_{\alpha=1}^{3} \Big(S_\alpha \varphi_\alpha(z) - \frac{\lambda_\alpha}{\varphi_\alpha(z)} \Big) \sigma_{4-\alpha} \,.$$

This Lax matrix satisfies the classical quadratic reflection equation. More precisely, the following statement holds.

The Lax matrix satisfies the classical quadratic reflection equation:

$$\{L_1^{\mathrm{ZhV}}(z), L_2^{\mathrm{ZhV}}(w)\} = \frac{1}{2c}\,[L_1^{\mathrm{ZhV}}(z) L_2^{\mathrm{ZhV}}(w), r_{12}(z-w)] +$$

$$+ \frac{1}{2c}\, L_2^{\mathrm{ZhV}}(w) r_{12}(z+w) L_1^{\mathrm{ZhV}}(z) - \frac{1}{2c}\, L_1^{\mathrm{ZhV}}(z) r_{12}(z+w) L_2^{\mathrm{ZhV}}(w)$$

identically in z, w and provides the Poisson brackets for BC_1 Sklyanin algebra.

Intermediate summary:

Quadratic reflection eq.:	???	gauge equivalence $\longleftrightarrow$	rel. Zh-V gyrostat
	$\downarrow$ non-rel. limit		$\downarrow$ non-rel. limit
Linear reflection eq.	Calogero-Inoz.	gauge equivalence $\longleftrightarrow$	non-rel Zh-V gyrostat

What is "???" ?

A candidate for ??? is the Ruijsenaars-van Diejen model. One degree of freedom case:

$$H^{8\mathrm{vD}} = v(\eta,q)\bar{v}(\bar{\eta},q)e^{p/c} + v(\eta,-q)\bar{v}(\bar{\eta},-q)e^{-p/c} - 2\sum_{a=0}^{3}\nu_a\bar{\nu}_a\wp(q+\omega_a)\,,$$

where $p,q \in \mathbb{C}$ are canonically conjugated momenta and coordinate of particle, $\eta,\bar{\eta},c,\nu_a,\bar{\nu}_a \in \mathbb{C}$, $a = 0,...,3$ are constants, $\wp(x)$ – is the Weierstrass elliptic function on an elliptic curve $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ with elliptic moduli τ and ω_a are four half-periods $0, 1/2, 1/2+\tau/2, \tau/2$. The function v is defined as

$$v(\eta,q|\nu) = v(\eta,q) = \sum_{a=0}^{3}\nu_a \exp(4\pi\imath\eta\partial_\tau\omega_a)\phi(2\eta,q+\omega_a)\,, \quad \phi(x,q) = \frac{\vartheta'(0)\vartheta(x+q)}{\vartheta(x)\vartheta(q)}$$

$$\bar{v} = \bar{v}(\bar{\eta},q|\bar{\nu})\,.$$

It has 8 independent coupling constants among 10: η, $\bar{\eta}$, ν_a, $\bar{\nu}_a$, $a = 0,1,2,3$ since ν_a and $\bar{\nu}_a$ are defined up to overall multiplication.

O. Chalykh introduced $2n \times 2n$ Lax matrix. We study 2×2 case. Perform the gauge transformation

$$\mathcal{L}^{\mathrm{Ch}}(z) \to \mathcal{L}(z) = \begin{pmatrix} e^{p/4c} & 0 \\ 0 & -e^{-p/4c} \end{pmatrix} \mathcal{L}^{\mathrm{Ch}}(z) \begin{pmatrix} e^{-p/4c} & 0 \\ 0 & -e^{p/4c} \end{pmatrix}.$$

Then the Lax matrix of the van Diejen system is represented in the symmetric form:

$$\boxed{\mathcal{L}(z) = L(z)\bar{L}(z)},$$

where

$$L(z) = \begin{pmatrix} v(\eta, q)e^{p/2c} & v(z, q) \\ \\ v(z, -q) & v(\eta, -q)e^{-p/2c} \end{pmatrix}$$

and

$$\bar{L}(z) = \begin{pmatrix} \bar{v}(\bar{\eta}, q)e^{p/2c} & \bar{v}(z, q) \\ \\ \bar{v}(z, -q) & \bar{v}(\bar{\eta}, -q)e^{-p/2c} \end{pmatrix}.$$

$L(z)$ and $\bar{L}(z)$ differ by only interchanging $\eta \leftrightarrow \bar{\eta}$, $\nu_a \leftrightarrow \bar{\nu}_a$.

It is interesting to notice that the expression $\operatorname{tr}\mathcal{L}(z) = \operatorname{tr}(L(z)\bar{L}(z))$ provides the same Hamiltonian as $\operatorname{tr} L(z) \operatorname{tr} \bar{L}(z)$.

Introduce

$$H_1^{4\mathrm{vD}} = \operatorname{tr} L(z) = v(\eta, q)e^{p/2c} + v(\eta, -q)e^{-p/2c}$$

and

$$\bar{H}_1^{4\mathrm{vD}} = \operatorname{tr} \bar{L}(z) = \bar{v}(\bar{\eta}, q)e^{p/2c} + \bar{v}(\bar{\eta}, -q)e^{-p/2c}\,.$$

The following relation holds true for $H^{8\mathrm{vD}}$ and $H_1^{4\mathrm{vD}}$, $\bar{H}_1^{4\mathrm{vD}}$:

$$\boxed{H^{8\mathrm{vD}} = H_1^{4\mathrm{vD}}\bar{H}_1^{4\mathrm{vD}} + \mathrm{const}}$$

Therefore, 4-constant model given by the Lax matrix $L(z)$ is also well defined. This can be verified directly.

The gauge transformation with the matrix $g(z)$ of 4-constants Lax matrix $L(z)$ provides the Lax matrix of the relativistic Zhukovsky-Volterra gyrostat:

$$g(z)L(z)g^{-1}(z) = L^{\mathrm{ZhV}}(z)$$

with the following **change of variables:**

$$S_0(p,q) = \frac{1}{2}\Big(\breve{v}(q,\eta)e^{p/2c} + \breve{v}(-q,\eta)e^{-p/2c}\Big)$$

and

$$S_\alpha(p,q|\eta,\nu_0,\nu_1,\nu_2,\nu_3) = S_\alpha(p,q) =$$

$$= \frac{c_\alpha}{2}\Big(\breve{v}(q,\eta)e^{p/2c} - \breve{v}(-q,\eta)e^{-p/2c}\Big)\varphi_\alpha(2q) + c_\alpha\breve{\nu}_0\varphi_\beta(2q)\varphi_\gamma(2q) + c_\alpha\varphi_\alpha(2q)\sum_{k=1}^{3}\breve{\nu}_k\varphi_k(2q)$$

for distinct $\alpha,\beta,\gamma \in \{1,2,3\}$, where the notation

$$\breve{v}(q,\eta) = \sum_{a=0}^{3}\breve{\nu}_a\varphi_a(2q,\eta+\omega_a) = v(\eta,q)\,.$$

is used. The identification of parameters is given by

$$\lambda_\alpha = \frac{\breve{\nu}_\alpha}{c_\alpha}\,, \quad \alpha = 1,2,3$$

with the constants c_α as previously.

The change of variables from p, q to $S_a(p,q)$, $a = 0, ..., 3$ provides the Poisson map between the canonical brackets $\{p, q\} = 1$ and the BC_1 Sklyanin algebra:

$$c\{S_\alpha, S_\beta\} = -i\varepsilon_{\alpha\beta\gamma} S_0 S_\gamma \,,$$

$$c\{S_0, S_\alpha\} = -i\varepsilon_{\alpha\beta\gamma} S_\beta S_\gamma (\wp_\beta - \wp_\gamma) + i\varepsilon_{\alpha\beta\gamma}(S_\beta \lambda_\gamma - \lambda_\beta S_\gamma)\,.$$

The Casimir functions C_1 and C_2

$$C_1 = S_1^2 + S_2^2 + S_3^2 \,,$$

$$C_2 = S_0^2 + \sum_{k=1}^{3} S_k^2 \wp_k + 2 S_k \lambda_k \,, \quad \wp_k = \wp(\omega_k)\,.$$

under this change of variables are mapped to the values

$$C_1 = \breve{\nu}_0^2 \,, \quad C_2 = \sum_{a=0}^{3} \breve{\nu}_a^2 \wp(\eta + \omega_a) - \sum_{a=1}^{3} \breve{\nu}_a^2 \wp(\omega_a)$$

What about original 8 constants van Diejen model?

Coupled Zhukovsky-Volterra gyrostats.

It is now straightforward to perform the gauge transformation to the 8-constant model given by $\mathcal{L}(z) = L(z)\bar{L}(z)$ Indeed,

$$g(z)\mathcal{L}(z)g^{-1}(z) = g(z)L(z)g^{-1}(z)\, g(z)\bar{L}(z)g^{-1}(z) = L^{\mathrm{ZhV}}(z)\bar{L}^{\mathrm{ZhV}}(z)\,,$$

where the matrix $L^{\mathrm{ZhV}}(z)$ is given by

$$L^{\mathrm{ZhV}}(z) = \sigma_0 \bar{S}_0 + \sum_{\alpha=1}^{3} \Big(\bar{S}_\alpha \varphi_\alpha(z) - \frac{\bar{\lambda}_\alpha}{\varphi_\alpha(z)} \Big) \sigma_{4-\alpha}\,.$$

Expressions for $\bar{S}_\alpha(p,q|\bar{\eta},\bar{\nu}_a)$ are the same as for $S_\alpha(p,q|\eta,\nu_a)$ but with another set of constants. That is, $\bar{S}_\alpha(p,q) = S_\alpha(p,q|\bar{\eta},\bar{\nu}_a)$. The Poisson brackets $\{S_a, S_b\}$ and $\{\bar{S}_a, \bar{S}_b\}$ have the form as previously:

$$c\{\bar{S}_\alpha, \bar{S}_\beta\} = -i\varepsilon_{\alpha\beta\gamma}\bar{S}_0\bar{S}_\gamma\,,$$

$$c\{\bar{S}_0, \bar{S}_\alpha\} = -i\varepsilon_{\alpha\beta\gamma}\bar{S}_\beta\bar{S}_\gamma(\wp_\beta - \wp_\gamma) + i\varepsilon_{\alpha\beta\gamma}(\bar{S}_\beta\bar{\lambda}_\gamma - \bar{\lambda}_\beta\bar{S}_\gamma)\,.$$

The mixed type brackets $\{S_a, \bar{S}_b\}$ can be computed as function of p, q but an answer in terms of S_a, $\bar{S}_b$ is unknown.

Classical spin chains:

$$\{\mathcal{L}_1^a(z), \mathcal{L}_2^a(w)\} = \frac{1}{c}\,[\mathcal{L}_1^a(z)\mathcal{L}_2^a(w), r_{12}(z-w)].$$

At each site we have its own Sklyanin algebra:

$$c\{\mathbb{S}_i^a, \mathbb{S}_j^a\} = -\imath\varepsilon_{ijk}\mathbb{S}_0^a\mathbb{S}_k^a\,,$$

$$c\{\mathbb{S}_0^a, \mathbb{S}_i^a\} = -\imath\varepsilon_{ijk}\mathbb{S}_j^a\mathbb{S}_k^a\big(\wp(\omega_j) - \wp(\omega_k)\big)\,.$$

Monodromy matrix

$$\mathbb{T}(z) = \mathcal{L}(z, \mathbb{S}^1)\mathcal{L}(z, \mathbb{S}^2)...\mathcal{L}(z, \mathbb{S}^n)\,.$$

Transfer-matrix has the property

$$\{\mathrm{tr}\,\mathbb{T}^k(z), \mathrm{tr}\,\mathbb{T}^l(w)\} = 0,$$

which provides integrability.

At each site we have a relativistic Euler top. One can perform the gauge transformation, which parameterize each Sklyanin algebra $\mathbb{S}^a$ by a pair of canonical variables $\{\mathbf{p}_a, \mathbf{q}_a\} = 1$.

What kind of model do we get for the XYZ chain in terms of $\mathbf{p}_a, \mathbf{q}_a$?

Namely, we have

$$\mathbb{S}_0^a = \frac{1}{2}\Big(\frac{\vartheta(2\mathbf{q}_a - \eta)}{\vartheta(2\mathbf{q}_a)}\, e^{\mathbf{p}_a/2c} + \frac{\vartheta(2\mathbf{q}_a + \eta)}{\vartheta(2\mathbf{q}_a)}\, e^{-\mathbf{p}_a/2c}\Big)\,,$$

$$\mathbb{S}_1^a = \frac{1}{2}\frac{\theta_4(0)}{\vartheta'(0)}\Big(\frac{\theta_4(2\mathbf{q}_a - \eta)}{\vartheta(2\mathbf{q}_a)}\, e^{\mathbf{p}_a/2c} - \frac{\theta_4(2\mathbf{q}_a + \eta)}{\vartheta(2\mathbf{q}_a)}\, e^{-\mathbf{p}_a/2c}\Big)\,,$$

$$\mathbb{S}_2^a = \frac{\imath}{2}\frac{\theta_3(0)}{\vartheta'(0)}\Big(\frac{\theta_3(2\mathbf{q}_a - \eta)}{\vartheta(2\mathbf{q}_a)}\, e^{\mathbf{p}_a/2c} - \frac{\theta_3(2\mathbf{q}_a + \eta)}{\vartheta(2\mathbf{q}_a)}\, e^{-\mathbf{p}_a/2c}\Big)\,,$$

$$\mathbb{S}_3^a = \frac{1}{2}\frac{\theta_2(0)}{\vartheta'(0)}\Big(\frac{\theta_2(2\mathbf{q}_a - \eta)}{\vartheta(2\mathbf{q}_a)}\, e^{\mathbf{p}_a/2c} - \frac{\theta_2(2\mathbf{q}_a + \eta)}{\vartheta(2\mathbf{q}_a)}\, e^{-\mathbf{p}_a/2c}\Big)\,.$$

These are the generators of the classical Sklyanin algebras.

The change of variables provides the Poisson map between the canonical Poisson structure for the variables $\mathbf{p}_a$, $\mathbf{q}_a$ and the classical Sklyanin algebras for $\mathbb{S}^a$:

$$c\{\mathbb{S}_i^a, \mathbb{S}_j^a\} = -\imath\varepsilon_{ijk}\mathbb{S}_0^a\mathbb{S}_k^a\,,$$

$$c\{\mathbb{S}_0^a, \mathbb{S}_i^a\} = -\imath\varepsilon_{ijk}\mathbb{S}_j^a\mathbb{S}_k^a\big(\wp(\omega_j) - \wp(\omega_k)\big)\,.$$

In this way we come to the **elliptic Ruijsenaars-Toda chain introduced by Adler, Shabat and Suris.** Equations of motion for this model were derived in the Newtonian form:

$$\frac{2\ddot{\mathbf{q}}_a}{\dot{\mathbf{q}}_a^2 - 1} =$$

$$= \dot{\mathbf{q}}_{a+1} f(\mathbf{q}_a, \mathbf{q}_{a+1}, \eta) - \dot{\mathbf{q}}_{a-1} f(\mathbf{q}_a, \mathbf{q}_{a-1}, \eta) + g(\mathbf{q}_a, \mathbf{q}_{a+1}, \eta) + g(\mathbf{q}_a, \mathbf{q}_{a-1}, \eta) - 4E_1(2\mathbf{q}_a)\,,$$

where η is a constant parameter, $E_1(x) = \vartheta'(x)/\vartheta(x)$ and

$$f(x, y, \eta) = E_1(x - y - \eta) + E_1(x + y + \eta) - E_1(x - y + \eta) - E_1(x + y - \eta)\,,$$

$$g(x, y, \eta) = E_1(x - y - \eta) + E_1(x + y + \eta) + E_1(x - y + \eta) + E_1(x + y - \eta)\,.$$

Particular case: elliptic Toda chain introduced by Krichever.

The equations for elliptic Toda model follow from those for the Ruijsenaars-Toda chain in the case $\eta = 0$ (then $f(x, y, 0) = 0$).

It is described by the following Hamiltonian:

$$H^{\mathrm{eToda}} = -\frac{1}{2}\sum_{a=1}^{n}\left(\log\frac{1}{\sinh^2(\mathbf{p}_a/2c)} + \log\Big(\wp(\mathbf{q}_{a-1} - \mathbf{q}_a) - \wp(\mathbf{q}_{a-1} + \mathbf{q}_a)\Big)\right),$$

For the elliptic Toda chain we get the following change of variables

$$\mathbf{S}_0^a = \frac{1}{2}\left(e^{\mathbf{p}_a/2c} + e^{-\mathbf{p}_a/2c}\right),$$

$$\mathbf{S}_1^a = \frac{1}{2}\frac{\theta_4(0)}{\vartheta'(0)}\left(\frac{\theta_4(2\mathbf{q}_a)}{\vartheta(2\mathbf{q}_a)}\, e^{\mathbf{p}_a/2c} - \frac{\theta_4(2\mathbf{q}_a)}{\vartheta(2\mathbf{q}_a)}\, e^{-\mathbf{p}_a/2c}\right),$$

$$\mathbf{S}_2^a = \frac{\imath}{2}\frac{\theta_3(0)}{\vartheta'(0)}\left(\frac{\theta_3(2\mathbf{q}_a)}{\vartheta(2\mathbf{q}_a)}\, e^{\mathbf{p}_a/2c} - \frac{\theta_3(2\mathbf{q}_a)}{\vartheta(2\mathbf{q}_a)}\, e^{-\mathbf{p}_a/2c}\right),$$

$$\mathbf{S}_3^a = \frac{1}{2}\frac{\theta_2(0)}{\vartheta'(0)}\left(\frac{\theta_2(2\mathbf{q}_a)}{\vartheta(2\mathbf{q}_a)}\, e^{\mathbf{p}_a/2c} - \frac{\theta_2(2\mathbf{q}_a)}{\vartheta(2\mathbf{q}_a)}\, e^{-\mathbf{p}_a/2c}\right).$$

The Lax matrices satisfy the classical quadratic exchange relation with the elliptic r-matrix and provides the same Sklyanin algebra

$$c\{\mathbf{S}_i^a, \mathbf{S}_j^a\} = -\imath\varepsilon_{ijk}\mathbf{S}_0^a\mathbf{S}_k^a\,,$$

$$c\{\mathbf{S}_0^a, \mathbf{S}_i^a\} = -\imath\varepsilon_{ijk}\mathbf{S}_j^a\mathbf{S}_k^a\big(\wp(\omega_j) - \wp(\omega_k)\big)$$

since the structure constants are independent of η. **Special values of the Casimir functions:**

$$\mathbf{C}_1^a = (\mathbf{S}_1^a)^2 + (\mathbf{S}_2^a)^2 + (\mathbf{S}_3^a)^2 = 0\,, \qquad \mathbf{C}_2^a = (\mathbf{S}_0^a)^2 + \sum_{k=1}^{3}(\mathbf{S}_k^a)^2\wp(\omega_k) = 1$$

XYZ chain with boundaries

Following Sklyanin's construction consider the monodromy matrix:

$$\mathcal{T}^{\mathrm{openXYZ}}(z) = K^{+}(z)\mathcal{T}(z)K^{-}(z)\mathcal{T}^{-1}(-z)\,,$$

where $\mathcal{T}(z)$ satisfies the quadratic classical exchange relation

$$c\{\mathcal{T}_1(z), \mathcal{T}_2(w)\} = [\mathcal{T}_1(z)\mathcal{T}_2(w), r_{12}(z-w)]$$

and $K^{\pm}(z)$ are 2×2 matrices solving the classical quadratic reflection equation

$$[K_1^{\pm}(z)K_2^{\pm}(w), r_{12}(z-w)] + K_2^{\pm}(w)r_{12}(z+w)K_1^{\pm}(z) - K_1^{\pm}(z)r_{12}(z+w)K_2^{\pm}(w) = 0$$

with the elliptic r-matrix from. Then the classical transfer-matrix $\mathrm{tr}\,\mathcal{T}^{\mathrm{openXYZ}}(z)$ is a generating function of commuting Hamiltonians since

$$\{\mathrm{tr}\,\mathcal{T}^{\mathrm{openXYZ}}(z), \mathrm{tr}\,\mathcal{T}^{\mathrm{openXYZ}}(w)\} = 0\,.$$

It is possible to perform the gauge transformation from the open XYZ chain with boundaries. This provides the elliptic open Toda chain with boundaries:

$$H^{\text{openToda}} = \sum_{a=1}^{n} \log \frac{1}{\sinh^2(\mathbf{p}_a/2c)} + \sum_{a=2}^{n} \log \Big(\wp(\mathbf{q}_{a-1} - \mathbf{q}_a) - \wp(\mathbf{q}_{a-1} + \mathbf{q}_a) \Big) -$$

$$- \log \Big(\nu_0^+ \coth \big(\frac{\mathbf{p}_1}{2c} \big) - \tilde{f}^+(\mathbf{q}_1) \Big) - \log \Big(\nu_0^- \coth \big(\frac{\mathbf{p}_n}{2c} \big) - \tilde{f}^-(\mathbf{q}_n) \Big) .$$

where

$$\big(\tilde{f}^+(\mathbf{q}_1)\big)^2 = \sum_{k=0}^{3} (\bar{\nu}_k^+)^2 \wp(\mathbf{q}_1 + \omega_k) - \sum_{k=1}^{3} (\nu_k^+)^2 \wp(\omega_k) ,$$

$$\big(\tilde{f}^-(\mathbf{q}_n)\big)^2 = \sum_{k=0}^{3} (\bar{\nu}_k^-)^2 \wp(\mathbf{q}_n + \omega_k) - \sum_{k=1}^{3} (\nu_k^-)^2 \wp(\omega_k)$$

with

$$\begin{pmatrix} \bar{\nu}_0^+ \\ \bar{\nu}_1^+ \\ \bar{\nu}_2^+ \\ \bar{\nu}_3^+ \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \nu_1^+ \\ \nu_2^+ \\ \nu_3^+ \end{pmatrix} .$$

Consider two particular cases.

Example 1: pure open chain. Let $\nu_k^\pm = 0$, $k = 1, 2, 3$ and $\nu_0^\pm = 1$. From viewpoint of the open XYZ chain this case corresponds to trivial K-matrices $K^\pm(z) = \sigma_0$, and the gauge transformed K-matrices are trivial as well: $\tilde{K}^\pm(z) = \sigma_0$. Then

$$H^{\text{pure open}} = \log \frac{1}{\sinh(\mathbf{p}_1/2c)\cosh(\mathbf{p}_1/2c)} + \log \frac{1}{\sinh(\mathbf{p}_n/2c)\cosh(\mathbf{p}_n/2c)} +$$

$$+ \sum_{a=2}^{n-1} \log \frac{1}{\sinh^2(\mathbf{p}_a/2c)} + \sum_{a=2}^{n} \log\Big(\wp(\mathbf{q}_{a-1} - \mathbf{q}_a) - \wp(\mathbf{q}_{a-1} + \mathbf{q}_a)\Big) .$$

This Hamiltonian differs from the closed elliptic Toda chain by the absence of interaction between the first and the last particles, and by type of dependence on momenta for these two particles on the boundaries.

Example 2: only boundary terms. Now let it be vice versa: $\nu_0^{\pm} = 0$. We obtain

$$H^{\text{boundary}} = \sum_{a=1}^{n} \log \frac{1}{\sinh^2(\mathbf{p}_a/2c)} + \sum_{a=2}^{n} \log \Big(\wp(\mathbf{q}_{a-1} - \mathbf{q}_a) - \wp(\mathbf{q}_{a-1} + \mathbf{q}_a) \Big) -$$

$$-\frac{1}{2} \log \Big(\sum_{k=0}^{3} (\bar{\nu}_k^+)^2 \wp(\mathbf{q}_1 + \omega_k) - \sum_{k=1}^{3} (\nu_k^+)^2 \wp(\omega_k) \Big) -$$

$$-\frac{1}{2} \log \Big(\sum_{k=0}^{3} (\bar{\nu}_k^-)^2 \wp(\mathbf{q}_n + \omega_k) - \sum_{k=1}^{3} (\nu_k^-)^2 \wp(\omega_k) \Big) .$$

In this case the kinetic part (dependence on momenta) is the same as in the closed elliptic Toda chain, but there are additional external fields for the first and the last particles on the boundaries.

Thank you!

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