On weak solvability of initial–boundary value problems for thermoviscoelastic fluids

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Voigt termo–model

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \mathrm{Div}\,\Big(\mu(I_2(v))\mathcal{E}(v)\Big) - \varkappa \frac{\partial \triangle v}{\partial t} + \nabla\theta + \nabla p = f;$$

$$\mathrm{div}\ v = 0; \qquad v\,|_{t=0} = v_0; \qquad v\,|_{[0,T]\times\partial\Omega} = 0;$$

$$\frac{\partial \theta}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial \theta}{\partial x_i} - \chi\triangle\theta = \Big(\mu(I_2(v))\mathcal{E}(v) + \varkappa\frac{\partial \mathcal{E}(v)}{\partial t}\Big) : \mathcal{E}(v) + g;$$

$$\theta\,|_{t=0} = \theta_0; \qquad \theta\,|_{[0,T]\times\partial\Omega} = 0.$$

- $v(t,x)$ is unknown vector-function of velocity;
- $\theta(t,x)$ is unknown temperature;
- $p(t,x)$ is unknown pressure of the fluid;
- f is a given density of external forces;
- g is a given source of external heat;
- μ is the viscosity coefficient;
- $\varkappa > 0$ is the retardation coefficient;
- $\chi > 0$ is the thermal conductivity coefficient;
- $\mathcal{E}(v)$ is the strain rate tensor with components $\mathcal{E}_{ij} = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$;
- $I_2^2(v) = \mathcal{E}(v) : \mathcal{E}(v)$;
- $A : B = \sum_{i,j=1}^{n} a_{ij} \cdot b_{ij}$ (for matrices $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^n$);
- $\mathrm{Div}\ A = \Big(\sum_{j=1}^{n} \frac{\partial a_{1j}(t,x)}{\partial x_j}, \cdots, \sum_{j=1}^{n} \frac{\partial a_{nj}(t,x)}{\partial x_j}\Big)$

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Functional spaces

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain. We denote by $C_0^\infty(\Omega)$ the space of functions defined on Ω which belong to the class C^∞, take on values in $\mathbb{R}^n$ and have a compact support in Ω.

$$\mathcal{V} = \{v(x) = (v_1, \dots, v_n) \in C_0^\infty(\Omega) : \operatorname{div} v = 0\};$$

$$H = \text{ is the closure } \mathcal{V} \text{ with respect to the norm of space } L_2(\Omega);$$

$$V = \text{ is the closure } \mathcal{V} \text{ with respect to the norm of space } W_2^1(\Omega);$$

$$V^* \text{ denote the space conjugate to the space } V;$$

$$W_1 = \{v : v \in C([0, T]; V),\ v' \in L_2(0, T; V^*)\};$$

$$W_2 = \{\theta : \theta \in L_p(0, T; W_p^1(\Omega)),\ \theta' \in L_1(0, T; W_p^{-1}(\Omega))\}.$$

Weak solution

Definition

A weak solution of the initial–boundary value problem is a pair $(v;\theta)$, where

$$v \in W_1 = \{v : v \in C([0,T];V),\ v' \in L_2(0,T;V^*)\},$$

$$\theta \in W_2 = \{\theta : \theta \in L_p(0,T;W_p^1(\Omega)),\ \theta' \in L_1(0,T;W_p^{-1}(\Omega))\},$$

which satisfying initial conditions $v|_{t=0} = v_0$ and $\theta|_{t=0} = \theta_0$, and equalities

$$\frac{d}{dt}\int_\Omega v\,\varphi\,dx - \int_\Omega \sum_{i,j=1}^{n} v_i v_j \frac{\partial \varphi_j}{\partial x_i}\,dx + \int_\Omega \mu(I_2(v))\mathcal{E}(v) : \mathcal{E}(\varphi)\,dx +$$

$$+\varkappa \frac{\partial}{\partial t}\int_\Omega \mathcal{E}(v) : \mathcal{E}(\varphi)\,dx = \langle f, \varphi\rangle \quad \text{for all } \varphi \in V \text{ and a.e. } t \in [0,T],$$

$$\frac{d}{dt}\int_\Omega \theta\,\phi\,dx - \int_\Omega \sum_{i,j=1}^{n} v_i \theta_j \frac{\partial \phi_j}{\partial x_i}\,dx + \chi\int_\Omega \mathcal{E}(\theta) : \mathcal{E}(\phi)\,dx =$$

$$= \int_\Omega \left(\mu(I_2(v))\mathcal{E}(v) : \mathcal{E}(v)\right) : \phi\,dx + \varkappa\int_\Omega \left(\frac{\partial \mathcal{E}(v)}{\partial t} : \mathcal{E}(v)\right) : \phi\,dx +$$

$$+\langle g, \phi\rangle \quad \text{for all } \phi \in C_0^\infty(\Omega) \text{ and a.e. } t \in [0,T].$$

Main Result

The viscosity function $\mu : \mathbb{R}^n \to \mathbb{R}$ have to be continuously differentiable and satisfy the inequalities

$$\text{a)} \quad 0 < C_1 \le \mu(s) \le C_2 < \infty; \qquad \text{b)} \quad |s\mu'(s)| \le C_3 < \infty;$$
$$\text{c)} \quad -s\mu'(s) \le \mu(s) \text{ as } \mu'(s) < 0.$$

Main Theorem

Let $f \in L_2(0, T; V^*)$, $g \in L_1(0, T; H_p^{-2(1-1/p)}(\Omega))$, $v_0 \in V$, $\theta_0 \in W_p^{1-2/p}(\Omega)$, and the viscosity coefficient μ satisfies conditions $a) - c)$. Then in the case $n = 2$ for $1 < p < 4/3$ and in the case $n = 3$ for $1 < p < 5/4$ a weak solution to the initial–boundary value problem exists.

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Proof Sketch of Main Theorem
• First Step

We consider the initial–boundary value problem with the fixed $\theta \in W_2$:

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \text{Div}\,[\mu(I_2(v))\mathcal{E}(v)] - \varkappa \frac{\partial \triangle v}{\partial t} + \nabla \theta + \nabla p = f; \tag{1}$$

$$\text{div}\ v = 0; \qquad v\,|_{t=0} = v_0; \qquad v\,|_{[0,T]\times\partial\Omega} = 0. \tag{2}$$

Theorem 2.

Let $f \in L_2(0, T; V^*)$, $v_0 \in V$ and the viscosity coefficient μ satisfies conditions $a) - c)$. Then problem (1)–(2) has at least one weak solution $v \in W_1$, for which the following estimate hold

$$\|v\|_{W_1} \leqslant R_1, \qquad R_1 = R_1(T, \|f\|_{L_2(0,T;V^*)}, \|v_0\|_V).$$

Proof Sketch of Theorem 2

Let's introduce operators using the following equalities

$$J : V \to V^*, \quad \langle Jv, \varphi \rangle = \int_\Omega v\varphi \, dx \quad v \in V, \varphi \in V;$$

$$A : V \to V^*, \quad \langle Av, \varphi \rangle = \int_\Omega \nabla v : \nabla \varphi \, dx, \quad v \in V, \varphi \in V;$$

$$B : L_4(\Omega) \to V^*, \quad \langle B(v), \varphi \rangle = \int_\Omega \sum_{i,j=1}^{n} v_i v_j \frac{\partial \varphi_j}{\partial x_i} \, dx, \quad v \in L_4(\Omega), \varphi \in V;$$

$$D : V \to V^*, \quad \langle Dv, \varphi \rangle = \int_\Omega \mu(I_2(v))\mathcal{E}(v) : \nabla \varphi \, dx, \quad v \in V, \varphi \in V.$$

The integral equality from the definition of a weak solution can be written in the following form

$$(\varkappa A + J)v' + D(v) - B(v) = f.$$

Let's introduce the following operators

$$L : W_1 \to L_2(0, T; V^*) \times V, \qquad L(v) = ((\varkappa A + J)v' + D(v), v|_{t=0});$$
$$K : W_1 \to L_2(0, T; V^*) \times V, \qquad K(v) = (B(v), 0).$$

Then the problem under consideration (1)–(2) is equivalent to the following operator equation:

$$L(v) - K(v) = (f, v_0).$$

Proof Sketch of Theorem 2

Lemma 1

1. The nonlinear mapping $L : W_1 \to L_2(0, T; V^*) \times V$ is invertible. The inverse operator $L^{-1} : L_2(0, T; V^*) \times V \to W_1$ is continuous.
2. The mapping $K : W_1 \to L_2(0, T; V^*) \times V$ is completely continuous.

Let's introduce an auxiliary family of operator equations.

$$L(v) - \eta K(v) = \eta(f, v_0), \quad \text{where } \eta \in [0, 1]. \tag{3}$$

Lemma 2

If $v \in W_1$ is a solution to the equation (3) for some $\eta \in [0, 1]$, then the following a priori estimate holds:

$$\|v\|_{W_1} \leqslant R_1, \quad R_1 = R_1(T, \|f\|_{L_2(0,T;V^*)}, \|v_0\|_V).$$

Due to the a priori estimate obtained above, all solutions of the family of equations (3) lie in the ball $B_{R_1+1} \subset W_1$ centered at zero. The Leray–Schauder degree is defined:

$$\deg_{LS}(I - \eta L^{-1}[K(v) + (f, v_0)], B_R, 0).$$

By the properties of homotopy invariance and normalization of the degree, we obtain

$$\deg_{LS}(I - L^{-1}[K(v) + (f, v_0)], B_R, 0) = \deg_{LS}(I, B_R, 0) = 1. \qquad \square$$

• Second Step

We consider the initial–boundary value problem with the fixed $v \in W_1$:

$$\frac{\partial \theta}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial \theta}{\partial x_i} - \chi \triangle \theta = \left[\mu(I_2(v))\mathcal{E}(v) + \varkappa \frac{\partial \mathcal{E}(v)}{\partial t}\right] : \mathcal{E}(v) + g; \tag{4}$$

$$\theta \mid_{t=0} = \theta_0; \qquad \theta \mid_{[0,T]\times\partial\Omega} = 0. \tag{5}$$

Theorem 3.

Let $g \in L_1(0, T; H_p^{-2(1-1/p)}(\Omega))$, $\theta_0 \in W_p^{1-2/p}(\Omega)$, $v \in W_1$ and the viscosity coefficient μ satisfies conditions $a) - c)$. Then in the case $n = 2$ for $1 < p < 4/3$ and in the case $n = 3$ for $1 < p < 5/4$ the initial–boundary value problem (4)–(5) has at least one weak solution for which the following estimate hold

$$\|\theta\|_{W_2} \leq R_2\Big(\|g\|_{L_1(0,T;H_p^{-2(1-1/p)}(\Omega))} + \|\frac{\partial v}{\partial t}\|^2_{L_2(0,T;H)} + \|\theta_0\|_{W_p^{1-2/p}(\Omega)}\Big).$$

Proof Sketch of Theorem 3

Let's rewrite the equation (4) in the following form

$$\frac{\partial}{\partial t}(A^{-1/2}\theta) + \sum_{i=1}^{n} v_i A^{-1/2}\frac{\partial \theta}{\partial x_i} + \chi A^{1/2}\theta = \Phi_1 + \Phi_2, \qquad (6)$$

where $\Phi_1 = A^{-1/2}g$, $\Phi_2 = A^{-1/2}[(\mu(I_2(v))\mathcal{E}(v) + \varkappa\frac{\partial \mathcal{E}(v)}{\partial t}] : \mathcal{E}(v))$ and A is a linear operator in $L_p(\Omega)$ generated by the differential equation $A(\theta) = -\Delta\theta$.

Consider the following problem

$$\frac{\partial}{\partial t}(A^{-1/2}\theta) + \chi A^{1/2}\theta = \Phi_2, \qquad \theta|_{t=0} = 0.$$

Let $z = A^{-1/2}\theta$ and $\Phi_2 = A^{-\epsilon}A^{-(1/2-\epsilon)}([(\mu(I_2(v))\mathcal{E}(v) + \varkappa\frac{\partial \mathcal{E}(v)}{\partial t}] : \mathcal{E}(v)) = A^{-\epsilon}\tilde{g}(t)$. Then we have

$$z' + Az = A^{-\epsilon}\tilde{g}(t), \quad z(0) = 0,$$

where $\tilde{g}(t)$ belongs to $L_1(0, T; L_p(\Omega))$.

With help of fractional powers method we obtain the solution

$$z(t) = \int_0^t exp(-(t-s)A)A^{-\epsilon}\tilde{g}(s)\, ds$$

with $z \in L_p(0, T; W_p^2(\Omega)) \cap W_1^1(0, T; L_p(\Omega))$. It provides a solution $\theta_1 = A^{1/2}z$.

Proof Sketch of Theorem 3

Substituting $\theta = \theta_1 + A^{-1/2}u$ in (6) and rewrite it the following form

$$u' + Au = \Phi, \quad u|_{t=0} = A^{-1/2}\theta_0 = u_0,$$

where $\Phi = \Phi_1 - A^{-1/2}\dfrac{\partial}{\partial x_i}(v_i\theta_1) + A^{-1/2}\dfrac{\partial}{\partial x_i}(v_i A^{1/2}u)$ and Φ belongs to $L_p(0, t; L_p(\Omega))$.
This gives us the solvability of the initial–boundary problem (4)–(5).

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• Third Step

Consider the sequence (v^m, θ^m), $m = 0, 1, 2, \dots$, defined as follows.
Let v^0 and θ^0 denote the initial values v_0 and θ_0 for v and θ, respectively. Let (v^m, θ^m) be known.

Then, first, v^{m+1} is found as a weak solution to the problem

$$\frac{\partial v^{m+1}}{\partial t} + \sum_{i=1}^{n} v_i^{m+1} \frac{\partial v^{m+1}}{\partial x_i} - \mathrm{Div}\, [\mu(I_2(v^{m+1}))\mathcal{E}(v^{m+1})] - \varkappa \frac{\partial \Delta v^{m+1}}{\partial t} + \nabla \theta^m + \nabla p = f;$$

$$\mathrm{div}\, v^{m+1} = 0; \quad v^{m+1}|_{t=0} = v_0; \quad v^{m+1}|_{[0,T]\times\partial\Omega} = 0.$$

Then, for given v^{m+1} the function θ^{m+1} is found as a weak solution to the problem

$$\frac{\partial \theta^{m+1}}{\partial t} + \sum_{i=1}^{n} v_i^{m+1} \frac{\partial \theta^{m+1}}{\partial x_i} - \chi \triangle \theta^{m+1} = [\mu(I_2(v^{m+1}))\mathcal{E}(v^{m+1}) + \varkappa \frac{\partial \mathcal{E}(v^{m+1})}{\partial t}] : \mathcal{E}(v^{m+1}) + g;$$

$$\theta^{m+1}|_{t=0} = \theta_0; \quad \theta^{m+1}|_{[0,T]\times\partial\Omega} = 0.$$

Fourth Step

From the results of Theorem 3 it follows:

Lemma

The sequence $\{\theta^m\}$ is relatively compact in $L_p(0, T; L_p(\Omega))$, where p satisfies the conditions of Theorem 3.

From the results of Theorem 2 (without loss of generality) it follows:
v^m converges weakly in $L_2(0, T; V)$, and $\dfrac{\partial v^m}{\partial t}$ converges weakly in $L_2(0, T; V^*)$.
By Simon's Theorem (passing to subsequences if necessary), $v^m \to v$ strongly in $C([0, T], L_4(\Omega))$.

Lemma

The sequence v^m converges strongly in $L_2(0, T; V)$ to $v \in W_1$.

The strong convergence of v^m in $L_2(0, T; V)$ and θ^m in $L_p(0, T; L_p(\Omega))$ and the weak convergence of v^m to v in $L_2(0, T; V)$ and θ^m to θ in $L_p(0, T; W_p^1(\Omega))$, where p satisfies the conditions of Theorem 3, implies the possibility of passage to the limit in all terms of integral equalities for v^m and θ^m. This completes the proof of Main Theorem.

Bingham termo–model

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \operatorname{Div}\sigma + \nabla p = f;$$

$$\sigma = \begin{cases} 2\mu(\theta)\mathcal{E}(v) + \tau^* \frac{\mathcal{E}(v)}{|\mathcal{E}(v)|}, & \text{for } |\mathcal{E}(v)| \neq 0; \\ |\sigma| \leq \tau^*, & \text{for } |\mathcal{E}(v)| = 0; \end{cases}$$

$$\operatorname{div} v = 0; \qquad v\,|_{t=0} = v_0;$$

$$\frac{\partial \theta}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial \theta}{\partial x_i} - \chi \triangle \theta = \mu(\theta)\mathcal{E}(v) : \mathcal{E}(v) + g;$$

$$\theta\,|_{t=0} = \theta_0; \qquad \theta\,|_{[0,T]\times\partial\Omega} = 0.$$

- σ — deviator of the stress tensor;
- τ^* — yield–stress of fluid;

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Functional spaces

Let $\Omega = \prod_{i=1}^{n}(0, l_i) \subset \mathbb{R}^n$, $n = 2, 3$. By C^∞_{per} we denote the space of periodic vector functions with values in $\mathbb{R}^n$ and periods l_i. We introduce the set

$$\Phi = \Big\{\phi \in C^\infty_{per} : \int_\Omega \phi\, dx = 0,\ \operatorname{div}\phi = 0\Big\}.$$

$$H = \text{ is the closure } \mathcal{V} \text{ with respect to the norm of space } L_2(\Omega);$$
$$V = \text{ is the closure } \mathcal{V} \text{ with respect to the norm of space } W_2^1(\Omega);$$

$$W_1 = \{v \in L_2(0,T;V) \cap L_\infty(0,T;H), \quad v' \in L_{4/3}(0,T;V^*)\};$$
$$W_2 = \{\theta : \theta \in L_p(0,T;W_p^1(\Omega)),\ \theta' \in L_1(0,T;W_p^{-1}(\Omega))\}.$$

Weak solution

Definition

A weak solution of the Bingham problem is a triple $(v;\sigma;\theta)$, where

$$v \in W_1 = \{v \in L_2(0,T;V) \cap L_\infty(0,T;H), \quad v' \in L_{4/3}(0,T;V^*)\},$$

$$\sigma \in L_2(0,T;L_2(\Omega)),$$

$$\theta \in W_2 = \{\theta : \theta \in L_p(0,T;W_p^1(\Omega)),\ \theta' \in L_1(0,T;W_p^{-1}(\Omega))\},$$

which for all $\varphi \in V$, $\phi \in C_{per}^\infty$ and a.e. $t \in [0,T]$ satisfying equalities

$$\langle v', \varphi\rangle - \sum_{i,j=1}^{n} \int_\Omega v_i v_j \frac{\partial \varphi_j}{\partial x_i}\, dx + \int_\Omega \sigma : \mathcal{E}(\varphi)\, dx = \int_\Omega f\, \varphi\, dx,$$

$$\frac{d}{dt}\int_\Omega \theta\phi\, dx - \int_\Omega \sum_{i,j=1}^{n} v_i\theta_j \frac{\partial \phi_j}{\partial x_i}\, dx + \chi \int_\Omega \mathcal{E}(\theta) : \mathcal{E}(\phi)\, dx = \int_\Omega \big(\mu(\theta)\mathcal{E}(v) : \mathcal{E}(v)\big) : \phi\, dx + \langle g, \phi\rangle,$$

and also Bingham relation and initial conditions $v|_{t=0} = v_0$ and $\theta|_{t=0} = \theta_0$.

Main Result

Main Theorem

Let $f \in L_2(0, T; V^*)$, $g \in L_1(0, T; H_p^{-2(1-1/p)}(\Omega))$, $v_0 \in V$, $\theta_0 \in W_p^{1-2/p}(\Omega)$, and the viscosity coefficient $\mu \in C^2(\mathbb{R})$ and $0 < \mu \leq C$. Then in the case $n = 2$ for $1 < p < 4/3$ and in the case $n = 3$ for $1 < p < 5/4$ a weak solution to the Bingham problem exists.

Proof Sketch of Main Theorem

• First Step

We consider the problem with the fixed $\theta \in W_2$:

$$\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \operatorname{Div} \sigma + \nabla p = f; \tag{7}$$

$$\sigma = \begin{cases} 2\mu(\theta)\mathcal{E}(v) + \tau^* \frac{\mathcal{E}(v)}{|\mathcal{E}(v)|}, & \text{если } |\mathcal{E}(v)| \neq 0; \\ |\sigma| \leq \tau^*, & \text{если } |\mathcal{E}(v)| = 0; \end{cases} \tag{8}$$

$$\operatorname{div} \, v = 0; \qquad v \,|_{t=0} = v_0. \tag{9}$$

Theorem 5.

Let $f \in L_2(0, T; V^*)$, $v_0 \in V$ and the viscosity coefficient $\mu \in C^2(\mathbb{R})$ and $0 < \mu \leq C$. Then problem (7)–(9) has at least one weak solution $v \in W_1$, for which the following estimate hold

$$\|v\|_{W_1} \leqslant R_1, \qquad R_1 = R_1(T, \|f\|_{L_2(0,T;V^*)}, \|v_0\|_V).$$

Proof Sketch of Theorem 5

To prove this theorem, the rheological relation of the Bingham model is approximated by the following non-Newtonian relation:

$$\sigma = 2\mu\mathcal{E}(v) + \tau^* \frac{\mathcal{E}(v)}{\max\{\delta, |\mathcal{E}(v)|\}}, \qquad \delta > 0.$$

With this approximation of the rheological relation, the unknown σ can be eliminated from the problem formulation, and the problem can be considered solely for determining the velocity v. In this case, the integral equality (18) is also approximated using the term:

$$\delta \int_\Omega A^2 v \, A\varphi \, dx.$$

Based on the theory of the Leray–Schauder topological degree, the existence of a weak solution to the approximating problem is proved, and then a limit passage to the solution of the original problem is performed, which allows us to obtain the result of Theorem 5.

• Second Step

We consider the problem with the fixed $v \in W_1$ and $\tilde{\theta} \in W_2$:

$$\frac{\partial \theta}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial \theta}{\partial x_i} - \chi \triangle \theta = \mu(\tilde{\theta}) \mathcal{E}(v) : \mathcal{E}(v) + g; \tag{10}$$

$$\theta \mid_{t=0} = \theta_0. \tag{11}$$

Theorem 6.

Let $g \in L_1(0, T; H_p^{-2(1-1/p)}(\Omega))$, $\theta_0 \in W_p^{1-2/p}(\Omega)$, $v \in W_1$ and the viscosity coefficient $\mu \in C^2(\mathbb{R})$ and $0 < \mu \leq C$. Then in the case $n = 2$ for $1 < p < 4/3$ and in the case $n = 3$ for $1 < p < 5/4$ the initial–boundary value problem (4)–(5) has at least one weak solution for which the following estimate hold

$$\|\theta\|_{W_2} \leq R_2 \Big(\|g\|_{L_1(0,T;H_p^{-2(1-1/p)}(\Omega))} + \| \frac{\partial v}{\partial t} \|^2_{L_2(0,T;H)} + \|\theta_0\|_{W_p^{1-2/p}(\Omega)} \Big).$$

• Last Step

Consider the sequence (v^m, θ^m), $m = 0, 1, 2, \dots$, defined as follows. Let v^0 and θ^0 denote the initial values v_0 and θ_0 for v and θ from (9) and (11), respectively. Suppose (v^m, θ^m) is known. Then, first, the pair (v^{m+1}, σ^{m+1}) is found as a weak solution to problem (7)–(9); subsequently, with v^{m+1} determined, θ^{m+1} is found as a weak solution to problem (10)–(11).

From the strong convergence of v^m in $L_2(0, T; V^1)$ and θ^m in $L_p(0, T; L_p(\Omega))$, together with the weak convergence of v^m to v in $L_2(0, T; V^1)$, σ^m to σ in $L_2(0, T; L_2(\Omega))$, and θ^m to θ in $L_p(0, T; W_p^1(\Omega))$, where p satisfies the conditions of Theorem 6, it follows that one can pass to the limit in all terms of the integral equalities for v^m, σ^m, and θ^m. This completes the proof of Main Theorem.

Thank you for your attention!