Volume of hyperbolic octahedron with $\overline{3}$-symmetry

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Introduction

Calculating volumes of polyhedra is a classical problem, that has been well known since Euclid and remains relevant nowadays. This is partly due to the fact that the volume of a fundamental polyhedron is one of the main geometrical invariants for a 3-dimensional manifold.

Every 3-manifold can be presented by a fundamental polyhedron. That means we can pair-wise identify the faces of some polyhedron to obtain a 3-manifold. Thus the volume of 3-manifold is the volume of prescribed fundamental polyhedron.

It is known that regular hyperbolic octahedron with all vertices at infinity is a fundamental polyhedron for Whitehead link manifold. On the other hand, the minimal volume hyperbolic manifold and many others can be obtained by Dehn surgery along Whitehead link. Thus, hyperbolic octahedra can serve as fundamental polyhedra for a wide class of 3-manifolds, including hyperbolic manifolds of small volume. The latter seems especially interesting if we arrange hyperbolic manifolds in order of volume increasing.

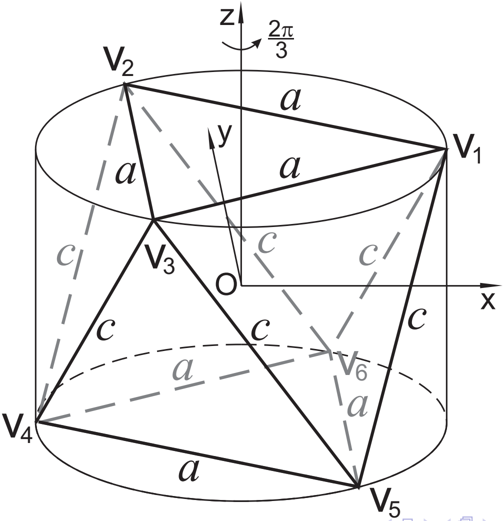
It is difficult problem to find the exact volume formulas for hyperbolic polyhedra of prescribed combinatorial type. It was done for hyperbolic tetrahedron of general type, but for general hyperbolic octahedron it is an open problem.

Nevertheless, if we know that a polyhedron has a symmetry, then the volume calculation is essentially simplified. Firstly this effect was shown by Lobachevskij. He found the volume of an ideal tetrahedron, which is symmetric by definition.

R.V. Galiulin, S.N. Mikhalev and I.Kh. Sabitov found the volumes of Euclidean octahedra with all possible types of symmetry, except the trivial one. The volumes of spherical octahedron with mmm or $2|m$-symmetry were given by N. Abrosimov, M. Godoy and A. Mednykh. The volume of hyperbolic octahedron with mmm-symmetry was obtained by N. Abrosimov and G. Baigonakova.

Definition

An octahedron has $\overline{3}$-symmetry if it admits the antipodal involution and order 3 rotation.



Criterion of existence and relations between angles and lengths

Proposition

A hyperbolic octahedron $\mathcal{O} = \mathcal{O}(a, c)$, admitting $\overline{3}$-symmetry, with edge lengths a, c is exist if and only if

$$3\,\mathrm{ch}\,c - \mathrm{ch}\,a - 2 > 0\,.$$

Proposition

$$\begin{aligned} \cos A &= \frac{(\mathrm{ch}\,c - \mathrm{ch}\,a - 1)\sqrt{\mathrm{ch}\,a - 1}}{\sqrt{(1 + 2\,\mathrm{ch}\,a)(2\,\mathrm{ch}^2\,c - \mathrm{ch}\,a - 1)}}, \\ \cos C &= \frac{1 - \mathrm{ch}\,c + \mathrm{ch}\,a\;\mathrm{ch}\,c - \mathrm{ch}^2\,c}{2\,\mathrm{ch}^2\,c - \mathrm{ch}\,a - 1}. \end{aligned} \tag{1}$$

We differentiate the volume as a composite function

$$\begin{aligned} \frac{\partial V}{\partial a} &= \frac{\partial V}{\partial A}\frac{\partial A}{\partial a} + \frac{\partial V}{\partial C}\frac{\partial C}{\partial a}, \\ \frac{\partial V}{\partial c} &= \frac{\partial V}{\partial A}\frac{\partial A}{\partial c} + \frac{\partial V}{\partial C}\frac{\partial C}{\partial c}. \end{aligned} \tag{2}$$

By Schläfli formula we have

$$dV = -\sum_{\theta} \frac{\ell_\theta}{2}\, d\theta = -3\, a\, dA - 3\, c\, dC,$$

hence $\dfrac{\partial V}{\partial A} = -3\, a, \quad \dfrac{\partial V}{\partial C} = -3\, c.$

Taking into account that $\dfrac{\partial A}{\partial a} = \dfrac{\partial A}{\partial \operatorname{ch} a} \operatorname{sh} a$, $\dfrac{\partial A}{\partial c} = \dfrac{\partial A}{\partial \operatorname{ch} c} \operatorname{sh} c$ and analogous equations for angle C, we use relations (1). We obtain

$$\frac{\partial A}{\partial a} = -\frac{F}{(1+2\operatorname{ch} a)\Delta}, \qquad \frac{\partial A}{\partial c} = -\frac{G}{\Delta},$$
$$\frac{\partial C}{\partial a} = -\frac{G}{\Delta}, \qquad \frac{\partial C}{\partial c} = -\frac{H}{\Delta},$$

where

$$\begin{aligned} F &= 1 + 2\operatorname{ch} a + 2\operatorname{ch}^2 a + \operatorname{ch}^3 a - 2\operatorname{ch} c - 2\operatorname{ch} a \operatorname{ch} c - \operatorname{ch}^2 c \\ &\quad - \operatorname{ch} a \operatorname{ch}^2 c + \operatorname{ch}^2 a \operatorname{ch} c - 4\operatorname{ch}^2 a \operatorname{ch}^2 c + 3\operatorname{ch}^3 c\,, \\ G &= \operatorname{sh} a \operatorname{sh} c\,(-1 + 2\operatorname{ch} a), \\ H &= (1 - \operatorname{ch} a)(1 + \operatorname{ch} a + 2\operatorname{ch} c\,(-1 + \operatorname{ch} c)), \\ \Delta &= (\operatorname{ch} 2c - \operatorname{ch} a)\sqrt{(-2 - \operatorname{ch} a + 3\operatorname{ch} c)(\operatorname{ch} a + \operatorname{ch} c)}. \end{aligned}$$

Remark

In the domain of existence of the octahedron $\mathcal{O}$ with $\overline{3}$-symmetry

$$\Omega = \{(\operatorname{ch} a, \operatorname{ch} c) : \operatorname{ch} a > 1,\ \operatorname{ch} c > 1,\ 3 \operatorname{ch} c - \operatorname{ch} a - 2 > 0\}$$

the quantities G and Δ are positive, H is negative, and F changes the sign.

We substitute all the expressions into formulas (2). Finally we get

$$\begin{aligned} f(a,c) &:= \frac{\partial V}{\partial a} = \frac{3\,(a\,F + (1 + 2 \operatorname{ch} a)\, c\, G)}{(1 + 2 \operatorname{ch} a)\Delta}, \\ g(a,c) &:= \frac{\partial V}{\partial c} = \frac{3\,(a\,G + c\,H)}{\Delta}. \end{aligned} \tag{3}$$

The boundary of Ω consists of two rays $\{\operatorname{ch} a = 1, \operatorname{ch} c \geq 1\}$ and $\{\operatorname{ch} a \geq 1, 3\operatorname{ch} c = \operatorname{ch} a + 2\}$. The octahedron degenerate on each of these rays into the line segment or planar hexagon. Hence, $V = 0$ on the boundary of Ω.

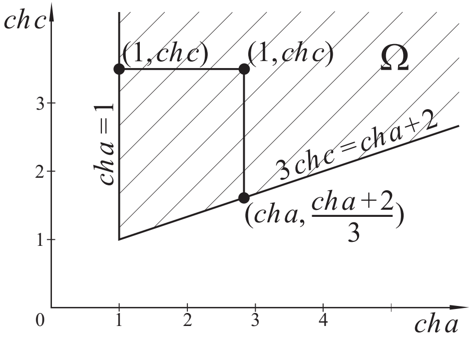


Рис.: Domain of existence Ω of octahedron $\mathcal{O} = \mathcal{O}(a, c)$

Now we integrate the differential form $dV = f(a, c)\,da + g(a, c)\,dc$ along horizontal or vertical segment from the boundary of Ω to the point $(\operatorname{ch} a, \operatorname{ch} c)$. Thus we obtain the main theorem.

Theorem

The volume of hyperbolic octahedron $\mathcal{O} = \mathcal{O}(a, c)$ with $\overline{3}$-symmetry is given by each of the following two formulas

$$\text{(i)} \qquad V = \int_0^a f(a, c)\, da\,,$$

$$\text{(ii)} \qquad V = \int_{arch(\frac{\operatorname{ch} a+2}{3})}^c g(a, c)\, dc\,,$$

where

$$f(a, c) = \frac{3\,(a\,F + (1 + 2\operatorname{ch} a)\,c\,G)}{(1 + 2\operatorname{ch} a)\Delta}, \quad g(a, c) = \frac{3\,(a\,G + c\,H)}{\Delta},$$

$$F = 1 + 2\operatorname{ch} a + 2\operatorname{ch}^2 a + \operatorname{ch}^3 a - 2\operatorname{ch} c - 2\operatorname{ch} a\operatorname{ch} c - \operatorname{ch}^2 c$$
$$- \operatorname{ch} a\operatorname{ch}^2 c + \operatorname{ch}^2 a\operatorname{ch} c - 4\operatorname{ch}^2 a\operatorname{ch}^2 c + 3\operatorname{ch}^3 c\,,$$

$$G = \operatorname{sh} a\operatorname{sh} c\,(-1 + 2\operatorname{ch} a),$$

$$H = (1 - \operatorname{ch} a)(1 + \operatorname{ch} a + 2\operatorname{ch} c\,(-1 + \operatorname{ch} c)),$$

$$\Delta = (\operatorname{ch} 2c - \operatorname{ch} a)\sqrt{(-2 - \operatorname{ch} a + 3\operatorname{ch} c)(\operatorname{ch} a + \operatorname{ch} c)}.$$

Thank you for attention!