On the number of cycles of small length in Star graph

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The Star graph $S_n = Cay(Sym_n, ST), n \geqslant 2$ is a Cayley graph on the symmetric group Sym_n with the generating set of transpositions $ST = \{t_i \in Sym_n, 1 < i \leqslant n\}$ exchanging i'th element of the permutation with the first one. Graph S_n, $n \geqslant 3$, is bipartite, therefore contains only even cycles of lengths C_l, where $6 \leqslant l \leqslant n!$ [1] and has the diameter $D = \lfloor \frac{3(n-1)}{2} \rfloor$.

This graph has a well described distance structure. The number and the structure of vertices on every distance layer d, where $1 \leqslant d \leqslant D$, from the identity vertex is known [2]. In current work we use this approach to study the number of cycles of lengths $2d$, $3 \leqslant d \leqslant D$, constructed from two shortest paths to the vertex at distance d from the identity vertex. The study of such cycles is closely related to the method proposed to solve the First Passage Percolation problem on graphs [3,4].

Any permutation $\pi \in Sym_n$ can be represented uniquely in terms of non-intersecting cycles, i.e.

$$\pi = (1\, \pi_2^1 \dots \pi_{l_1}^1)(\pi_1^2 \dots \pi_{l_2}^2) \dots (\pi_1^k \dots \pi_{l_k}^k).$$

Denote the cycle of length r containing the element "1" as $r - CO$ and not containing it as $r - CN$. It is easy to show the vertices on the distance layer d may have either

1. only a $(d+1) - CO$;
2. a $k - CO$, $1 \leqslant k \leqslant d-2$ and $t \geqslant 1$ items of $l_i - CN$, where $1 \leqslant i \leqslant t$, such that $d = t + (k-1) + \sum_{i=1}^{t} l_i$.

The first kind of vertices have a unique shortest path to the identity vertex. For the second kind the number of results has been obtained for distance $3 \leqslant d \leqslant D$.

Theorem 1 *The number of cycles of length $2d$ passing through the vertices with only $(d-1) - CN$ is*

$$N_{C_1} = \frac{d-2}{2}(n-1)(n-2)\dots(n-d+1).$$

Theorem 2 *The number of cycles of length $2d$ passing through the vertices with $k - CO$ and one $l - CN$, over all $k + l = d$, is*

$$N_{C_2} = \frac{d(d-3)}{2}(n-1)(n-2)\dots(n-d+1).$$

Theorem 3 *The number of cycles of length $2d$ passing through the vertices with $k - CO$ and $t \geqslant 2$ items of $l_i - CN$ is at least*

$$N_{C_3} \geqslant \left(((t-1)!)^2 \prod_{i=1}^{t} l_i \right)^{\min(2,k)} (n-1)(n-2)\dots(n-d+t).$$

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