Schur rings and Cayley schemes of finite non-abelian groups

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(joint work with Ilya Ponomarenko)

Let G be a finite group. There exists the well-known one-to-one correspondence between S-rings (Schur rings) $\mathcal{A}(G, \mathcal{S})$ and Cayley schemes $\mathcal{X}(G, \mathcal{R})$ over G. It is based on the bijection between partitions $\mathcal{S}$ of G (such that $\mathcal{S}$ generates S-ring over G) and partitions $\mathcal{R}$ of $G \times G$ (such that $\mathcal{R}$ gives Cayley scheme over G).

If Γ is a permutation group with $G_{right} \leq \Gamma \leq \mathrm{Sym}(G)$ and $\mathcal{S}$ is the set of orbits of the stabilizer of the identity $e = e_G$ in Γ, then $\mathbb{Z}$-submodule $\mathcal{A} = \mathcal{A}(\Gamma, G) = \mathrm{Span}\{\underline{X} : \ X \in \mathcal{S}\}$ of the group ring $\mathbb{Z}G$ is an S-ring as it was observed by Schur. Following Pöschel an S-ring $\mathcal{A}$ over G is said to be *schurian* if there exists a suitable permutation group Γ such that $\mathcal{A} = \mathcal{A}(\Gamma, G)$. The schurity of Cayley schemes can be defined similarly. Moreover, it is very helpful to our purposes to treat the following criterion of schurity of Cayley schemes (and, subsequently, of corresponding S-rings). The automorphism group of a scheme $\mathcal{X} = \mathcal{X}(G, \mathcal{R})$ is $\mathrm{Aut}(\mathcal{X}) = \{\gamma \in \mathrm{Sym}(G) \mid R^\gamma = R, R \in \mathcal{R}\}$. A scheme $\mathcal{X}$ is schurian iff it coincides with the corresponding Cayley scheme over $\mathrm{Aut}(\mathcal{X})$.

A finite group G is called a *Schur group* if every S-ring over G is schurian. We prove that every non-abelian Schur group G is metabelian and the number of distinct prime divisors of the order of G is at most 7.