Linear perfect codes in Doob graphs

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A connected regular graph is called *distance regular* if every bipartite subgraph generated by two cocentered spheres of different radius is biregular. A set of vertices of a graph or any other discrete metric space is called an *e-perfect code*, or simply a *perfect code*, if the vertex set is partitioned into the radius-e balls centered in the code vertices. The perfect codes in distance regular graphs are objects that are highly interesting from the point of view of both coding theory and algebraic combinatorics. On one hand, these codes are error correcting codes that attain the sphere-packing bound ("perfect" means "extremely good"). On the other hand, they possess algebraic properties that are connected with the algebraic properties of the distance regular graph; a perfect code is a some kind of divisor of the graph. It may safely be said that the most important class of distance regular graphs, for coding theory, is the Hamming graphs $H(n,q)$. The Doob graph $D(m,n)$ is a distance regular graph of diameter $2m+n$ with the same parameters as the Hamming graph $H(2m+n,4)$. In [1], Koolen and Munemasa constructed 1-perfect codes in the Doob graphs of diameter 5.

We study the existence of linear, over the rings $\mathrm{GR}(4^2)$ and $\mathbb{Z}_4$, 1-perfect codes in Doob graphs. We define the Doob graphs as a Cayley graph of the additive group of a module over the ring $\mathrm{GR}(4^2)$ or $\mathbb{Z}_4 = \mathrm{GR}(4)$. A submodule of this module is called a linear (over $\mathrm{GR}(4^2)$) or additive (over $\mathbb{Z}_4$) code. We construct linear and additive codes in Doob graphs and derive restrictions on the parameters of a Doob graph that can contain an additive 1-perfect code, in terms of parameters Γ, Δ of the factorgroup $\mathbb{Z}_2^\Gamma \times \mathbb{Z}_4^\Delta$ of cosets of the code.

Theorem 1. *Assume that there is an additive* 1*-perfect code in the Doob graph* $D(m,n'+n'')$ *with the structure of the module* $\mathbb{Z}_4^{2m} \times \mathbb{Z}_2^{2n'} \times \mathbb{Z}_4^{n''}$. *Then* $n'' \neq 1$ *and for some even* $\Gamma \geq 0$ *and integer* $\Delta \geq 2$: *(1)* $2m+n'+n'' = (2^{\Gamma+2\Delta}-1)/3$, *(2)* $3n'+n'' = 2^{\Gamma+\Delta}-1$, *(3)* $n'' \leq 2^\Delta - 1$.

Theorem 2. *For every* m, n', *and* n'' *satisfying the statement of Theorem 1 with even* Δ, *there is a* 1*-perfect code in the Doob graph* $D(m,n'+n'')$ *with the structure* $\mathbb{Z}_4^{2m} \times \mathbb{Z}_2^{2n'} \times \mathbb{Z}_4^{2n''}$.

The case of odd Δ remains unsolved with one exception, for which a code is constructed, $\Delta = 3$, $\Gamma = n' = 0$, $m = n'' = 7$.

Theorem 3. *Linear* 1*-perfect codes in the Doob graph* $D(m,n)$ *with the structure* $(\mathrm{GR}(4^2))^m \times (\mathrm{GF}(2^2))^n$ *exist if and only if for some integers* $\gamma \geq 0$ *and* $\delta > 0$, $n = (4^{\gamma+\delta}-1)/3$ *and* $m = (4^{\gamma+2\delta} - 4^{\gamma+\delta})/6$.

[1] J. H. Koolen and A. Munemasa. Tight 2-designs and perfect 1-codes in Doob graphs. *J. Stat. Plann. Inference*, 86(2):505–513, 2000.