Combinatorics of Space-Fillers

Graphs and Groups,
Cycles and Coverings
Novosibirsk

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September 24, 2014

Tilings: Definition

- A space filler is a (convex) polytope that admits a tiling of space with congruent copies.
- $P \subset R^d$ is a convex, bounded d-dimensional polytope.
- A collection of polytopes

$$T = \{P_1, P_2, \ldots, P_i, \ldots\}$$

is called a tiling if they are located in space $\mathbb{R}^d$ so that
(1) Any P_i and P_j do not overlap (packing condition):

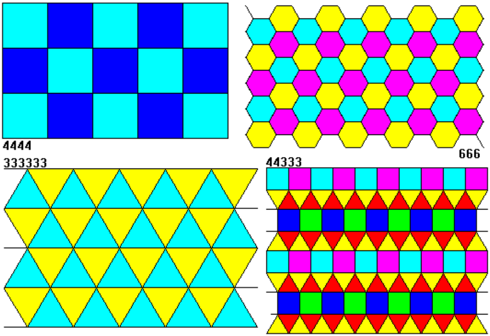
$$Int(B_i) \cap Int(P_j) \neq \emptyset$$

(2) The collection T covers space $\mathbb{R}^d$ (covering condition):

$$\cup_i P_i = \mathbb{R}^d.$$

- Polytopes of a tiling are called tiles (or cells).

Tilings: Examples

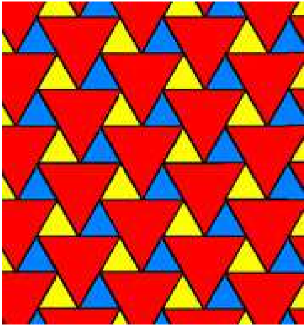


Tilings: restrictions

- Locally finite tilings:
  every ball intersects finite number of tiles.
- Homogenous tilings:
  For a tiling T there are positive r and R s.t. any cell of T contains a ball with radius r and is contained in a ball with radius R
- Face-to-face property = polyhedral complex:
  Each two cells
  either do not intersect at all
  or they have the whole face in common

Tilings: Restrictions II

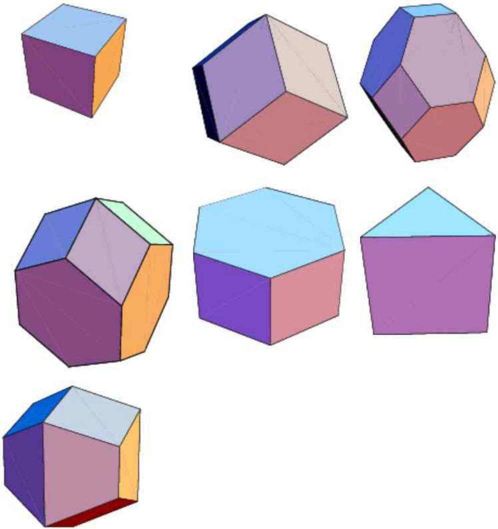
Face-to-face feature



Space Fillers

- A (convex) polytope P is called space-filler if it admits a tiling with congruent copies.
- A tiling by a space-filler is called a *monohedral* tiling.
- A monohedral tiling is
  locally finite;
  homogenuous.
- We will consider just face-to-face monohedral tilings.

Space-fillers: examples

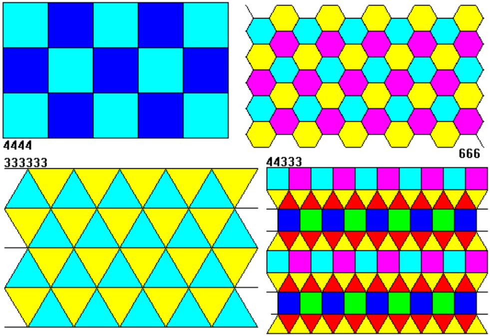


Space-fillers: results and problems

- $d = 2$ Full Clsssification is completed
  any triangle, any quadrangle,
  several classes of hexagons,
  several classes of pentagons (the most difficult and just recently solved part, Bagina, Sugimoto, independently).
- No general theory of space-fillers exists
- Killer-problem: Let $f(P)$ be the number of faces in a 3-space-filler.
  Is there some upper bound $c < \infty$ for $f(P)$: $f(P) < c$?
- In a class of not strictly convex space-fillers no upper bound c for all P.
- There is a space-filler with 38 faces (P.Engel).

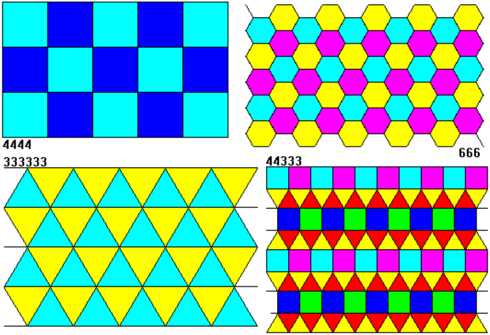
Tilings with transitive groups

- A tiling T is called isohedral if its symmetry group Sym(T) operates transitively on the set of tiles.
- Three first tilings of four are isohedral.



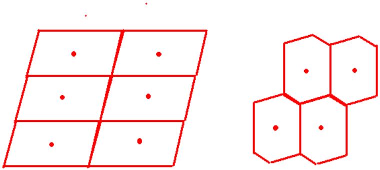
Transitive group of translations: Parallelohedra

- Particular case: a transitive group consists of translations
- Space-filler which tiles space by means translations is called a parallelohedron (1885, Fedorov).
- The theory of parallelohedra has been developed by Minkowski, Voronoi, Delone, Alexandrov, Venkov, Ryshkov, P. McMullen etc.



Parallelohedra

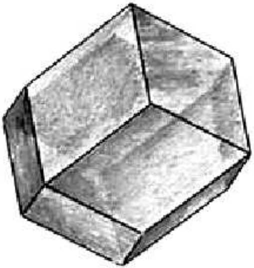
- $d = 2$



- In plane there are 2 combinatorial types of parallelohedra:
  (1) parallelogram and
  (2) c.-s. hexagon

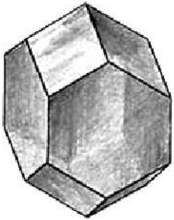
3D-parallelohedra = the Fedorov solids

- In 3-space there are 5 combinatorial types: Parallelepiped, c.-s. hexagonal prism, rhombic dodecahedron, elongated dodecahedron, truncated octahedron (Fedorov, 1889)
- Rhombic dodecahedron:

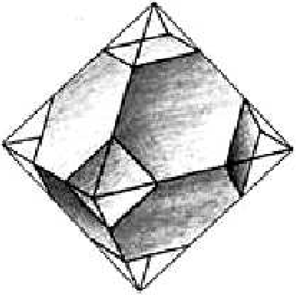


3D-parallelohedra = the Fedorov solids

- Elongated dodecahedron:



- Primitive parallelohedron: Truncated octahedron = 3D-permutahedron:

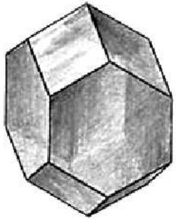


Parallelohedra: How many combinatorial types?

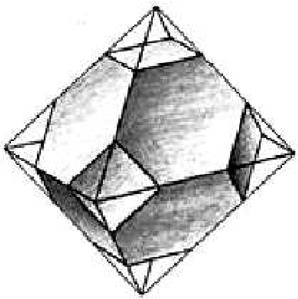
- a d-parallelohedron is called *primitive* if in each vertex of tiling exactly $d+1$ parallelohedra meet
  Otherwise parallelohedron is *non-primitive*
- A d-permutahedron is a primitive parallelohedron
  A cube is non-primitive
- $d=2$: 1 primitive and 1 non-primitive
- $d=3$: 1 primitive 4 non-primitive (Fedorov, 1885)
- $d=4$: 3 primitive and 49 non-primitive (Delone)
  1 missed by Delone found by Stogrin (1969)
- $d=5$: 222 primitive (Ryshkov and Baranovski 1972, Grishukhin and Engel)
  10^5 non-primitive (Engel, Dutour, Garber)
- $d=6$: 10^8 combinatorial types (Engel).
- $d>6$: ????

Minkowski's theorem on parallelohedra

- Minkowski's theorem (1897) :
- Let P be a parallelohedron, then:
  (1) P is centrosymmetrical
  (2) all facets of P are centrosymmetrical
  (3) all belts consist of 4 or 6 facets.
- Remark. (3) $\Leftrightarrow$ projection of P along (d-2)-face (ridge) is either parallelogram or centrally symmetrical hexagon



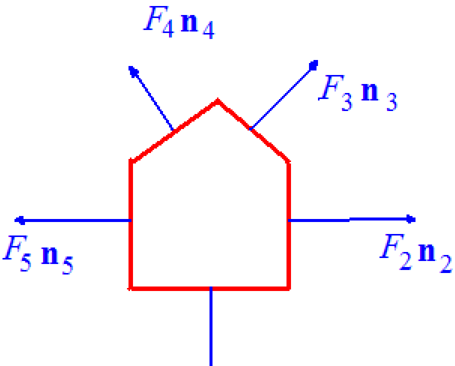
Minkowski's theorem on parallelohedra II



- In Minkowski's thm the basic point is to prove:
  (1) a parallelohedron is centrosymmetrical.
- (1) easily implies (2) (symmetricity of hyperfaces) and
  (3) all belts are either 4- or 6-gonal.
- In order to prove p.(1) Minkowski has discovered and proved
  a theorem on convex polyhedra.

Hedgehog of a polyhedron

- Let P be compact (= closed bounded) convex polyhedron in $\mathbb{E}^d$ with k $(d-1)$-faces (=facets)
- $\mathbf{n}_1, \ldots, \mathbf{n}_k$ are unit outward normals to facets;
- $F_1, \ldots, F_k$ are "areas" ($(d-1)$-volumes) of facets.
- Set of vectors $F_1\mathbf{n}_1, \ldots, F_k\mathbf{n}_k$ is called hedgehog of P.



Minkowski's theorem on convex polyhedra

- A hedgehog of a convex bounded polyhedron fulfils two conditions:
  (1) it does not lie in a hyperplane
  (2) $\sum_{j=1}^{k} F_j \mathbf{n}_j = 0$
- The Minkowski theorem on polyhedra (1897) says that the inverse is also true:

Theorem

Given set $= \{\mathbf{F}_1 \ldots, \mathbf{F}_k\}$, of vectors
let $\mathcal{F}$ fulfil the following properties:
(1) The set $\mathcal{F}$ does not lie in one plane
(2) $\sum \mathbf{F}_i = 0$.
Then there exists a convex polyhedron P such that
the set $\mathcal{F}$ is its hedgehog
the P is determined uniquely up to translation.

Some corollaries of the Minkowski Thm on polyhedra

- If hedgehog $\mathcal{F}(P)$ of a polyhedron P is centrosymmetrical then P is centrosymmetrical too.
- Therefore, if each facet in a polyhedron P has a parallel facet of same area, P is centrosymmetrical
- In a parallelohedron each facet has a parallel and congruent facet. Therefore, by the Mink thm on polyhedra, a parallelohedron is centrally symmetrical too.

Venkov's theorem

Theorem (B.A.Venkov, 1954, the inverse thm to Minkowski)

If a convex d-polyhedron P satisfies
(1) is centrally symmetrical,
(2) each facet is centrally symmetrical
(3) any belt is either 4- or 6 -gonal
then P is a parallelohedron

- P. McMullen independently (1981);
- Venkov's Thm follows from the Extension thm (Criterion for stereohedra)(2000 - N.D.; 2003 - N.D., Makarov V.S.)

On the number of facets in a parallelohedron

Theorem (Minkowski, 1897)

The number f_{d-1} of facets in d-parallelohedron does not exceed

$$f_{d-1} \le 2(2^d - 1) \quad (1)$$

The upper bound (1) is non-refinable.

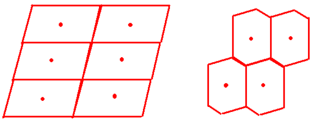
-
- For any dimension d, there are parallelohedra with $f_{d-1} = 2(2^d - 1)$
- for d-parallelohedron $2d \le f_{d-1} \le 2(2^d - 1)$

Standard faces

- Consider face-to-face tiling T by parallelohedra P, P_1,...
- Given face $F \subset P \in T$, *star $St(F)$* is called a set of all $P_i \in T$ s.t. $P_i \supset F$
- Face $F \subset P$ is called *standard* if $St(F) \ni P'$ s.t.

$$F = P \cap P'$$

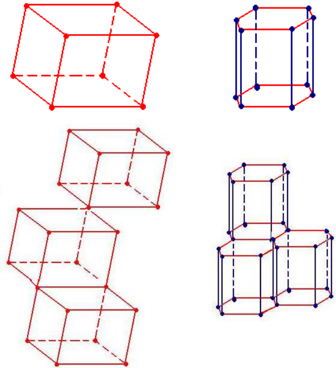
- otherwise a face is *non-standard*



- In parallelogram all sides and vertices are standard faces
- In a hexagon only sides are standard faces, vertices are non-standard.
- In parallelohedron all facets are standard

Standard faces (examples)

- in d-parallelepiped: faces of all dimens. are standard
- in 3D- hexagonal prism:
  2-faces and 12 horizontal edges are standard
  all vertices and vertical edges are non-standard



Face index

- Let face F belong to n parallelohedra, i.e. St F contains n parallelohedra
- Define n as *valency* of F: $n(F) = n$
- $\nu(F) = \frac{1}{n(F)}$ is defined *index* of F
- For facet F^{d-1} it is obvious: $\nu(F^{d-1}) = \frac{1}{2}$

Why are belts short?

- **Claim** Let F be $d - 2$ -face,
  then the valence $n(F)$ can be either 3 or 4;
  $n(F) = 3$ if F is non-standard face and
  $n(F) = 4$ if F is standard one.
  $n(F)$ never can be > 4
- Claim immediately implies quite new proof of that any belt has length 6 (non-standard $d - 2$-face case) or length 4 (standard $d - 2$-face case)

The index theorem

Theorem (On index, N.D.)

$$\sum_{All\ standard\ faces \subset P} \nu(F) = 2^d - 1 \qquad (1)$$

- From (1) Minkowski's upper bound for f_{d-1} follows immediately:
$$f_{d-1} \leq (2^d - 1) \qquad (2)$$
- {All stand.faces}= {Facets}∪ {stand. i-faces, $i < d - 1$}
- (1)$\Rightarrow 2^d - 1 =$ $\sum_{\{\text{Facets}\}} \nu 5(F) + \sum_{\{\text{Stand} F^i, i<d-1\}} \nu(F^i) =$ $\sum_{\{\text{Facets}\}} \frac{1}{2} + \sum_{\{\text{stand} F^i, i<d-1\}} \nu(F^i) = \frac{1}{2} f_{d-1} + \ldots \Rightarrow$
- $f_{d-1} = 2(2^d - 1) - 2 \sum_{\{\text{stand} F^i, i<d-1\}} \nu(F^i) \Rightarrow (2)$
- Equality in (2) occurs iff in tiling no standard faces of dim $< d - 1$

On degree of k-face of a parallelohedron

Theorem (Sasha Magazinov)

Let F^k be a k-face of a parallelohedron, $n(F^k)$ its degree,and $\nu(F^k)$ its index, then

$$n(F^k) \leq 2^{d-k}$$

$$\nu(F^k) \geq \frac{1}{2^{d-k}}.$$

- Value of the theorem is that we got the same upper bound as if we already know that the dimension of lattice $\Lambda := \mathbb{Z}(O_1, \ldots, O_n)$ is equal to $d - k$.

Isohedral tilings

- Let group $G \subset Iso(d)$ operate transitively on tiles of an isohedral tiling. Recall: such tiles are called stereohedra.
- G is crystallographic group (i.e. discrete and has compact fundamental domain).
- Schoenflis-Bieberbach theorem: G has subgroup T of translations of finite index h.
- For any dimension $d \neq 2, 4, 8$ $h \leq 2^d!$ (the order of the d-cube's group)
- Let f_{d-1} be the number of facets of the d-stereohedron, then (Delone and Sandakova)

$$f_{d-1} \leq 2(2^d - 1) + (h - 1)2^d.$$

Stereohedra: d=2 and 3

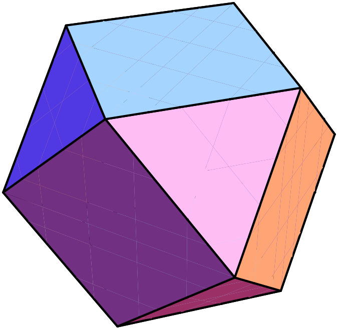
- $d = 2$ $f = f_1 \leq 6$
  Delone-Sandakova upperbound: $h \leq 12$
  $f \leq 2(4-1) + (12-1)4 = 50$
- $d = 3$, $f \geq 38$ (P.Engel's example)
  Delone and Sandakova: $h \leq 48$,
  $f \leq 2(8-1) + (48-1)8 = 390$
  A.Tarasov: $f \leq 378$
- $38 \leq \max f \leq 378$.

Combinatorial tile and Danzer's problem

- Assume in a tiling T (locally finite, homogenous, and face-to-face) all cells are convex polytopes of the same combinatorial type,
  then the tiling is called *monotypic*
- A combinatorial type of polytopes in a monotypic tiling is called a combinatorial tile.
- Ludvig Danzer (1975): Given 3-polytope P, is there a monotypic tiling with tiles isomorphic to P?
- In other words, prove or disprove that any 3-polytope is a combinatorial tile?

Combinatorial Non-tiles

- Combinatorial non-tile is a type s.t. no monotypic tilings with the given type.
- There has been established an infinite series of combinatorial non-tiles
- An example of combinatorial non-tile is the truncated cube



Why the truncated cube is a non-tile?

- Prove that the TC is a combinatorial tile
- Assume the contrary: the TC is a combinatorial tile.
- Given a tiling by TCs, a vertex V, consider all tiles sharing the V.
- The intersection of the tiles of the tiling at vertex V with a "small" sphere is a complex C on the sphere.
- all 2-faces of the C are quadrangles.
- Since in the TC each edge is shared by a triangular and quadrangular faces, at each edge always even number of tiles meet.
- Therefore, a vertex of C on the sphere is of even degree $\deg V \geq 4$.
- Euler's equation $v - e + f = 2$ implies the following inequality:

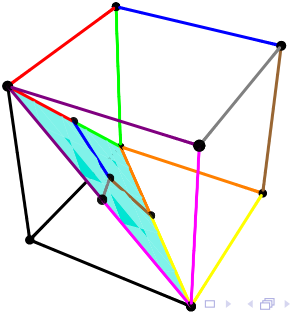
$$\frac{4f}{4} - \frac{4f}{2} + f \geq 2 \qquad (1)$$

Infinite Series of Non-tiles

- By means of this idea one can construct many types of combinatorial non-tiles
- Start with a simple 3-polyope Q without simplicial faces.
- Steinitz's surgery $I(Q)$: is the cutting of all vertices by planes through midpoints of edges meeting at a vertex
- A 3-polytope $P = I(Q)$ is non-tile.
- In particular, let Q_n be a prism over n-gon, $n \geq 4$. Then $P_n = I(Q_n)$ is a non-tile.
- At the same time we are to show that a prism Q_n is a tile in Euclidean space for any $n \leq 3$.

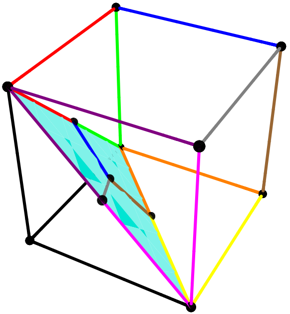
Combinatorial Tiles: Construction

- There is a good connection between monotypic tilings of d-space and equifaceted convex $(d+1)$-polytopes (E.Schulte)
- Let Q be a convex $(d+1)$-polytope with f_d isomorphic facets (d-faces) and with at least one *simplicial* vertex V
- Take a d-simplex S with $d+1$ vertices as the end-points of $d+1$ edges meeting at the V.



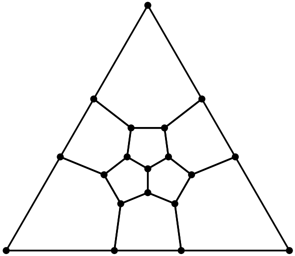
Combinatorial Tiles: Construction II

- The central projection from the V of the boundary $\partial(P)$ of Q into the P induces a face-to face tessellation of S by $f_d - (d+1)$ convex d-polytopes isomorphic to a facet of Q.



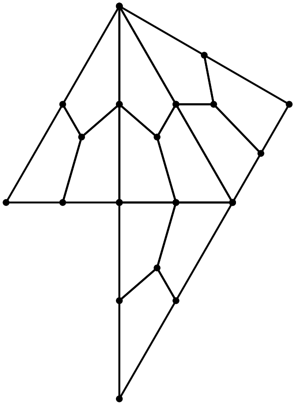
Combinatorial Tiles: Construction III

- In case of the dodecahedron we get a tiling of the simplex S as follows



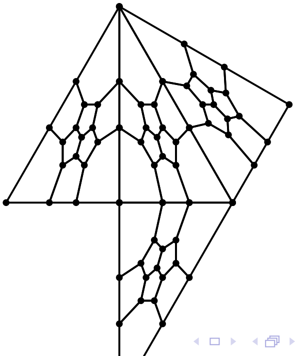
Combinatorial Tiles: Construction IV

- Next step is to affine transform the simplex S with its content into some Coxeter simplex S_C (with angles π/k)
- Move S_C by reflections in facets of S_C and pave the whole space by images of S_C with its content



Combinatorial Tiles: Construction VI

- This construction induces a so-called crystallographic tiling by isomorphic polytopes.
- The fundamental domain of the group is the simplex S_C which is paved by exactly by $f_d - d$ non-equivalent but isomorphic d-polytopes, where f_d is the number of facets in $(d+1)$-polytope Q.



Number of faces in combinatorial tile

- In $d = 3$ case there are combinatorial tiles with as large number of faces as you like.
- For any positive integer $n \geq 3$ consider
$$Q := C_n \times C_n,$$
where C_n is a regular n-gon.
- Q is a simple 4-polytope whose all facets (3-faces) are regular prisms over n-gons.
- The n-gonal prism for arbitrary n is a combinatorial tile of 3-space.
- For Q we have $f_3 = 2n$
- Coxeter simplex S_C is paved by $2n - 4$ pairwise non-congruent n-gonal prisms.

Open problems

- What is the maximal number of faces for a 3-space-filler, i.e. a convex polyhedron that allows for a tiling of Euclidean 3-space by congruent copies?
- What is the maximal number of faces for a stereohedron, i.e. a cinvex polytope that admits an isohedral face-to-face tiling of 3-space?
  For non-convex or non-strictly convex no upperbound exist.
- Find a criterion for combinatorial tiles.