On groups all of whose undirected Cayley graphs of bounded valency are integral

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Setting

For a group G and subset $S \subseteq G$, $1 \notin S$, the Cayley digraph $\mathrm{Cay}(G, S)$ is the digraph whose vertex set is G and (x, y) is an arc if and only if $yx^{-1} \in S$.

We regard $\mathrm{Cay}(G, S)$ as an undirected graph when $S = S^{-1}$, and use the term Cayley graph.

The spectrum of a matrix is the set of its eigenvalues.

The spectrum of a graph is the spectrum of its adjacency matrix.

Definition

A group G is called Cayley integral if every undirected Cayley graph $\mathrm{Cay}(G, S)$ of G has integral spectrum.

Motivation

Finite abelian Cayley integral groups have already been determined:

Theorem (Klotz, Sander 2010)

If G is an abelian Cayley integral group, then G is isomorphic to one of the following:

$$\mathbb{Z}_2^n,\ \mathbb{Z}_3^n,\ \mathbb{Z}_4^n,\ \mathbb{Z}_2^m \times \mathbb{Z}_3^n,\ \mathbb{Z}_2^m \times \mathbb{Z}_4^n,\ (m \geq 1, n \geq 1)$$

WHAT ARE THE FINITE NON-ABELIAN CAYLEY INTEGRAL GROUPS?

Theorem (Abdollahi and Jazaeri 2014; Ahmady et al. 2014+)

The only finite non-abelian Cayley integral groups are S_3, Dic_{12} and $Q_8 \times E_{2^n}$, where $n \geq 0$.

The main result

HOW TO GENERALIZE CAYLEY INTEGRAL GROUPS FURTHER?

Let us study groups G for which we require $\mathrm{Cay}(G, S)$ to be integral only when $|S|$ is bounded by a constant. Formally, for $k \in \mathbb{N}$, we set

Definition

$$\mathcal{G}_k = \{G : \mathrm{Cay}(G, S) \text{ is integral whenever } |S| \leq k\}.$$

Theorem (E., Kovács, 2014+)

Every class $\mathcal{G}_k$ consists of the Cayley integral groups if $k \geq 6$. Furthermore, $\mathcal{G}_4$ and $\mathcal{G}_5$ are equal, and consist of the following groups:

(1) *the Cayley integral groups,*

(2) *the generalized dicyclic groups $Dic(E_{3^n} \times \mathbb{Z}_6)$, where $n \geq 1$.*

Generalized dicyclic groups

Let A be an abelian group with a unique involution $x \in A$.

Definition

The generalized dicyclic group over A is $Dic(A) = \langle A, y \rangle$, where $y^2 = x$ and $a^y = a^{-1}$ for all $a \in A$.

One can see that $A \lhd Dic(A) = A\langle y \rangle$ and $|Dic(A)| = 2|A|$.

Some important special cases:

- $A = \mathbb{Z}_n$ gives rise to the dicyclic group Dic_{2n}.
- $A = \mathbb{Z}_{2^n}$ gives rise to the generalized quaternion group $Q_{2^{n+1}}$.

In particular if $A = \mathbb{Z}_4 = \langle i \rangle$, then we get $Q_8 = \langle i, j \rangle$, the quaternion group.

Basic properties of $\mathcal{G}_k$

Lemma

The following hold for every $G \in \mathcal{G}_k$ if $k \geq 2$.

(i) *For every $x \in G$, the order of x is in $\{1, 2, 3, 4, 6\}$.*

(ii) *For every subgroup $H \leq G$, $H \in \mathcal{G}_k$.*

(iii) *For every $N \trianglelefteq G$ such that $|N| \mid k$, $G/N \in \mathcal{G}_l$, where $l = k/|N|$.*

Proof:

(i) Take $S = \{g, g^{-1}\}$, where $g \in G$ is not an involution or let S consist of two involtuions. Then components of Cay(G, S) are cycles.

(ii) Is clear.

(iii) Goes by inflating Cayley graphs of G/N using Kronecker product of (adjacency) matrices.

Basic properties of groups in $\mathcal{G}_k$

Lemma

Let $G \in \mathcal{G}_k$, and $N \trianglelefteq G$, N is abelian and $|N|$ is odd. Then $G/N \in \mathcal{G}_k$.

Unlike the Cayley integral groups, the class $\mathcal{G}_k$ is generally not closed under taking homomorphic images:

Consider for example $G = \mathbb{Z}_4 \rtimes \mathbb{Z}_4 = \langle a \rangle \rtimes \langle b \rangle$, where $a^b = a^{-1}$. Although G is in $\mathcal{G}_2$, the factor $G/\langle b^2 \rangle \cong D_8$ is not.

Spectrum of graphs with a semiregular group

Let Γ be a graph, and let $H \leq \operatorname{Aut}\Gamma$ an abelian semiregular group of automorphisms with m orbits on the vertex set. Fix m verices $v_1, \ldots, v_m$, a complete set of representatives of H-orbits.

Definition

The symbol of Γ relative to H and the m-tuple $(v_1, \ldots, v_m)$ is the $m \times m$ array

$$\mathbf{S} = (S_{ij})_{i,j \in \{1,\ldots,m\}}, \text{ where } S_{ij} = \{x \in H \ : \ v_i \sim v_j^x \text{ in } \Gamma\}.$$

Definition

For an irreducible character χ of H let $\chi(\mathbf{S})$ be the $m \times m$ complex matrix defined by

$$(\chi(\mathbf{S}))_{ij} = \begin{cases} \sum_{s \in S_{ij}} \chi(s) & \text{if } S_{ij} \neq \emptyset \\ 0 & \text{otherwise,} \end{cases} \quad i,j \in \{1,\ldots,m\}.$$

Spectrum of graphs with a semiregular group

Theorem (Kovács, Marušič, Malnič, Miklavič, 2014+)

The spectrum of Γ is the union of eigenvalues of $\chi(\mathbf{S})$, where χ runs over the set of all irreducible characters of H.

Using this theorem we have proved:

Lemma

Let $G \in \mathcal{G}_k$, and $N \trianglelefteq G$, N is abelian and $|N|$ is odd. Then $G/N \in \mathcal{G}_k$.

Lemma

The group $Dic(E_{3^n} \times \mathbb{Z}_6)$ is in $\mathcal{G}_5$ for every $n \geq 0$.

Nilpotent groups in $\mathcal{G}_k, \ k \geq 4$

Proposition

Every p-group in $\mathcal{G}_k$ is Cayley integral if $k \geq 4$. Namely, they are one of the following: E_{3^m}, $E_{2^n} \times \mathbb{Z}_4^m$, $Q_8 \times E_{2^n}$, where $m, n \geq 0$.

In order to prove this first we show that the minimal non-abelian subgroup of such a group can only be Q_8.Then we use the following theorem:

Theorem (Janko, 2007)

If G is a 2-group whose minimal nonabelian subgroups are isomorphic to Q_8, then $G \cong Q_{2^m} \times E_{2^n}$, where $m \geq 3, n \geq 0$.

Since every nilpotent group is the direct product of its Sylow subgroups, we have obtained the following corollary:

Corollary

Every nilpotent group in $\mathcal{G}_k$ is Cayley integral if $k \geq 4$.

Minimal non-abelian p-groups in $\mathcal{G}_k, \ k \geq 4$

A finite group G is said to be minimal non-abelian if it is non-abelian, but all proper subgroups of G are abelian.

Theorem (Rédei, 1947)

Let G be a minimal non-abelian p-group. Then G is one of the following:

(i) Q_8;

(ii) $\langle a, b \mid a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}} \rangle$, *where* $m \geq 2$ *(metacyclic);*

(iii) $\langle a, b, c \mid a^{p^m} = b^{p^n} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$, *where* $m + n \geq 3$ *if* $p = 2$ *(non-metacyclic).*

Minimal non-abelian p-groups in $\mathcal{G}_k,\ k \geq 4$

Corollary

The minimal non-abelian groups of exponent at most 4 are the following groups:

(i) Q_8;

(ii) $D_8 = \langle a, b \mid a^4 = b^2 = 1, a^b = a^{-1} \rangle$,
$H_2 = \langle a, b \mid a^4 = b^4 = 1, a^b = a^{-1} \rangle$ *(metacyclic);*

(iii) $H_{16} = \langle a, b, c \mid a^4 = b^2 = c^2 = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$,
$H_{32} = \langle a, b, c \mid a^4 = b^4 = c^2 = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$,
$H_{27} = \langle a, b, c \mid a^3 = b^3 = c^3 = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$
(non-metacyclic).

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Non-niloptent groups in $\mathcal{G}_k, \ k \geq 4$

Proposition

Suppose that $G \in \mathcal{G}_k,\ k \geq 4$, and G is not nilpotent. Then $G \cong S_3$ or $Dic(E_{3^n} \times \mathbb{Z}_6)$ for some $n \geq 0$.

In order to prove this we used the following lemma:

Lemma

Suppose that $G \in \mathcal{G}_k,\ k \geq 4$, and $3 \mid |G|$. Then G has a normal Sylow 3-subgroup.

Proof of the main theorem

Let $G \in \mathcal{G}_k,\ k \geq 4$.

- If G is nilpotent, then G is Cayley integral by

Corollary

Every nilpotent group in $\mathcal{G}_k$ is Cayley integral if $k \geq 4$.

- If G is not nilpotent, then we apply an earlier

Proposition

Suppose that $G \in \mathcal{G}_k,\ k \geq 4$, and G is not nilpotent. Then $G \cong S_3$ or $Dic(E_{3^n} \times \mathbb{Z}_6)$ for some $n \geq 0$.

As seen earlier, these groups are in $\mathcal{G}_5$. However, they are not in $\mathcal{G}_k,\ k \geq 6$, except for S_3 and $Dic(\mathbb{Z}_6) = Dic_{12}$.

What about $\mathcal{G}_3$?

This class of groups may even be too wide for a "nice" characterization, since

- For example, all 3-groups of exponent 3 are in $\mathcal{G}_3$.
- For 2-groups in $\mathcal{G}_3$ we have proved the following proposition:

Proposition

Let G be a non-abelian 2-group of exponent 4. Then $G \in \mathcal{G}_3$ if and only if every minimal non-abelian subgroup of G is isomorphic to Q_8, H_2 or H_{32}.

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