On the Deza graphs with disconnected second neighbourhood

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All graphs in this paper will be undirected, without loops and multiple edges. We will denote the adjacency of vertices u and v by $u \sim v$. The distance between vertices u and v, denoted by $d(u, v)$, is the length of the shortest path from u to v, or ∞ if no such path exists. The *neighborhood* $\Gamma_1(v) \equiv \Gamma(v)$ of vertex v, *second neighborhood* $\Gamma_2(v)$ of vertex v and *common neighborhood* $\Gamma(uv)$ of two vertices are defined by $\Gamma(v) := \{u \ : \ u \sim v\}$, $\Gamma_2(v) := \{u \ : \ d(v,u) = 2\}$, $\Gamma(uv) := \Gamma(u) \cap \Gamma(v)$. A graph is a strongly regular graph (SRG) if it is connected, regular and, for distinct vertices u and v, $|\Gamma(uv)| = \lambda$ if $u \sim v$, and μ otherwise. In this paper we will consider the following generalization of strongly regular graphs. Let n, k, b, and a be integers such that $0 \le a \le b \le k < n$. A graph $\Gamma = (V, E)$ is called an (n, k, b, a)-*Deza graph* (DG) if $|V| = n$ and for $u, v \in V$,

$$|\Gamma(uv)| = \begin{cases} a \text{ or } b, & \text{if } u \neq v; \\ k, & \text{if } u = v. \end{cases}$$

The only difference between a strongly regular graph and a Deza graph is that the size of $\Gamma(uv)$ does not necessarily depend on whether $u \sim v$. In this paper we will consider Deza graphs of diameter 2 only. A subgraph Γ' of a graph Γ is called *induced subgraph* if for all two vertices u, v $u \sim v$ in Γ' if and only if $u \sim v$ in Γ. In this paper we will use the term "subgraph" while meaning "induced subgraph". Also, for any set of vertices and for induced subgraph on this set we will use the same notation. Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be graphs. The *composition* $\Gamma_1[\Gamma_2]$ of Γ_1 and Γ_2 is a graph with vertex set $V_1 \times V_2$, and adjacency defined by $(u_1, u_2) \sim (v_1, v_2)$ iff $u_1 \sim v_1$ or $(u_1 = v_1$ and $u_2 \sim v_2)$. Strongly regular graphs with disconnected second neighborhood were classified in [1, Lemma 3.1].

Lemma 1 *[1, Lemma 3.1] Let Γ be a strongly regular graph. For any $u \in V(\Gamma)$, if $\Gamma_2(u)$ is disconnected, then it contains no edges and Γ is a complete multipartite graph (with parts of the same size $s > 2$).*

The following question is naturally arises. What are Deza graphs of diameter 2 with disconnected second neighborhood? In this paper we study the "extremal" cases of edge or coedge regular Deza graphs and also case of vertex transitive Deza graphs. The main results of this paper are the following theorems.

Theorem 1 *A vertex transitive Deza graph with disconnected second neighborhood is edge regular or coedge regular.*

Theorem 2 *Let Γ be a coedge regular Deza graph of diameter 2. If there exists $u \in \Gamma$ such that $\Gamma_2(u)$ is disconnected then Γ is either a complete multipartite graph with parts of the same size $s > 2$ or its 2-clique-extension $\Gamma[K_2]$.*

Theorem 3 *Let Γ be an edge regular Deza graph of diameter 2. If there exists $u \in \Gamma$ such that $\Gamma_2(u)$ is disconnected then Γ is either a complete multipartite graph with parts of the same size $s > 2$ or $\Gamma \cong G_1[G_2]$ where G_1 is a strongly regular graph with $\lambda = \mu$ and G_2 is a coclique of size $s \ge 2$.*

References

[1] Gardiner A.D., Godsil C.D., Hensel A.D., Royal G.F., Second neighbourhoods of strongly regular graphs, *Discrete Mathematics* **103** (1992) 161–170.