On the Deza graphs with disconnected second neighbourhood

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Edge and coedge regular graphs

We consider undirected graphs without loops and multiple edges.

For a graph Γ and its an arbitrary vertex x define the *i-th neighbourhood* $\Gamma_i(x) := \{y \mid y \in V(\Gamma), d(x,y) = i\}$ of the vertex x.

For a graph Γ and its an arbitrary vertex x define *neighbourhood* $\Gamma(x) \equiv \Gamma_1(x)$ of the vertex x.

A graph Γ is called *regular* of valency k, if for all $x \in \Gamma \quad |\Gamma(x)| = k$.

A graph Γ is called *edge regular* with parameters (v, k, λ), if Γ has v vertices, and for any pair of vertices $x, y \in \Gamma$ the following holds

$$|\Gamma(x) \cap \Gamma(y)| = \begin{cases} k, & \text{if } x = y; \\ \lambda, & \text{if } x \sim y. \end{cases}$$

A graph Γ is called *coedge regular* with parameters (v, k, μ), if Γ has v vertices, and for any pair of vertices $x, y \in \Gamma$ the following holds

$$|\Gamma(x) \cap \Gamma(y)| = \begin{cases} k, & \text{if } x = y; \\ \mu, & \text{if } x \neq y \text{ and } x \not\sim y. \end{cases}$$

Strongly regular and Deza graphs

A graph Γ is called strongly regular (SRG) with parameters (v, k, λ, μ), if Γ has v vertices, and for any pair of vertices $x, y \in \Gamma$ the following holds

$$|\Gamma(x) \cap \Gamma(y)| = \begin{cases} k, & \text{if } x = y; \\ \lambda, & \text{if } x \sim y; \\ \mu, & \text{if } x \neq y \text{ and } x \nsim y. \end{cases}$$

A graph Δ is called a Deza graph with parameters (v, k, b, a) (usually $a \leq b$), if Δ has v vertices, and for any pair of vertices $x, y \in \Delta$ the following holds

$$|\Delta(x) \cap \Delta(y)| = \begin{cases} k, & \text{if } x = y; \\ a \text{ or } b, & \text{if } x \neq y. \end{cases}$$

Deza graphs are natural generalization of strongly regular graphs.

A Deza graph Δ is called a strictly Deza graph, if Δ has diameter 2, and is not SRG.

Preliminary results and problem

Problem 1

Classify strongly regular graphs which contain a vertex with disconnected second neigbourhood.

Theorem (Gardiner A.D., Godsil C.D., Hensel A.D., Royle G.F. 1992)

Let Γ *be a strongly regular graph. If there is* $u \in V(\Gamma)$*, such that* $\Gamma_2(u)$ *is disconnected, then* $\Gamma_2(u)$ *contains no edges and* Γ *is a complete multipartite graph with* $s \geq 2$ *parts of the same size* $t > 2$*.*

Problem 2

Classify strictly Deza graphs which contain a vertex with disconnected second neigbourhood.

In this work we consider "extremal" cases of coedge regular and edge regular strictly Deza graphs and also the case of vertex transitive Deza graphs.

Composition of graphs

Definition

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be graphs. The composition $\Gamma_1[\Gamma_2]$ of graphs Γ_1 and Γ_2 is the graph with vertex set $V_1 \times V_2$ and the adjacency rule

$$(u_1, v_1) \sim (u_2, v_2) \Leftrightarrow u_1 \sim u_2 \ \mathit{OR} \ (u_1 = u_2 \ \mathit{AND} \ v_1 \sim v_2)$$

Construction 1 of Deza graphs

Proposition

Let Γ be a complete multipartite graph with $s \geq 2$ parts of the same size $t > 2$. Denote $D(t,s) := \Gamma[K_2]$ then

1. *$D(t,s)$ is a strictly Deza graph with parameters*

$$(2ts,\ 2t(s-1)+1,\ 2t(s-1),\ 2t(s-2)+2);$$

2. *$D(t,s)$ is a coedge regular graph;*
3. *$\forall x \in D(t,s)$ the second neigbourhood of x is a disconnected graph.*

Remark

$D(t,s)$ is a vertex transitive graph.

Construction 2 of Deza graphs

Proposition

Let Γ_1 be an (n, k, λ, μ) strongly regular graph with $\lambda = \mu$ and Γ_2 be an n'-coclique, where $n' \geq 2$, then

1. $\Gamma_1[\Gamma_2]$ *is a strictly Deza graph with parameters*
$$(nn', kn', kn', \lambda n');$$
2. $\Gamma_1[\Gamma_2]$ *is an edge regular graph;*
3. $\forall x \in \Gamma_1[\Gamma_2]$ *the second neigbourhood of x is a disconnected graph.*

There is a not vertex transitive SRG with parameters $(45, 12, 3, 3)$. So, Contruction 2 gives the infinite series of not vertex transitive edge regular strictly Deza graphs.

The problem of existence of SRG with $\lambda = \mu$ is open in general case.

Example: $(153, 96, 60, 60)$ is the smallest(w.r.t. number of vertices) set of parameters of SRG with $\lambda = \mu$ for which the existence of a graph is unknown.

Our results

Theorem 1 (Vertex transitive case)

Let Γ be a vertex transitive strictly Deza graph. If there is $u \in V(\Gamma)$, such that $\Gamma_2(u)$ is disconnected, then Γ is either edge regular or coedge regular.

Theorem 2 (Coedge regular case)

Let Γ be a coedge regular strictly Deza graph. If there is $u \in V(\Gamma)$, such that $\Gamma_2(u)$ is disconnected, then $\Gamma \cong D(t, s)$ with appropriate values of parameters.

Theorem 3 (Edge regular case)

Let Γ be an edge regular strictly Deza graph. If there is $u \in V(\Gamma)$, such that $\Gamma_2(u)$ is disconnected, then $\Gamma \cong \Gamma_1[\Gamma_2]$ where Γ_1 is a strongly regular graph with $\lambda = \mu$ and Γ_2 is a coclique of size $s \geq 2$.

Open problems

Open problem 1

Are there strictly Deza graphs which are a not vertex transitive, a not edge regular and a not coedge regular, and contain a vertex with disconnected second neigbourhood?

If the previous problem has positive solution, then

Open problem 2

Classify strictly Deza graphs which are a not vertex transitive, a not edge regular and a not coedge regular, and contain a vertex with disconnected second neigbourhood.

Thank you for your attention!