On multiplicities of eigenvalues of the Star graphs

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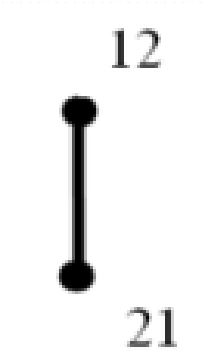
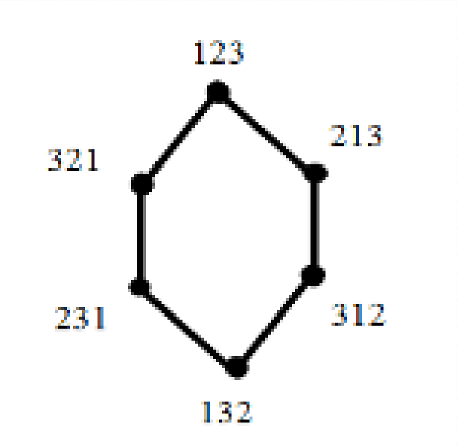
The Star graphs S_n = Cay(Sym_n , t)

are the Cayley graphs on Sym_n with the generating set $t = \{(1, i), i \in \{2, \ldots, n\}\}$.

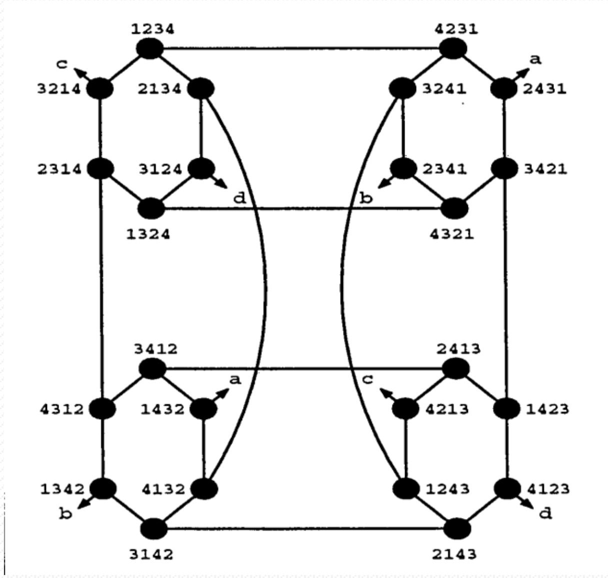
S2



S3



S4



K .Qiu, S.K. Das " Interconnection networks and their eigenvalues " (2003)
A.Abdollahi, E.Vatandoost "Which Cayley graphs are integral?" (2009)

Multiplicities of eigenvalues of the Star graphs S_n,
$2 \leq n \leq 6$, which were calculated by GAP.

	-5	-4	-3	-2	-1	0	1	2	3	4	5
2					1		1				
3				1	2		2	1			
4			1	6	3	4	3	6	1		
5		1	12	28	4	30	4	28	12	1	
6	1	20	105	120	30	168	30	120	105	20	1

R.Krakovski, B.Mohar "Spectrum of Cayley Graphs on the Symmetric Group generated by transposition" (2012)

Theorem (The spectrum of S_n)

Let $n \geq 2$ be an integer. For each integer $\pm(n - k)$ for all $1 \leq k \leq n\text{-}1$ are eigenvalues of S_n with multiplicity at least $\binom{n-2}{k-1}$. If $n \geq 4$, then 0 is an eigenvalue of S_n with multiplicity at least $\binom{n-1}{2}$.

Note that $\pm(n-1)$ is a simple eigenvalue of S_n since the graph is $(n-1)$-regular, bipartite, and connected.

Chapuy – Feray combinatorial approach

Representation theory of groups

A representation of a group G on a vector space V over a field K is a group homomorphism ρ: G → GL(V), where GL(V) is the general linear group on V, such that

$$\rho(g1g2) = \rho(g1)\,\rho(g2)\text{ , for all } g1, g2 \in G.$$

A partition of a positive integer n is a way of writing n as a sum of positive integers.

P(n) - the partition function.

$\lambda \in$ P(n) - a partition of n.

V_λ - the irreducible module associated with the partition λ.

B.E. Sagan "The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions" (2001)

The regular representation C[Sym_n] of Sym_n is decomposed into irreducible submodules as follows:

$$\mathbf{C}\ [Sym_n] = \oplus_{\lambda \in P(n)} dim(V_\lambda) V_\lambda$$

The regular representation C[Sym_n] in the representation theory is called group algebra of Sym_n.

Standard Young tableau

For n = 5. P(5) = 7. Partitions:

(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)

$\lambda = (3, 2)$, $\dim(V_\lambda) = 5$

4	5	
1	2	3

3	5	
1	2	4

3	4	
1	2	5

2	4	
1	3	5

2	5	
1	3	4

4	5	
1	2	3

$\lambda = (3, 2)$

$c(4) = y - x = 2 - 1 = 1$

$c(3) = y - x = 1 - 3 = -2$

$c(5) = y - x = 2 - 2 = 0$

Jucys-Murphy elements

The Jucys–Murphy elements $J_2, \ldots, J_n$ in the group algebra $\mathbb{C}[Sym_n]$ of the symmetric group Sym_n, are defined as a sum of transpositions by the formula:

$$J_1 = 0, \qquad J_2 = (1, 2),$$

$$J_i = (1, i) + (2, i) + \ldots + (i-1, i),$$

$$i \in \{2, \ldots, n\}.$$

Transpositions can be represented as matrices, for example:

$$(1,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (1,3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (2,3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

then J_1 represents a zero matrix and

$$J_2 = (1,2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad J_3 = (1,3) + (2,3) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

A. Jucys " Symmetric polynomials and the center of the symmetric group ring." (1974)

Theorem

Let $\lambda \in$ P(n). Then there exists a basis (v) of the irreducible module V_λ, indexed by standard Young tableaux of shape λ, such that for all i $\in$ {2, …,n}, one has:

$$\boxed{J_i v = \mathbf{c}(i) v}$$

G.Chapuy, V.Feray "A note on a Cayley graph of Sym_n" (2012)

Corollary

The spectrum of S_n contains only integers. The multiplicity mul(k) of an integer k is given by:

$$mul(k) = \sum_{\lambda \in P(n)} dim(V_\lambda) I_\lambda(k)$$

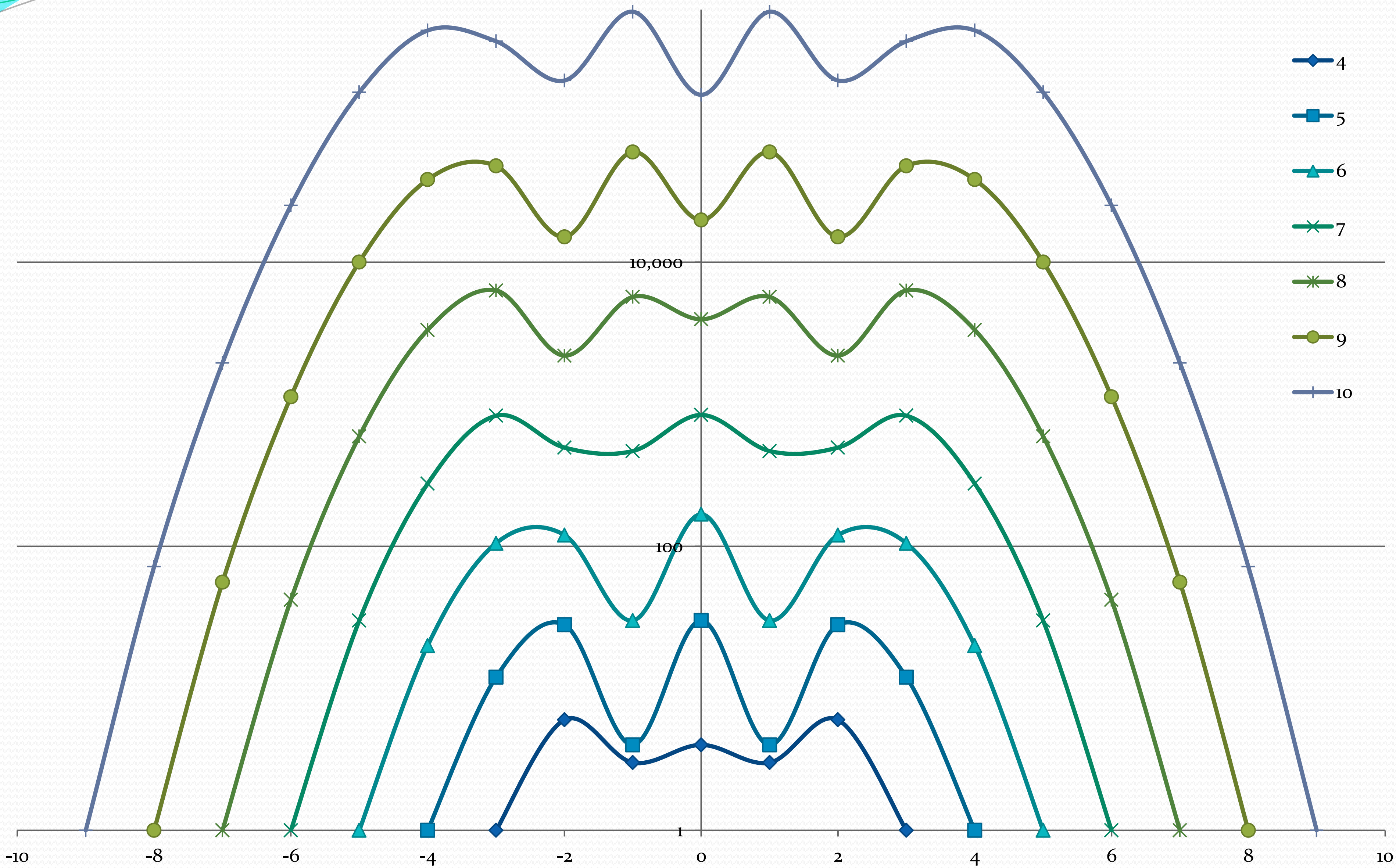
$I_\lambda(k)$ is the number of standard Young tableaux of shape, satisfying c(n) = k.

Results

We present multiplicities of eigenvalues of S_n for $4 \leq n \leq 10$, which were calculated by using a combinatorial approach.

	0	1 (-1)	2 (-2)	3 (-3)	4(-4)	5 (-5)	6 (-6)	7 (-7)	8 (-8)	9 (-9)
2		1								
3		2	1							
4	4	3	6	1						
5	30	4	28	12	1					
6	168	30	120	105	20	1				
7	840	468	495	830	276	30	1			
8	3960	5691	2198	6321	3332	595	42	1		
9	19782	59624	15064	47544	38108	10024	1128	56	1	
10	150640	579078	189936	358764	427392	156780	25104	1953	72	1

$$\sum_{k=-(n-1)}^{n-1} \mathrm{mul(k)} = \mathrm{n!}$$



10,000
100
1
-10
-8
-6
-4
-2
0
2
4
6
8
10
4
5
6
7
8
9
10

Thanks for attention!