Cycles in triangle-free graphs with large chromatic number

A. Kostochka
University of Illinois at Urbana-Champaign, Urbana
and Sobolev Institute of Mathematics, Novosibirsk, Russia
Joint work with B. Sudakov and J. Verstraete

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Colorings

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A graph G is k-colorable if has a k-coloring.
The chromatic number, $\chi(G)$, of a graph G is the smallest k such that G is k-colorable.
A graph G is k-chromatic if $\chi(G) = k$. If in addition, $\chi(G') < k$ for every proper subgraph G' of G, then G is k-critical.

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Examples: $\chi(K_n) = n$, $\chi(C_5) = 3$. Both are critical.

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Theorem 2 [Mihok and Schiermeyer, 2004]. Every k-chromatic graph G has cycles of at least $\frac{k}{2}-1$ even lengths.

A conjecture

Conjecture 1 [Erdős, 1992]. For every $\varepsilon > 0$, there exists $k_0(\varepsilon)$ such that for $k \geq k_0(\varepsilon)$, every triangle-free k-chromatic graph contains more than $k^{2-\epsilon}$ odd cycles of different lengths.

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Similar conjecture without odd.

Theorem 3 [Sudakov and Verstraete, 2008]. Every graph G of average degree k and girth at least five contains cycles of $\Omega(k^2)$ consecutive even lengths.

Theorem 4 [Sudakov and Verstraete, 2011]. Every n-vertex triangle-free graph of independence number at most $\frac{n}{k}$ contains cycles of $\Omega(k^2 \log k)$ consecutive lengths.

A question

A question that Erdős should have asked
(but probably never asked):

How short can be the longest cycle in a triangle-free k-chromatic graph?

In other words,

What is the smallest circumference of a triangle-free k-chromatic graph?

Main result

Theorem 5 [A.K.–B.S.–J.V, 2014]. For each $\varepsilon > 0$, there exists $k_0(\varepsilon)$ such that for $k \geq k_0(\varepsilon)$, every triangle-free k-chromatic graph G contains a cycle of length at least $(\frac{1}{4} - \varepsilon)k^2 \log k$ as well as cycles of at least $(\frac{1}{64} - \varepsilon)k^2 \log k$ consecutive lengths.

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Example 1: Bohman and Keevash and Fiz Pontiveros, Griffiths and Morris independently constructed a k-chromatic triangle-free graph with at most $(4 + o(1))k^2 \log k$ vertices as $k \to \infty$, refining the earlier construction of Kim.

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Example 2: (a) Graph $K_{2k,n-2k}$ is triangle-free and has min. degree $2k$ and cycles of $2k - 1$ distinct lengths.
(b) Graph $H(n, k)$ obtained from $K_{2k,n-2k}$ by splitting a vertex of degree $n - 2k$ into two of degree $(n - 2k)/2$ and joining them by an edge is nonbipartite, triangle-free and has min. degree $2k$ and cycles of $4k - 1$ distinct lengths.

Hereditary Properties

Let $n_{\mathcal{P}}(k)$ denote the smallest possible order of a k-chromatic graph in a hereditary property $\mathcal{P}$.

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Let $\alpha \geq 1$ and let $f : [3, \infty) \to \mathbf{R}^+$. Then f is α-bounded if f is non-decreasing and for each $y \geq x \geq 3$,

$$y^\alpha f(x) \geq x^\alpha f(y), \qquad \text{i.e.} \qquad \frac{f(y)}{f(x)} \leq \left(\frac{y}{x}\right)^\alpha.$$

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Theorem 6 [K.-S.-V]. For all $\varepsilon > 0$ and $\alpha, m \geq 1$, there exists $k_1 = k_1(\varepsilon, \alpha, m)$ such that the following holds. If $\mathcal{P}$ is a hereditary property of graphs with $n_{\mathcal{P}}(k) \geq f(k)$ for $k \geq m$ and some α-bounded function f, then for $k \geq k_1$, every k-chromatic graph $G \in \mathcal{P}$ contains
(i) a cycle of length at least $(1 - \varepsilon)f(k)$ and
(ii) cycles of at least $(1 - \varepsilon)f(\frac{k}{4})$ consecutive lengths.

H-free graphs

For each graph H, property of H-free graphs is hereditary.
Together with Ramsey bounds, this yields:

Theorem 7 [K.-S.-V]. If $k > r \geq 3$ and G is a k-chromatic K_{r+1}-free graph, then G contains cycles of $\Omega(k^{\frac{r}{r-1}})$ consecutive lengths.

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Similarly, if C_ℓ denotes the cycle of length ℓ, then for large k every C_{2s+1}-free k-chromatic graph has cycles of $\Omega(k^{s+1} \log k)$ consecutive lengths.

Proof steps: 1. Many lengths from a long length

Lemma 1 [Verstraete]. Let H be a graph comprising a cycle with a chord. Let (A, B) be a nontrivial partition of $V(H)$. Then H contains A, B-paths of every positive length less than $|H|$, unless H is bipartite with bipartition (A, B).

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Lemma 2. Let $k \geq 4$ and $\mathcal{Q}$ be a hereditary class of graphs. Let $h(k, \mathcal{Q})$ denote the smallest possible length of a longest cycle in any k-chromatic graph in $\mathcal{Q}$. Then every $4k$-chromatic graph in $\mathcal{Q}$ contains cycles of at least $h(k, \mathcal{Q})$ consecutive lengths.

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Lemma 3. Let G be a k-critical graph and let S be a vertex cut of G. Then for any component H of $G - S$,

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Let $c(G)$ denote the circumference of G.

Lemma 4. Let $k \geq 4$. For every k-chromatic graph G, there is a graph G^* and an edge $e^* \in E(G^*)$ such that
(a) $G^* - e^* \subset G$ and $\chi(G^* - e^*) \geq k - 1$,
(b) G^* is 3-connected,
(c) $c(G^*) \leq c(G)$.

Proof steps: 3. A lemma on α-bounded functions

Lemma 5. Let $\alpha \geq 1$, $x_0 \geq 3$ and let f be α-bounded. Then the function

$$g(x) = \frac{xf(x)}{x + f(x_0)}$$

is $(\alpha + 1)$-bounded, $g(x) \leq x$ for $x \in [3, x_0]$, and $g(x) \leq f(x)$ for all $x \in [3, \infty)$.

Conjecture: Let n_k be the minimum number of vertices in a k-chromatic triangle-free graph. Then for every k-chromatic triangle-free graph G,

$$c(G) \geq n_k - o(n_k).$$