Schur rings and DCI-groups

István Kovács

FAMNIT and IAM, University of Primorska
Koper, Slovenia

G2C2, 2014, Novosibirsk, Russia
September 26, 2014

1. Schur rings

Let H be a finite group with multiplicative notation and identity element e.

The **group algebra** $\mathbb{Q}H$ has operations:

$$\begin{aligned} \sum_{x \in H} a_x x + \sum_{x \in H} b_x x &= \sum_{x \in H} (a_x + b_x) x \\ \sum_{x \in H} a_x x \cdot \sum_{x \in H} b_x x &= \sum_{x,y \in H} (a_y b_{y^{-1}x}) x. \end{aligned}$$

For $S \subseteq H$ let $\underline{S} := \sum_{x \in H} a_x x$ where $a_x = 1$ if $x \in S$ and 0 otherwise, such element is called **simple quantity**.

Definition (Wielandt)

A subalgebra $\mathcal{A}$ of $\mathbb{Q}H$ is a **Schur ring** (for short **S-ring**) over H if the following hold:

(S1) $\mathcal{A} = \langle \underline{T_0}, \ldots, \underline{T_r} \rangle$, $T_i \subseteq H$ for all $i \in \{0, \ldots, r\}$,

(S2) $T_0 = \{e\}$, and the sets T_i form a partition of H.

(S3) For every $i \in \{0, \ldots, r\}$, there exists $j \in \{0, \ldots, r\}$ such $T_i^{-1} := \{g^{-1} : g \in T_i\} = T_j$.

The sets T_i are the **basic sets** of $\mathcal{A}$, and their cardinality is the **rank** of $\mathcal{A}$.

Definition (Wielandt)

A subalgebra $\mathcal{A}$ of $\mathbb{Q}H$ is a **Schur ring** (for short **S-ring**) over H if the following hold:

(S1) $\mathcal{A} = \langle \underline{T_0}, \dots, \underline{T_r} \rangle$, $T_i \subseteq H$ for all $i \in \{0, \dots, r\}$,

(S2) $T_0 = \{e\}$, and the sets T_i form a partition of H.

(S3) For every $i \in \{0, \dots, r\}$, there exists $j \in \{0, \dots, r\}$ such $T_i^{-1} := \{g^{-1} : g \in T_i\} = T_j$.

The sets T_i are the **basic sets** of $\mathcal{A}$, and their cardinality is the **rank** of $\mathcal{A}$.

Example:

$H = C_6 = \langle g \rangle$,

$\mathcal{A} = \big\langle \underline{e}, \underline{g, g^3, g^5}, \underline{g^2, g^4} \big\rangle$.

$\mathcal{A}$ is a subalgebra of $\mathbb{Q}H$ (closed under $\cdot$):

$$\underline{g, g^3, g^5} \cdot \underline{g, g^3, g^5} \quad = \quad 3\,\underline{e} + 3\,\underline{g^2, g^4}$$

As all axioms (S1)-(S3) hold, $\mathcal{A}$ is an S-ring over C_6 of rank 3.

Theorem (Schur)

Let $H_R \leq G \leq \mathrm{Sym}(H)$. Then $V(H, G_e) := \langle \underline{T} : T \in \mathrm{Orb}(G_e) \rangle$ is a subalgebra of $\mathbb{Q}H$.

$H_R \leq \mathrm{Sym}\, H$ denotes the group of right multiplications:

$$x \mapsto xh \text{ for every } x, h \in H.$$

For $G \leq \mathrm{Sym}(H)$, G_e denotes the **point stabilizer** of e in G.

Theorem (Schur)

Let $H_R \leq G \leq \mathrm{Sym}(H)$. Then $V(H, G_e) := \langle \underline{T} : T \in \mathrm{Orb}(G_e) \rangle$ is a subalgebra of $\mathbb{Q}H$.

As all axioms (S1)-(S3) hold, the subalgebra $V(H, G_e)$ is an S-ring over H (**transitivity module**).

* In fact, $\langle \underline{e}, \underline{g, g^3, g^5}, \underline{g^2, g^4} \rangle = V(C_6, G_e)$, where $G \leq \mathrm{Sym}(C_6)$ consists of all permutations that preserve the partition

$$e, g^2, g^4 \mid g, g^3, g^5.$$

Theorem (Schur)

Let $H_R \le G \le \mathrm{Sym}(H)$. Then $V(H, G_e) := \langle \underline{T} : T \in \mathrm{Orb}(G_e) \rangle$ is a subalgebra of $\mathbb{Q}H$.

As all axioms (S1)-(S3) hold, the subalgebra $V(H, G_e)$ is an S-ring over H (**transitivity module**).

- In fact, $\left\langle \underline{e}, \underline{g, g^3, g^5}, \underline{g^2, g^4} \right\rangle = V(C_6, G_e)$, where $G \le \mathrm{Sym}(C_6)$ consists of all permutations that preserve the partition

$$e, g^2, g^4 \mid g, g^3, g^5.$$

Wielandt's example:

$H = \mathbb{Z}_p \times \mathbb{Z}_p$, $p \geq 5$ a prime, and

$$\mathcal{A} = \Big\langle \underline{e}, \underline{P_1 \setminus \{e\}}, \underline{P_2 \setminus \{e\}}, \underline{P_3 \setminus \{e\}}, \underline{H \setminus \cup_{i=1}^3 P_i} \Big\rangle,$$

where $P_1, P_2, P_3 \leq H$ are pairwise distinct subgroups of order p.

It can be proved that there exists no permutation group G of H such that

1. $H_R \leq G$,
2. The orbits of G_e on H are:

$$\{e\}, P_1 \setminus \{e\}, P_2 \setminus \{e\}, P_3 \setminus \{e\}, H \setminus \cup_{i=1}^3 P_i.$$

* Schur rings are equivalent to **Cayley schemes,** i.e., association schemes having a regular automorphism group.

Wielandt's example:

$H = \mathbb{Z}_p \times \mathbb{Z}_p$, $p \geq 5$ a prime, and

$$\mathcal{A} = \Big\langle \underline{e}, \underline{P_1 \setminus \{e\}}, \underline{P_2 \setminus \{e\}}, \underline{P_3 \setminus \{e\}}, \underline{H \setminus \cup_{i=1}^3 P_i} \Big\rangle,$$

where $P_1, P_2, P_3 \leq H$ are pairwise distinct subgroups of order p.

It can be proved that there exists no permutation group G of H such that

1. $H_R \leq G$,
2. The orbits of G_e on H are:

$$\{e\}, P_1 \setminus \{e\}, P_2 \setminus \{e\}, P_3 \setminus \{e\}, H \setminus \cup_{i=1}^3 P_i.$$

$*$ Schur rings are equivalent to **Cayley schemes,** i.e., association schemes having a regular automorphism group.

Definition

An S-ring $\mathcal{A}$ over a group H is **Schurian** if $\mathcal{A} = V(H, G_e)$ for some $H_R \leq G \leq \mathrm{Sym}(H)$.

A finite group H is a **Schur group** if every S-ring over H is Schurian.

Problem (Pöschel)

Classify the finite Schur groups.

Known results about Schur groups:

- C_n is a Schur group iff $n \in \{p^k, pq^k, 2pq^k, pqr, 2pqr\}$, where p, q, r are distinct primes, and $k \geq 0$ is an integer (Evdokimov, K, Ponomarenko, 2013).
- $C_p^n, n > 1$ is a a Schur group iff $p^n \in \{4, 8, 9, 16, 27, 32\}$ (Pech, Reichard, 2009).
- If H is an abelian Schur group which is neither cyclic nor elementary abelian, then H is isomorphic to one of the following groups:
  (1) $C_2 \times C_{2^k},\ C_{2p} \times C_{2^k},\ C_2^2 \times C_{p^k},\ C_2^2 \times C_{pq},\ C_2^4 \times C_p,$
  (2) $C_3 \times C_{3^k},\ C_6 \times C_{3^k},\ C_3^2 \times C_q,\ C_3^2 \times C_{2q},$
  where p, q are distinct primes, $p \neq 2$, and $k \geq 1$ is an integer (Evdokimov, K, Ponomarenko, 2014+).
- If H is a non-abelian Schur group, then it is metaabelian (Ponomarenko, Vasiljev 2014+).

2. Sub-, quotient- and generated S-rings

Definition

Given two S-rings $\mathcal{A}$ and $\mathcal{B}$ over H, $\mathcal{A}$ is called an **sub-S-ring** (or a **fusion**) of $\mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$.

A subgroup $K \le H$ is an $\mathcal{A}$-**subgroup** if $\underline{K} \in \mathcal{A}$.

Definition

Given an S-ring $\mathcal{A}$ over H, a normal $\mathcal{A}$-subgroup K, the **quotient S-ring of $\mathcal{A}$ by K** is the S-ring over H/K defined by

$$\mathcal{A}_{H/K} := \langle \pi_{H/K}(T) : T \text{ a basic set of } \mathcal{A} \rangle,$$

where $\pi_{H/K}$ is the natural homomorphism from H to H/K.

2. Sub-, quotient- and generated S-rings

Definition

Given two S-rings $\mathcal{A}$ and $\mathcal{B}$ over H, $\mathcal{A}$ is called an **sub-S-ring** (or a **fusion**) of $\mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$.

A subgroup $K \leq H$ is an $\mathcal{A}$-**subgroup** if $\underline{K} \in \mathcal{A}$.

Definition

Given an S-ring $\mathcal{A}$ over H, a normal $\mathcal{A}$-subgroup K, the **quotient S-ring of $\mathcal{A}$ by K** is the S-ring over H/K defined by

$$\mathcal{A}_{H/K} := \langle \pi_{H/K}(T) : T \text{ a basic set of } \mathcal{A} \rangle,$$

where $\pi_{H/K}$ is the natural homomorphism from H to H/K.

Definition

The S-ring **generated by given elements** $\eta_1, \ldots, \eta_k \in \mathbb{Q}H$ is defined as

$$\langle\langle \eta_1, \ldots, \eta_k \rangle\rangle = \bigcap_{\eta_1, \ldots, \eta_k \in \mathcal{A}} \mathcal{A}.$$

3. Automorphisms and isomorphisms of S-rings

For a subset $S \subset H$, $e \notin S$, the **Cayley graph** $\mathrm{Cay}(H, S)$ has vertex set H and arc set $\{(x, sx) : x \in H, s \in S\}$ (thus $H_R \leq \mathrm{Aut}(\mathrm{Cay}(H, S))$).

If $S = S^{-1}$, $\mathrm{Cay}(H, S)$ is regarded as an undirected graph.

Definition (Klin)

Ler $\mathcal{A} = \langle T_0, \ldots, T_r \rangle$ be an S-ring over a group H. The **automorphism group** of $\mathcal{A}$ is defined by

$$\mathrm{Aut}(\mathcal{A}) = \mathrm{Aut}(\mathrm{Cay}(H, T_1)) \cap \cdots \cap \mathrm{Aut}(\mathrm{Cay}(H, T_r)).$$

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be S-rings over the groups H and K respectively. A **combinatorial isomorphism** from $\mathcal{A}$ to $\mathcal{B}$ is a bijection f from H to K such that

$$\{\mathrm{Cay}(H, T) : T \in \mathrm{Bsets}(\mathcal{A})\}^f = \{\mathrm{Cay}(H, T) : T \in \mathrm{Bsets}(\mathcal{B})\}.$$

For an S-ring $\mathcal{A}$ over H, let

$$\mathbf{Iso}(\mathcal{A}) = \{f \in \mathrm{Sym}(H) : f : \mathcal{A} \to \mathcal{B} \text{ comb. iso.}\}.$$

We say that $f \in \mathrm{Iso}(\mathcal{A})$ is **normalized** if $e^f = e$, notation: $\mathrm{Iso}_1(A)$.

There are some natural sets of combinatorial isomorphisms:

(1) $\mathrm{Aut}(\mathcal{A})$,

(2) $\mathrm{Aut}(H)$,

(3) $\mathrm{Aut}(\mathcal{A})\,\mathrm{Aut}(H)$.

Definition

An S-ring $\mathcal{A}$ over a group H is a **CI-S-ring** if it satisfies the second condition of the proposition.

There are some natural sets of combinatorial isomorphisms:

(1) $\mathrm{Aut}(\mathcal{A})$,

(2) $\mathrm{Aut}(H)$,

(3) $\mathrm{Aut}(\mathcal{A})\,\mathrm{Aut}(H)$.

Definition

An S-ring $\mathcal{A}$ over a group H is a **CI-S-ring** if it satisfies the second condition of the proposition.

Proposition

Let $G \leq \mathrm{Sym}(H)$ be a permutation group such that $H_R \leq G$ and and let $\mathcal{A} = V(H, G_e)$. Then the following are equivalent:

- $\mathrm{Iso}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A})\,\mathrm{Aut}(H)$.
- $\mathrm{Iso}_1(\mathcal{A}) = \mathrm{Aut}(\mathcal{A})_e\,\mathrm{Aut}(H)$.
- Every two regular subgroups of $\mathrm{Aut}(\mathcal{A})$, which are isomorphic to H, are conjugate in G.

4. DCI- and CI-groups

It is a trivial observation that, if $\sigma \in \mathrm{Aut}(H)$, then $\mathrm{Cay}(H,S) \cong \mathrm{Cay}(H,S^\sigma)$. However, the converse implication does not hold in general.

Definition

A Cayley garph $\mathrm{Cay}(H,S)$ is a **CI-graph** if $\mathrm{Cay}(H,S) \cong \mathrm{Cay}(H,T)$ implies that $T = S^\sigma$ for some automorphism $\sigma \in \mathrm{Aut}(H)$.

4. DCI- and CI-groups

It is a trivial observation that, if $\sigma \in \mathrm{Aut}(H)$, then $\mathrm{Cay}(H, S) \cong \mathrm{Cay}(H, S^\sigma)$. However, the converse implication does not hold in general.

Definition

A Cayley garph $\mathrm{Cay}(H, S)$ is a **CI-graph** if $\mathrm{Cay}(H, S) \cong \mathrm{Cay}(H, T)$ implies that $T = S^\sigma$ for some automorphism $\sigma \in \mathrm{Aut}(H)$.

Definition

A finite group H is called a **DCI-group** if every Cayley graph over H is a CI-graph; and a **CI-group** if every undirected Cayley graph over H is a CI-graph.

Problem (Babai)

Classify the finite DCI- and CI-groups.

Theorem (Li–Praeger–Xu)

If H is a DCI-group, then H is a coprime product (i.e., a direct product of groups of coprime orders). of groups from the following list:

$$\mathbb{Z}_p^e,\ \mathbb{Z}_4,\ Q,\ A_4,\ E(M,2),\ E(M,4),$$

where M is a direct product of elementary abelian groups of odd order and $E(M,k) := M \rtimes \langle z \rangle$ where z is an element of order k the action of which on M is defined as follows $m^z = m^{-1}$.

An analogous list exists for CI-groups (Li–Lu–Pálfy).

In order to finish the classification of *DCI*-groups one has to answer two basic questions:

- Which groups in the above list are *DCI*-groups?
- When a coprime product of two *DCI*-groups is a *DCI*-group?

Theorem (Li–Praeger–Xu)

If H is a DCI-group, then H is a coprime product (i.e., a direct product of groups of coprime orders). of groups from the following list:

$$\mathbb{Z}_p^e,\ \mathbb{Z}_4,\ Q,\ A_4,\ E(M,2),\ E(M,4),$$

where M is a direct product of elementary abelian groups of odd order and $E(M,k) := M \rtimes \langle z \rangle$ where z is an element of order k the action of which on M is defined as follows $m^z = m^{-1}$.

An analogous list exists for CI-groups (Li–Lu–Pálfy).

In order to finish the classification of *DCI*-groups one has to answer two basic questions:

- Which groups in the above list are *DCI*-groups?
- When a coprime product of two *DCI*-groups is a *DCI*-group?

The first question was answered affirmatively for the following groups:

- $\mathbb{Z}_4, Q_8$ and A_4 (Conder–Li).
- $D_{2p} = E(\mathbb{Z}_p, 2)$ (Babai).
- $E(\mathbb{Z}_p, 4)$ (Li-Pálfy).
- $\mathbb{Z}_p^e$ with $e \leq 4$ (Elspas–Turner; Godsil; Alspah–Nowitz; Dobson; Hirasaka–Muzychuk, Morris).
- $\mathbb{Z}_2^5$ (Conder–Li).
- $\mathbb{Z}_3^5$ (Spiga).

On the other hand,

- $\mathbb{Z}_2^6$ is **NOT** a CI-group (Nowitz)
- $\mathbb{Z}_3^8$ is **NOT** a CI-group (Spiga)
- $\mathbb{Z}_p^e, p > 2$ is **NOT** CI-group if $e \geq 2p + 3$ (Somlai).

Concerning the second question, all known examples of *DCI*-groups are:

- $\mathbb{Z}_n, \mathbb{Z}_{2n}$ and $\mathbb{Z}_{4n}$ where n is square-free odd (Muzychuk).
- $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ where p is any prime (Dobson–Spiga)
- $Q \times \mathbb{Z}_p$ where $p > 2$ a prime (Somlai)
- $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ where p, q are distinct primes (K–Muzychuk)
- $\mathbb{Z}_p^3 \times \mathbb{Z}_q$ where p, q are primes such that $q > p^3$ (Somlai)

On the other hand,

- $\mathbb{Z}_2^6$ is **NOT** a CI-group (Nowitz)
- $\mathbb{Z}_3^8$ is **NOT** a CI-group (Spiga)
- $\mathbb{Z}_p^e, p > 2$ is **NOT** CI-group if $e \geq 2p + 3$ (Somlai).

Concerning the second question, all known examples of *DCI*-groups are:

- $\mathbb{Z}_n, \mathbb{Z}_{2n}$ and $\mathbb{Z}_{4n}$ where n is square-free odd (Muzychuk).
- $\mathbb{Z}_2^3 \times \mathbb{Z}_p$ where p is any prime (Dobson–Spiga)
- $Q \times \mathbb{Z}_p$ where $p > 2$ a prime (Somlai)
- $\mathbb{Z}_p^2 \times \mathbb{Z}_q$ where p, q are distinct primes (K–Muzychuk)
- $\mathbb{Z}_p^3 \times \mathbb{Z}_q$ where p, q are primes such that $q > p^3$ (Somlai)

5. Schur rings and DCI-groups

Theorem (Babai)

The following are equivalent for every Cayley graph $\mathrm{Cay}(H, S)$:

- *Every regular subgroup of* $\mathrm{Aut}(\mathrm{Cay}(H, S))$ *isomorphic to* H *is conjugate to* H_R *in* $\mathrm{Aut}(\mathrm{Cay}(H, S))$.
- $\mathrm{Cay}(H, S)$ *is a CI-graph.*

Proposition

Let $G \leq \mathrm{Sym}(H)$ be a permutation group such that $H_R \leq G$ and and let $\mathcal{A} = V(H, G_e)$. Then the following are equivalent:

- Every regular subgroup of G isomorphic to H is conjugate to H_R in G.
- $\mathrm{Iso}(\mathcal{A}) = \mathrm{Aut}(\mathcal{A})\,\mathrm{Aut}(H)$.
- $\mathrm{Iso}_1(\mathcal{A}) = \mathrm{Aut}(\mathcal{A})_e\,\mathrm{Aut}(H)$.

Proposition

For every subset $S \subset G \setminus \{e\}$,

$$\mathrm{Aut}(\mathrm{Cay}(G,S)) = \mathrm{Aut}(\langle\langle \underline{S} \rangle\rangle).$$

The Klin–Pöschel approach:

(1) Classify the S-rings over H.

(2) Select the non-CI-S-rings.

(3) For every non-CI-S-ring chek that whether it can be generated by a subset of H.

Proposition

For every subset $S \subset G \setminus \{e\}$,

$$\mathrm{Aut}(\mathrm{Cay}(G, S)) = \mathrm{Aut}(\langle\!\langle \underline{S} \rangle\!\rangle).$$

The Klin–Pöschel approach:

(1) Classify the S-rings over H.

(2) Select the non-CI-S-rings.

(3) For every non-CI-S-ring chek that whether it can be generated by a subset of H.

Example:

no.	basic sets
1.	$0 \mid 1,2,3,4,5,6,7$
2.	$0 \mid 1,3,5,7 \mid 2,4,6$
3.	$0 \mid 1,3,5,7,2,6 \mid 4$
4.	$0 \mid 1,3,5,7 \mid 2,6 \mid 4$
5.	$0 \mid 1,3,5,7 \mid 2 \mid 6 \mid 4$
6.	$0 \mid 1,5 \mid 3,7 \mid 2,6 \mid 4$
7.	$0 \mid 1,7 \mid 3,5 \mid 2,6 \mid 4$
8.	$0 \mid 1,7 \mid 3,5 \mid 2,6 \mid 4$
9.	$0 \mid 1,5 \mid 3,7 \mid 2 \mid 4 \mid 6$
10.	$0 \mid 1 \mid 2 \cdots \mid 7$

Table : S-rings over $\mathbb{Z}_8$

$\mathrm{Cay}(\mathbb{Z}_8, S)$ is a non-CI-graph iff $\langle\langle \underline{S} \rangle\rangle = \mathcal{A}_9$.

no.	basic sets
1.	$0 \mid 1,2,3,4,5,6,7$
2.	$0 \mid 1,3,5,7 \mid 2,4,6$
3.	$0 \mid 1,3,5,7,2,6 \mid 4$
4.	$0 \mid 1,3,5,7 \mid 2,6 \mid 4$
5.	$0 \mid 1,3,5,7 \mid 2 \mid 6 \mid 4$
6.	$0 \mid 1,5 \mid 3,7 \mid 2,6 \mid 4$
7.	$0 \mid 1,7 \mid 3,5 \mid 2,6 \mid 4$
8.	$0 \mid 1,7 \mid 3,5 \mid 2,6 \mid 4$
9.	$0 \mid 1,5 \mid 3,7 \mid 2 \mid 4 \mid 6$
10.	$0 \mid 1 \mid 2 \cdots \mid 7$

$\langle\!\langle \underline{1,2,5} \rangle\!\rangle = \mathcal{A}_9$, thus $\mathbb{Z}_8$ is not a DCI-group.

$\langle\!\langle \underline{S} \rangle\!\rangle \neq \mathcal{A}_9$ if $S^{-1} = S$, thus $\mathbb{Z}_8$ is a CI-group.

no.	basic sets
1.	$0 \mid 1,2,3,4,5,6,7$
2.	$0 \mid 1,3,5,7 \mid 2,4,6$
3.	$0 \mid 1,3,5,7,2,6 \mid 4$
4.	$0 \mid 1,3,5,7 \mid 2,6 \mid 4$
5.	$0 \mid 1,3,5,7 \mid 2 \mid 6 \mid 4$
6.	$0 \mid 1,5 \mid 3,7 \mid 2,6 \mid 4$
7.	$0 \mid 1,7 \mid 3,5 \mid 2,6 \mid 4$
8.	$0 \mid 1,7 \mid 3,5 \mid 2,6 \mid 4$
9.	$0 \mid 1,5 \mid 3,7 \mid 2 \mid 4 \mid 6$
10.	$0 \mid 1 \mid 2 \cdots \mid 7$

$\langle\!\langle \underline{1,2,5} \rangle\!\rangle = \mathcal{A}_9$, thus $\mathbb{Z}_8$ is not a DCI-group.

$\langle\!\langle \underline{S} \rangle\!\rangle \neq \mathcal{A}_9$ if $S^{-1} = S$, thus $\mathbb{Z}_8$ is a CI-group.

6. Which groups $\mathbb{Z}_p^n$ are DCI-groups?

Theorem (Djoković; Elspas–Turner)

$\mathbb{Z}_p$ is a DCI-group.

In the case of p-groups the Klin–Pöschel approach can be modified by searching in Step (1) only for p-S-rings.

Definition

An S-ring over a p-group is a p-S-ring if every basic set has cardinality a p-power.

Proposition

If H is a p-group and every p-S-ring over H is a CI-S-ring, then H is a DCI-group.

Wreath product of S-rings

For an S-ring $\mathcal{A}$ over H and $\mathcal{A}$-subgroup $K \leq H$, the **induced S-ring** of $\mathcal{A}$ on K is the S-ring over K defined by

$$\mathcal{A}_K = \langle T : T \in \text{Bsets}(\mathcal{A}) \text{ and } T \subset K \rangle.$$

Definition

Let $\mathcal{A}$ be an S-ring over a group H and N be an $\mathcal{A}$-subgroup such that N is normal in H. Then $\mathcal{A}$ is a **wreath product**, notation:

$$\mathcal{A} = \mathcal{A}_N \wr \mathcal{A}_{H/N}$$

if for every $T \in \text{Bsets}(\mathcal{A})$, if $T \not\subset N$, then T is a union of N-cosets.

Wreath product of S-rings

For an S-ring $\mathcal{A}$ over H and $\mathcal{A}$-subgroup $K \le H$, the **induced S-ring** of $\mathcal{A}$ on K is the S-ring over K defined by

$$\mathcal{A}_K = \langle T : T \in \mathrm{Bsets}(\mathcal{A}) \text{ and } T \subset K \rangle.$$

Definition

Let $\mathcal{A}$ be an S-ring over a group H and N be an $\mathcal{A}$-subgroup such that N is normal in H. Then $\mathcal{A}$ is a **wreath product**, notation:

$$\mathcal{A} = \mathcal{A}_N \wr \mathcal{A}_{H/N}$$

if for every $T \in \mathrm{Bsets}(\mathcal{A})$, if $T \not\subset N$, then T is a union of N-cosets.

Proposition

Up to combinatorial isomorphisms, there are two p-S-rings over $\mathbb{Z}_p^2$:

$$\mathbb{Q}\mathbb{Z}_p^2 \text{ and } \mathbb{Q}\mathbb{Z}_p \wr \mathbb{Q}\mathbb{Z}_p.$$

Theorem (Godsil; Alspah–Nowitz)

$\mathbb{Z}_p^2$ is a DCI-group.

Tensor product of S-rings

Definition

Let $\mathcal{A}$ be an S-ring over a group H and E, F be $\mathcal{A}$-subgroups such that $H = EF$ and $E \cap F = \{e\}$. Then $\mathcal{A}$ is a **tensor product**, notation:

$$\mathcal{A} = \mathcal{A}_E \otimes \mathcal{A}_F$$

if for every $T \in \mathrm{Bsets}(\mathcal{A})$, if $T \not\subset E \cup F$, then $T = RS$ where $R \in \mathrm{Bsets}(\mathcal{A}) \cap E$ and $S \in \mathrm{Bsets}(\mathcal{A}) \cap F$.

Proposition

If $\mathcal{A} = \mathcal{A}_N \wr \mathcal{A}_{H/N}$ such that $\mathcal{A}_N$ and $\mathcal{A}_{H/N}$ are CI-S-rings, then $\mathcal{A}$ is also a CI-S-ring.

Tensor product of S-rings

Definition

Let $\mathcal{A}$ be an S-ring over a group H and E, F be $\mathcal{A}$-subgroups such that $H = EF$ and $E \cap F = \{e\}$. Then $\mathcal{A}$ is a **tensor product**, notation:

$$\mathcal{A} = \mathcal{A}_E \otimes \mathcal{A}_F$$

if for every $T \in \mathrm{Bsets}(\mathcal{A})$, if $T \not\subset E \cup F$, then $T = RS$ where $R \in \mathrm{Bsets}(\mathcal{A}) \cap E$ and $S \in \mathrm{Bsets}(\mathcal{A}) \cap F$.

Proposition

If $\mathcal{A} = \mathcal{A}_N \wr \mathcal{A}_{H/N}$ such that $\mathcal{A}_N$ and $\mathcal{A}_{H/N}$ are CI-S-rings, then $\mathcal{A}$ is also a CI-S-ring.

Theorem (Spiga–Wang)

Up to combinatorial isomorphisms, there are 6 p-S-rings over $\mathbb{Z}_p^3$:

- $\mathbb{Q}\mathbb{Z}_p^3$,
- $\mathbb{Q}\mathbb{Z}_p^2 \wr \mathbb{Q}\mathbb{Z}_p$,
- $\mathbb{Q}\mathbb{Z}_p \wr \mathbb{Q}\mathbb{Z}_p^2$,
- $\mathbb{Q}\mathbb{Z}_p \wr \mathbb{Q}\mathbb{Z}_p \wr \mathbb{Q}\mathbb{Z}_p$,
- $\mathbb{Q}\mathbb{Z}_p \otimes (\mathbb{Q}\mathbb{Z}_p \wr \mathbb{Q}\mathbb{Z}_p)$,
- $V\left(\mathbb{Z}_p^3, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right)$.

Lemma

$\mathrm{Aut}\left(V\left(\mathbb{Z}_p^3, \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)\right)\right) = \mathbb{Z}_p^3 \rtimes \mathbb{Z}_p$, *which has a unique regular subgroup isomorphic to* $\mathbb{Z}_p^3$.

Theorem (Dobson)

$\mathbb{Z}_p^3$ *is a DCI-group.*

Lemma

$\mathrm{Aut}\left(V\left(\mathbb{Z}_p^3, \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)\right)\right) = \mathbb{Z}_p^3 \rtimes \mathbb{Z}_p$, *which has a unique regular subgroup isomorphic to* $\mathbb{Z}_p^3$.

Theorem (Dobson)

$\mathbb{Z}_p^3$ *is a DCI-group.*

Star product of S-rings

Definition (Hirasaka–Muzychuk)

Let $\mathcal{A}$ be an S-ring over a group H and E, F be $\mathcal{A}$-subgroups. Then $\mathcal{A}$ is a **star product**, notation:

$$\mathcal{A} = \mathcal{A}_E \star \mathcal{A}_F$$

if the following hold:

- $EF = H$ and $E \cap F$ is normal in F.
- For every $T \in \mathrm{Bsets}(\mathcal{A})$ such that $T \subseteq F \setminus E$, $T = T(E \cap F)$.
- For every basic set $T \in \mathrm{Bsets}(A)$ such that $T \subset H \setminus (E \cup F)$, $T = RS$, where $R \in \mathrm{Bsets}(E)$ and $S \in \mathrm{Bsets}(F)$.

The star product generalizes the tensor and wreath products:

$\mathcal{A}_E \star \mathcal{A}_F = \mathcal{A}_E \otimes \mathcal{A}_F$ when $E \cap F = \{e\}$,

$\mathcal{A}_E \star \mathcal{A}_F = \mathcal{A}_E \wr \mathcal{A}_{H/E}$ when $F = H$.

Star product of S-rings

Definition (Hirasaka–Muzychuk)

Let $\mathcal{A}$ be an S-ring over a group H and E, F be $\mathcal{A}$-subgroups. Then $\mathcal{A}$ is a **star product**, notation:

$$\mathcal{A} = \mathcal{A}_E \star \mathcal{A}_F$$

if the following hold:

- $EF = H$ and $E \cap F$ is normal in F.
- For every $T \in \mathrm{Bsets}(\mathcal{A})$ such that $T \subseteq F \setminus E$, $T = T(E \cap F)$.
- For every basic set $T \in \mathrm{Bsets}(A)$ such that $T \subset H \setminus (E \cup F)$, $T = RS$, where $R \in \mathrm{Bsets}(E)$ and $S \in \mathrm{Bsets}(F)$.

The star product generalizes the tensor and wreath products:

$\mathcal{A}_E \star \mathcal{A}_F = \mathcal{A}_E \otimes \mathcal{A}_F$ when $E \cap F = \{e\}$,

$\mathcal{A}_E \star \mathcal{A}_F = \mathcal{A}_E \wr A_{H/E}$ when $F = H$.

Theorem (Hirasaka–Muzychuk)

If $\mathcal{A}$ is an S-ring over $H = \mathbb{Z}_p^n$ such that $\mathcal{A} = \mathcal{A}_E \star \mathcal{A}_F$, and both $\mathcal{A}_E$ and $\mathcal{A}_{F/(E\cap F)}$ are CI-S-rings, then $\mathcal{A}$ is also a CI-S-ring.

Theorem (Hirasaka–Muzychuk; Morris)

$\mathbb{Z}_p^4$ is a DCI-group.

Definition (Leung–Man; Evdokimov–Ponomarenko)

Let $\mathcal{A}$ be an S-ring over a group H and E, F be $\mathcal{A}$-subgroups such that $E \le F$ and E is normal in H. Then $\mathcal{A}$ is a **generalized wreath product GWP** (or **wedge product**)

$$\mathcal{A} = \mathcal{A}_F \wr_{F/E} \mathcal{A}_{H/E}$$

if for every $T \in \mathrm{Bsets}(\mathcal{A})$ such that $T \not\subseteq F$, T is a union of E-cosets.

A GWP is **non-trivial** if $E \neq 1$ and $F \neq G$.

Example:

$H = \mathbb{Z}_8$.

$\mathcal{A} = \langle \underline{0}, \underline{1, 5}, \underline{3, 7}, \underline{2}, \underline{4}, \underline{6} \rangle$

$E = \{0, 4\}$ and $F = \{0, 2, 4, 6\}$.

$\mathcal{A}_F = \mathbb{Q}\mathbb{Z}_4$ and $\mathcal{A}_{H/E} = \mathbb{Q}\mathbb{Z}_4$

Note that, while both $\mathcal{A}_F$ and $\mathcal{A}_E$ are CI-S-rings, $\mathcal{A}$ is not.

Question

Let $\mathcal{A}$ be a Schurian p-S-ring over $\mathbb{Z}_p^5$ such that it is a non-trivial GWP of two CI-S-rings. Does this imply that $\mathcal{A}$ is a CI-S-ring?

Example:

$H = \mathbb{Z}_8$.

$\mathcal{A} = \langle \underline{0}, \underline{1,5}, \underline{3,7}, \underline{2}, \underline{4}, \underline{6} \rangle$

$E = \{0, 4\}$ and $F = \{0, 2, 4, 6\}$.

$\mathcal{A}_F = \mathbb{Q}\mathbb{Z}_4$ and $\mathcal{A}_{H/E} = \mathbb{Q}\mathbb{Z}_4$

Note that, while both $\mathcal{A}_F$ and $\mathcal{A}_E$ are CI-S-rings, $\mathcal{A}$ is not.

Question

Let $\mathcal{A}$ be a Schurian p-S-ring over $\mathbb{Z}_p^5$ such that it is a non-trivial GWP of two CI-S-rings. Does this imply that $\mathcal{A}$ is a CI-S-ring?

Every Schurian p-S-ring over $\mathbb{Z}_p^3$ is either a non-trivial GWP or it is normal with automorphism group of order at most p^4.

Definition

An S-ring $\mathcal{A}$ over H is **normal** if H_R is normal in $\mathrm{Aut}(\mathcal{A})$.

Theorem (K–Feng)

Every Schurian p-S-ring over $\mathbb{Z}_p^4$ is is either a non-trivial GWP or it is normal with automorphism group of order at most p^5.

THANK YOU FOR ATTENTION!

Every Schurian p-S-ring over $\mathbb{Z}_p^3$ is either a non-trivial GWP or it is normal with automorphism group of order at most p^4.

Definition

An S-ring $\mathcal{A}$ over H is **normal** if H_R is normal in $\mathrm{Aut}(\mathcal{A})$.

Theorem (K–Feng)

Every Schurian p-S-ring over $\mathbb{Z}_p^4$ is is either a non-trivial GWP or it is normal with automorphism group of order at most p^5.

THANK YOU FOR ATTENTION!

Every Schurian p-S-ring over $\mathbb{Z}_p^3$ is either a non-trivial GWP or it is normal with automorphism group of order at most p^4.

Definition

An S-ring $\mathcal{A}$ over H is **normal** if H_R is normal in $\mathrm{Aut}(\mathcal{A})$.

Theorem (K–Feng)

Every Schurian p-S-ring over $\mathbb{Z}_p^4$ is is either a non-trivial GWP or it is normal with automorphism group of order at most p^5.

THANK YOU FOR ATTENTION!