Arc-transitive Bicirculants

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Definitions

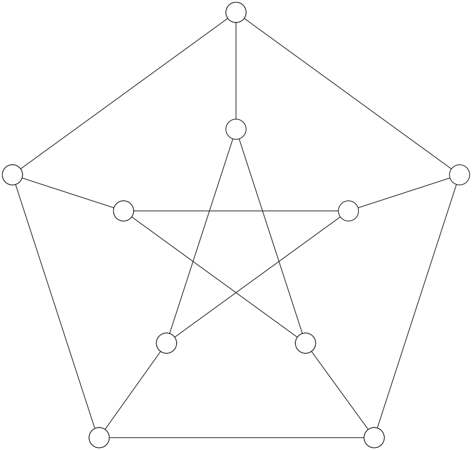
- An automorphism of a graph $X = (V, E)$ is an isomorphism of X with itself. Thus each automorphism α of X is a permutation of the vertex set V which preserves adjacency.
- A graph X is vertex-transitive if $\mathrm{Aut}(X)$ acts transitively on the set of vertices $V(X)$.
- A graph X is edge-transitive if its automorphism group acts transitively on edges.
- An s-arc in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i < s$, and also $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$.
- A graph X is arc-transitive (or symmetric) if its automorphism group acts transitively on arcs.

Cayley graphs

Let G be a finite group with identity element 1, and let $S \subset G \backslash \{1\}$ be such that $S^{-1} = S$. We define the **Cayley graph** $Cay(G, S)$ on the group G with respect to the connection set S, to be the graph with vertex set G, with edges of the form $\{g, gs\}$ for $g \in G$ and $s \in S$. A **circulant** of order n is a Cayley graph on a cyclic group of order n.

Every Cayley graph is vertex-transitive, but not every vertex-transitive graph is a Cayley graph. Smallest such example is the Petersen graph.

Petersen graph



Semiregular automorphisms

Semiregular permutation

An element of a permutation group is semiregular, more precisely (m, n)-semiregular, if it has m orbits of size n.

Circulant

A circulant is a graph admitting a semiregular automorphism with two orbits.

The classification of connected arc-transitive circulants has been obtained independently by Kovacs and Li in 2003.

Theorem (Kovacs, Li, 2003)

Let X be a connected arc-transitive circulant of order n. Then one of the following holds:

(i) $X \cong K_n$;

(ii) $X = \Sigma[\overline{K}_d]$, *where* $n = md$, $m, d > 1$ *and* Σ *is a connected arc-transitive circulant of order* m;

(iii) $X = \Sigma[\overline{K}_d] - d\Sigma$, *where* $n = md$, $d > 3$, $gcd(d, m) = 1$ *and* Σ *is a connected arc-transitive circulant of order* m;

(iv) X *is a normal circulant.*

Arc-transitive bicirculants

Bicirculant

A bicirculant is a graph admitting a semiregular automorphism with two orbits.

The vertices of a bicirculant graph can be labeled by x_i and y_i, $i \in \mathbb{Z}_n$, and its edge set can be partitioned into three subsets

$$\mathcal{L} = \cup_{i \in \mathbb{Z}_n} \{\{x_i, x_{i+l}\} \mid l \in L\},$$

$$\mathcal{M} = \cup_{i \in \mathbb{Z}_n} \{\{x_i, y_{i+m}\} \mid m \in M\},$$

$$\mathcal{R} = \cup_{i \in \mathbb{Z}_n} \{\{y_i, y_{i+r}\} \mid r \in R\},$$

where L, M, R are subsets of $\mathbb{Z}_n$ such that $L = -L$, $R = -R$, $M \neq \emptyset$ and $0 \notin L \cup R$. Such bicirculant is denoted by $BC_n[L, M, R]$.

Examples

The generalized Petersen graphs are bicirculants.

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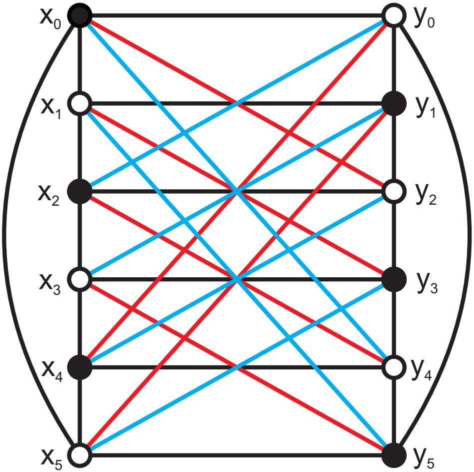

$$BC_6[\{\pm 1\}, \{0, 2, 4\}\ \{\pm 1\}]$$

Isomorphic bicirculants

Let L, M and R be subsets of $\mathbb{Z}_n$ such that $L = -L$, $R = -R$, $M \neq \emptyset$ and $0 \notin L \cap R$. Then we have:

$$BC_n[L, M, R] \cong BC_n[\lambda L, \lambda M + \mu, \lambda R] \ (\lambda \in \mathbb{Z}_n^*, \ \mu \in \mathbb{Z}_n)$$

with the isomorphism $\phi_{\lambda,\mu}$ given by:

$$\begin{aligned} \phi_{\lambda,\mu}(x_i) &= x_{\lambda i + \mu} \text{ and,} \\ \phi_{\lambda,\mu}(y_i) &= y_{\lambda i}. \end{aligned}$$

Therefore, we can without loss of generality assume that $0 \in M$.
Some graphs may have two (or more) different bicirculant representations, for example:

$$K_{6,6} - 6K_2 \cong BC_6[\{\pm 1\}, \{0, 2, 4\} \{\pm 1\}] \cong BC_6[\,\emptyset, \{0, 1, 2, 3, 4\}, \emptyset\,].$$

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Frucht, Graver and Watkins, 1971

There are only seven arc-transitive generalized Petersen graphs: $\mathrm{GP}(4,1)$, $\mathrm{GP}(5,2)$, $\mathrm{GP}(8,3)$, $\mathrm{GP}(10,2)$, $\mathrm{GP}(10,3)$, $\mathrm{GP}(12,5)$, and $\mathrm{GP}(24,5)$.

The classification of connected cubic arc-transitive bicirculants follows from results obtained in

- R. Frucht, J. E. Graver and M. E. Watkins, The groups of the generalized Petersen graphs, *Proc. Camb. Phil. Soc.* **70** (1971), 211–218.
- D. Marušič and T. Pisanski, Symmetries of hexagonal molecular graphs on the torus, *Croat. Chem. Acta* **73** (2000), 969–981.
- T. Pisanski, A classification of cubic bicirculants, *Discrete Math.* **307** (2007), 567–578.

Cubic arc-transitive bicirculants

Classification of cubic arc-transitive bicirculants

A connected cubic symmetric graph is a bicirculant if and only if it is isomorphic to one of the following graphs:

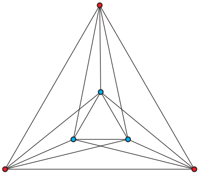
- the complete graph K_4,
- the complete bipartite graph $K_{3,3}$,
- the seven symmetric generalized Petersen graphs $GP(4,1)$, $GP(5,2)$, $GP(8,3)$, $GP(10,2)$, $GP(10,3)$, $GP(12,5)$, and $GP(24,5)$,
- the Heawood graph $F014A$, and
- a Cayley graph $Cay(D_{2n}, \{b, ba, ba^{r+1}\})$ on a dihedral group $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1 \rangle$ of order $2n$ with respect to the generating set $\{b, ba, ba^{r+1}\}$, where $n \geq 11$ is odd and $r \in \mathbb{Z}_n^*$ is such that $r^2 + r + 1 \equiv 0 \pmod{n}$.

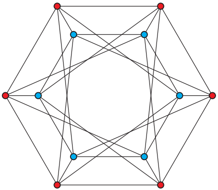
The classification of connected tetravalent arc-transitive bicirculants follows from results obtained in

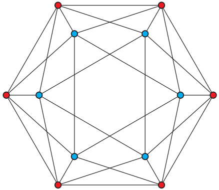
- I. Kovács, K. Kutnar and D. Marušič, Classification of edge-transitive rose window graphs, *J. Graph Theory* **65** (2010), 216–231.
- I. Kovács, B. Kuzman, A. Malnič and S. Wilson, Characterization of edge-transitive 4-valent bicirculants, *J. Graph Theory* **69** (2012), 441–463.
- I. Kovács, B. Kuzman and A. Malnič, On non-normal arc transitive 4-valent dihedrants, *Acta Math. Sinica, English Series* **26** (2010), 1485–1498.

Tabačjn graphs $T(n; a, b; r)$

The pentavalent generalization of the generalized Petersen graphs are the so-called *Tabačjn graphs*, that is, a Tabačjn graph is a bicirculant $BC_n[\{\pm 1\}, \{0, a, b\}, \{\pm r\}]$. (In the original notation for Tabačjn graphs, this graph would be denoted by $T(n; a, b; r)$).







Theorem (Arroyo, Guillot, Hubard, KK, O'Reilly and Šparl, 2014)

The complete graph $K_6 \cong T(3; 1, 2; 1)$, the complete bipartite graph minus a perfect matching $K_{6,6} - 6K_2 \cong T(6; 2, 4; 1)$, and the icosahedron $T(6; 1, 5; 2)$ are the only arc-transitive Tabačjn graphs.

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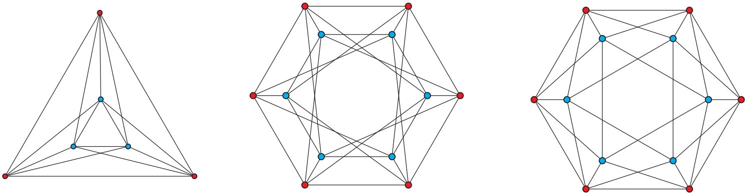

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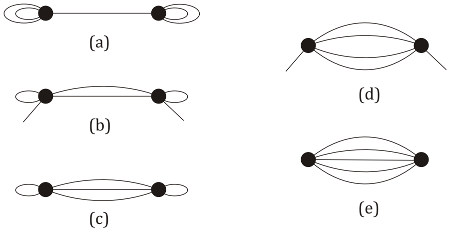
Tabachin

Pentavalent bicirculants

Multigraphs that can occur as quotient multigraphs of pentavalent bicirculants with respect to a $(2, n)$-semiregular automorphism ρ.



$|M| = 1$

Theorem (Kovacs, Malnič, Marušič, Miklavič, 2009)

Let $X = BC_n[L, M, R]$ be a pentavalent bicirculant with $|M| = 1$. Then X is not arc-transitive.

$|M| = 2$

Theorem

Let X be a connected pentavalent arc-transitive bicirculant $X = BC_n[L, M, R]$ with $|M| = 2$. Then either:

- $n = 6$ *and* $X \cong BC_6[\{\pm 1, 3\}, \{0, 2\}, \{\pm 1, 3\}] \cong K_{6,6} - 6K_2$, *or*
- $n = 8$ *and* $X \cong BC_8[\{\pm 1, 4\}, \{0, 2\}, \{\pm 3, 4\}]$

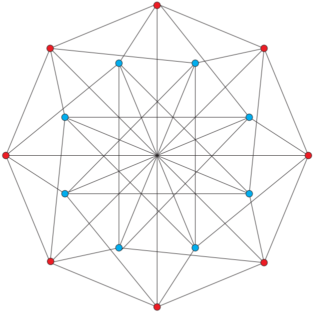


Figure : $BC_8[\{\pm 1, 4\}, \{0, 2\}, \{\pm 3, 4\}]$

Theorem

Let $X = BC_n[L, M, R]$ be a pentavalent bicirculant with $|M| = 4$. Then X is not arc-transitive.

Core-free bicirculants

Recall that the *core* of a subgroup K in a group G (denoted by $core_G(K)$) is the largest normal subgroup of G contained in K.

Definition

A bicirculant $X = BC_n[L, M, R]$ of order $2n$ is said to be *core-free* if there exists a $(2, n)$-semiregular automorphism $\rho \in \mathrm{Aut}(X)$ giving rise to the prescribed bicirculant structure of X such that the cyclic subgroup $\langle \rho \rangle$ has trivial core in $\mathrm{Aut}(X)$.

Theorem (Lucchini, 1998)

If H is a cyclic subgroup of a finite group G with $|H| \geq \sqrt{|G|}$, then H contains a non-trivial normal subgroup of G.

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Pentavalent arc-transitive graphs

Theorem (Guo and Feng, 2012)

Let X be a connected pentavalent (G, s)-transitive graph for some $G \leq Aut(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 5$ and one of the following holds:

(i) *For $s = 1$, $G_v \cong \mathbb{Z}_5, D_{10}$ or D_{20};*
(ii) *For $s = 2$, $G_v \cong F_{20}$, $F_{20} \times \mathbb{Z}_2$, A_5 or S_5;*
(iii) *For $s = 3$, $G_v \cong F_{20} \times \mathbb{Z}_4$, $A_4 \times A_5$, $S_4 \times S_5$ or $(A_4 \times A_5) \rtimes \mathbb{Z}_2$ with $A_4 \rtimes \mathbb{Z}_2 = S_4$ and $A_5 \rtimes \mathbb{Z}_2 = S_5$;*
(iv) *For $s = 4$, $G_v \cong \mathrm{ASL}(2,4)$, $\mathrm{AGL}(2,4)$, $A\Sigma L(2,4)$ or $AXL(2,4)$;*
(v) *For $s = 5$, $G_v \cong \mathbb{Z}_2^6 \rtimes XL(2,4)$.*

Therefore, if $X = BC_n[L, M, R]$ is core-free pentavalent bicirculant, which is $(Aut(X), 1)$-transitive, then $n < 40$, and if it is $(Aut(X), 2)$-transitive then $n < 240$.

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Let $X = BC_n[L, M, R]$ be a pentavalent bicirculant with $|M| = 3$. Then X is not 3-arc-transitive.

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Theorem

The only pentavalent arc-transitive bicirculants $BC_n[L, M, R]$ *with* $|M| = 3$ *are* $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}]$, $BC_6[\{\pm 1\}, \{0, 2, 4\}, \{\pm 1\}]$ *and* $BC_6[\{\pm 1\}, \{0, 1, 5\}, \{\pm 2\}]$. *Moreover, the first two are 2-transitive and the third one is 1-transitive.*

Proof.

If X is not core-free, then there exist a normal subgroup $N \leq \langle \rho \rangle$. The quotient graph X_N is a core-free pentavalent arc-transitive bicirculant, and X is a regular $\mathbb{Z}_m$-cover of X_N, where $|N| = m$. We then use the graph covering techniques to finish the proof. □

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$|M| = 5$

A pentavalent bicirculant $BC_n[\emptyset, M, \emptyset]$ is a Cayley graph on Dihedral group D_{2n}. Such graphs are also called dihedrants.

Lemma (Marušič, 2006)

Let X be a connected pentavalent 2-arc-transitive dihedrant. Then X is core-free if and only if X is isomorphic to the complete bipartite graph minus a matching $K_{6,6} - 6K_2$, or to the points-hyperplanes incidence graph of projective space $B(PG(2, 4))$.

To find all core-free pentavalent arc-transitive dihedrants, it suffices to check the graphs of order $2n$, where $n < 40$. With use of MAGMA, we obtain that the only such graphs are those mentioned in the previous lemma.

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$|M| = 5$

Proposition

Let X be a connected pentavalent arc-transitive bipartite dihedrant. Then X is isomorphic to one of the following graphs:

- $K_{6,6} - 6K_2$,
- $BC_{12}[\emptyset, \{0, 1, 2, 4, 9\}, \emptyset]$,
- $BC_{24}[\emptyset, \{0, 1, 3, 11, 20\}, \emptyset]$,
- $B(PG(2, 4))$,
- $Cay(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, ba^{r^3+r^2+r+1}\})$ where $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1\rangle$ and $r \in \mathbb{Z}_n^*$ such that $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod{n}$.

Pentavalent arc-transitive bicirculants

Theorem (Antončič, Hujdurović, KK, JACO, 2014)

A connected pentavalent bicirculant $BC_n[L, M, R]$ is arc-transitive if and only if it is isomorphic to one of the following graphs:

(i) $|M| = 1$: no graphs;

(ii) $|M| = 2$: $BC_6[\{\pm 1, 3\}, \{0, 2\}, \{\pm 1, 3\}]$ and $BC_8[\{\pm 1, 4\}, \{0, 2\}, \{\pm 3, 4\}]$;

(iii) $|M| = 3$: K_6, $K_{6,6} - 6K_2$, $BC_6[\{\pm 1\}, \{0, 1, 5\}, \{\pm 2\}]$;

(iv) $|M| = 4$: no graphs

(v) $|M| = 5$: $K_{6,6} - 6K_2$, $BC_{12}[\,\emptyset, \{0, 1, 2, 4, 9\}, \emptyset\,]$, $BC_{24}[\,\emptyset, \{0, 1, 3, 11, 20\}, \emptyset\,]$, $B(PG(2, 4))$ or $Cay(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, ba^{r^3+r^2+r+1}\})$ where $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1\rangle$, and $r \in \mathbb{Z}_n^*$ such that $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod{n}$.

Thank you!