Polygons and polyhedra over finite field.

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Regular stars of order n (Wildberger N. J.)

Definition:

A regular star of order n is a list of n distinct mutually non-parallel lines $[l_0, l_1, \ldots, l_{n-1}]$ such that $l_{k+1} = \Sigma_{l_k}(l_{k-1})$ for all k, with the convention that $l_k \equiv l_{k+n}$ for all k, and the convention that

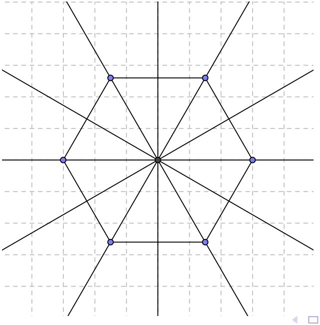
$$[l_0, l_1, \ldots l_{n-2}, l_{n-1}] \equiv [l_1, l_2, \ldots l_{n-1}, l_0]$$
$$[l_0, l_1, \ldots l_{n-2}, l_{n-1}] \equiv [l_{n-1}, l_{n-2} \ldots l_1, l_0]$$

Regular n-gons

Suppose that $[l_0, l_1, \ldots, l_{n-1}]$ is a regular star of order n with common intersection O. Choose a point A_0 on l_0 distinct from O. Define the sequence

$$A_0,\ A_2 = \sigma_{l_1}(A_0),\ A_4 = \sigma_{l_3}(A_2),\ A_6 = \sigma_{l_5}(A_4),\ \ldots$$

and so on, so that lies on l_{2k} by induction.
Connect the points by the lines sequentially.



Definition:

If list of lines $[l_0, l_1, \ldots, l_{n-1}]$ is a regular star of order n with the common intersection O of the lines, one obtains by this construction a set of n points(vertices)with the convention that the squared distances between its (adjacent) vertices are all equal modulo p, and lines which pass through these points (sides) is called a regular polygon over finite field F_p.

The following results are obtained by Norman Wildberger:

Theorem 2.1 (Order three star)
A regular star of order three exists precisely when the number 3 is a non-zero square.

Theorem 2.2 (Order five star)
A regular star of order five exists precisely when there is a non-zero number r satisfying the conditions

$$i)\ r^2 = 5,$$
$$ii)\ 2(5 - r)$$

is the square.

Theorem 2.3 (Order seven star)
A regular star of order five exists precisely when there is a non-zero number r for which

$$7 - 56s + 112s^2 - 64s^3 = 0$$

and such that $s(1 - s)$ is square

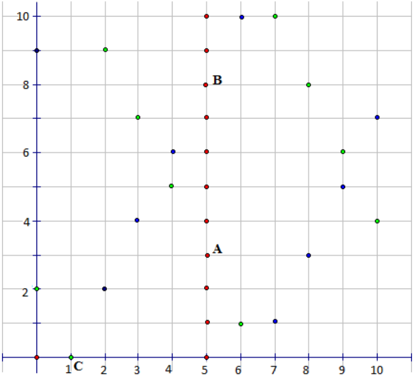
Regular triangle over finite field F_p

Let ΔABC be a regular triangle over $\mathbb{R}$, with the coordinates of the vertices

$$A(-\tfrac{1}{2};\tfrac{\sqrt{3}}{2}),\ B(-\tfrac{1}{2};-\tfrac{\sqrt{3}}{2}),\ C(1;0)$$

In the field F_{11}:

$$A(5;3),\ B(5;8),\ C(1;0)$$



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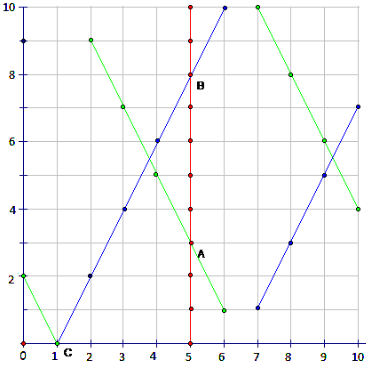
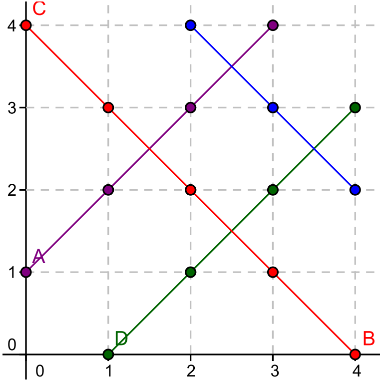
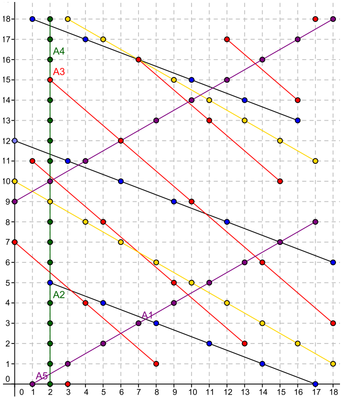


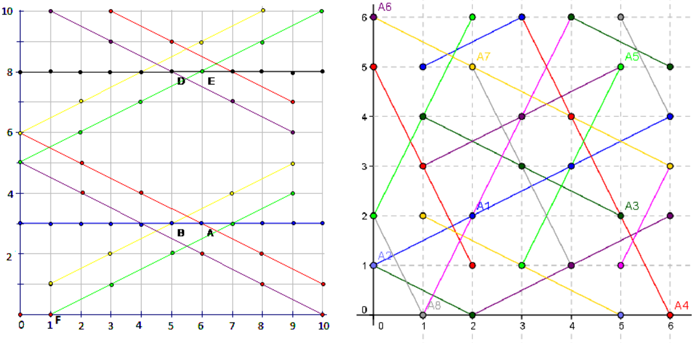
Square over F_5



Regular pentagon over F_{19}



Regular hexagon over F_{11} and regular octagon over F_7



a) b)

Irregular triangle with the coordinates

$$A_1(6;2),\ A_2(1;0),\ A_3(6,11)$$

$$|A_1A_2|^2 = 29,\ |A_2A_3|^2 = 146,\ |A_3A_1|^2 = 81$$

In the field F_{13}:

$$|A_1A_2|^2 \equiv |A_2A_3|^2 \equiv |A_3A_1|^2 \equiv 3 \pmod{13}$$

$\Delta A_1A_2A_3$ is regular triangle over finite field F_{13}

REGULAR HEXAHEDRON OVER FINITE FIELD F_p

Definition:

A regular hexahedron in F_p^3 is a set of points and lines which form the 6 squares over F_p, any two of which intersects along a common side and two common vertices or disjoint . Every vertex of the such squares is a vertex of the other two.

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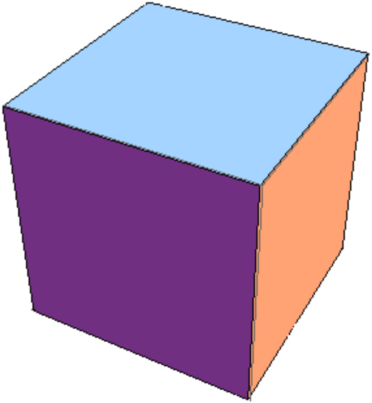
Theorem 1:

A regular hexahedron exists over any field F_p



REGULAR TETRAHEDRON OVER FINITE FIELD F_p

Definition:

A regular tetrahedron in F_p^3 is a set of four regular triangles over F_p intersecting pairwise along a common side.

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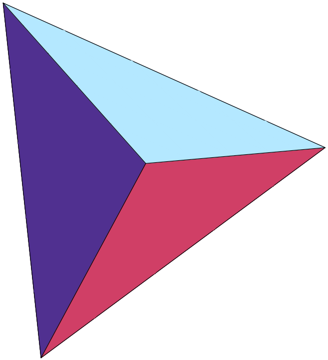
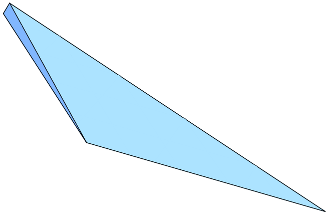
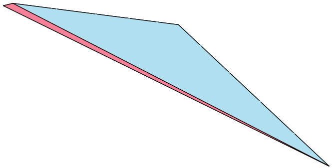
Regular tetrahedron in F_{47}^3



Regular tetrahedron in F_{47}^3



Regular tetrahedron in F_{73}^3



REGULAR OCTAHEDRON OVER FINITE FIELD F_p

Definition:

A regular octahedron in F_p^3 is a set of points and lines which form the 8 regular triangles over F_p, any two of which intersects along a common side and two common vertices or disjoint . Every vertex of the such triangles is a vertex of the other three.

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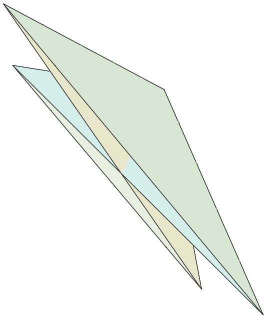
A regular octahedron in F_p^3 is a set of points and lines which form the 8 regular triangles over F_p, any two of which intersects along a common side and two common vertices or disjoint . Every vertex of the such triangles is a vertex of the other three.

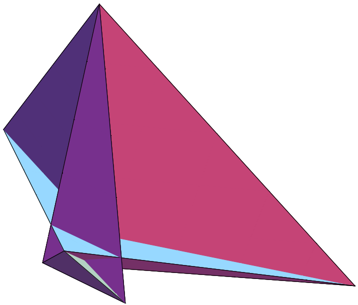
Theorem 3:

A regular octahedron exists over any field F_p

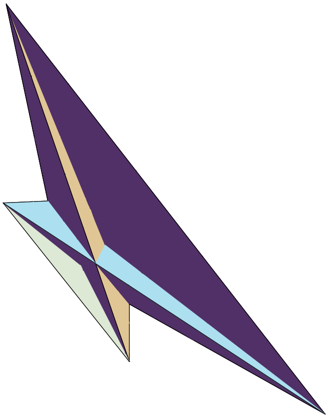
Regular tetrahedron in F_7^3



Regular tetrahedron in F_{17}^3



Regular tetrahedron in F_{41}^3



REGULAR DODECAHEDRON OVER FINITE FIELD F_p

Definition:

A regular dodecahedron in F_p^3 is a set of points and lines which form the 12 regular pentagons over F_p, any two of which intersects along a common side and two common vertices or disjoint . Every vertex of the such pentagons is a vertex of the other two.

Regular polyhedra over finite field F_p

REGULAR DODECAHEDRON OVER FINITE FIELD F_p

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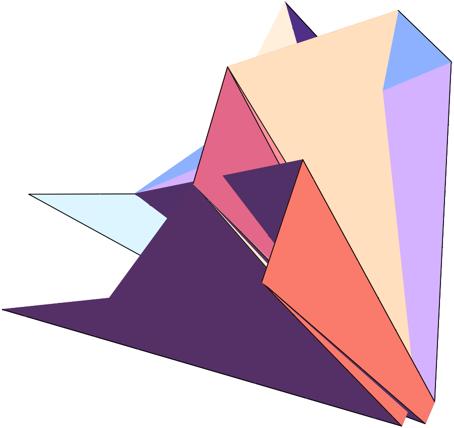
A regular dodecahedron in F_p^3 is a set of points and lines which form the 12 regular pentagons over F_p, any two of which intersects along a common side and two common vertices or disjoint . Every vertex of the such pentagons is a vertex of the other two.

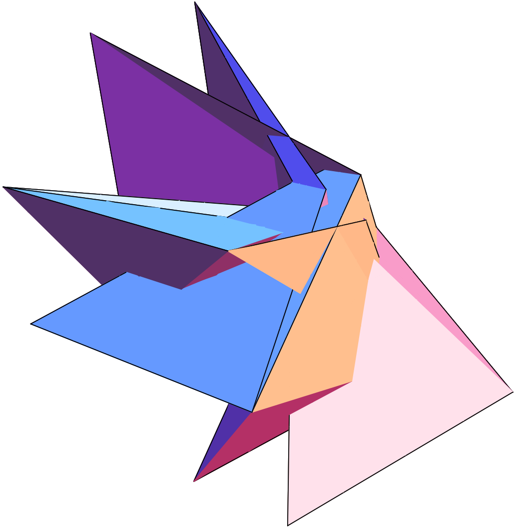
Theorem 4:

A regular dodecahedron exists over the finite field F_p for $p \equiv \{0, 1, 4\} \pmod 5$

Regular dodecahedron in F_{19}^3



Regular dodecahedron in F_{59}^3



REGULAR ICOSAHEDRON OVER FINITE FIELD F_p

Definition:

A regular icosahedron in F_p^3 is a set of points and lines which form the 20 regular triangles over F_p, any two of which intersects along a common side and two common vertices or has a one common vertex or disjoint . Every vertex of the such triangles is a vertex of the other four.

REGULAR ICOSAHEDRON OVER FINITE FIELD F_p

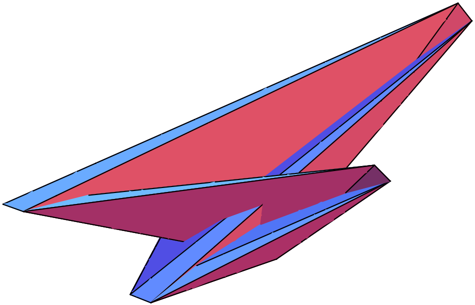
Definition:

A regular icosahedron in F_p^3 is a set of points and lines which form the 20 regular triangles over F_p, any two of which intersects along a common side and two common vertices or has a one common vertex or disjoint . Every vertex of the such triangles is a vertex of the other four.

Theorem 5:

A regular icosahedron exists over the finite field F_p for $p \equiv \{0, 1, 4\} \pmod 5$

Regular icosahedron in F_{29}^3



Regular icosahedron in F_{71}^3