On graphs with automorphism groups admitting a partition

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Introduction

1. A graph here is a finite connected multigraph without loops.
2. The genus of a graph is its cyclomatic number.
3. A graph is said to be γ-hyperelliptic if it is a two-fold covering of a genus γ graph.

Here we treat graphs as one dimensional analogue of Riemann surfaces.

Introduction

A finite group G is said to admit a *partition* if it can be expressed as a set-theoretic union of subgroups, with pairwise trivial intersections.

Given a compact Riemann surface M with automorphism group G_0, we can obtain a quotient surfaces M/G_i, where G_i are subgroups of G_0. Accola derived the formula relating the genera of M/G_i to the orders of G_i provided G_0 admits a partition. Taniguchi generalized this result to finite groups acting on a compact Hausdorff space.

Theorem (Taniguchi)

Let X be a graph on which a finite group G_0 acts, and assume that G admits a partition $\{G_1, G_2, \dots, G_s\}$. Then we have

$$(s-1)g(X) + |G_0|g(X/G_0) = \sum_{i=1}^{s} |G_i|g(X/G_i).$$

Main result

Theorem

Let X be a two-fold covering of a hyperelliptic graph Y of genus $g \geq 2$. Then X is γ-hyperelliptic for some $\gamma \leq \left[\frac{g-1}{2}\right]$.

How to prove:

1. We have $\varphi : X \to Y$ and $\psi : Y \to T$. The composite mapping $\Phi = \varphi \circ \psi$ is a harmonic morphism.
2. Denote by $\mathbf{X}$ and $\mathbf{T}$ graph of groups. The map $\Phi : X \to T$ can be naturally extended to the covering $\mathbf{\Phi} : \mathbf{X} \to \mathbf{T}$ of graph of groups.
3. Let $\mathrm{F} = \pi_1(\mathbf{X})$ and $\Gamma = \pi_1(\mathbf{T})$ be the fundamental groups and $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{T}}$ be the universal covering trees of graphs of groups $\mathbf{X}$ and $\mathbf{T}$ respectively.
4. By the Bass uniformization theorem there exists a lift of $\mathbf{\Phi}$ to an isomorphism $\widetilde{\mathbf{\Phi}} : \widetilde{\mathbf{X}} \to \widetilde{\mathbf{T}}$ between covering trees equivariant under the action of H and Γ on $\widetilde{\mathbf{X}}$ and $\widetilde{\mathbf{T}}$ respectively.

How to proof

5. We have $\mathbf{X} \cong \widetilde{\mathbf{X}}/\mathrm{H}$ and $\mathbf{T} \cong \widetilde{\mathbf{T}}/\Gamma$.
6. Replace the covering $\mathbf{\Phi} : \mathbf{X} \to \mathbf{T}$ by the covering $\mathbf{\Phi}' : \widetilde{\mathbf{X}}/\mathrm{H} \to \widetilde{\mathbf{X}}/\Gamma$ induced by the group inclusion $H < \Gamma$.
7. By Lemma, F is a normal subgroup of index 4 in Γ. Therefore the covering transformation group of $\mathbf{\Phi}'$ is $G_0 = \Gamma/H = D_4$.
8. D_4 admits a partition $\{G_1, G_2, G_3\}$ into three subgroups of order two.
9. Use $(s-1)g(X) + |G_0|g(X/G_0) = \sum_{i=1}^{s} |G_i|g(X/G_i)$.
   We get $g - 1 = g_1 + g_2$. The possible cases for g_1 and g_2 are

g_1	g_2
0	$g-1$
1	$g-2$
$\dots$	$\dots$
$\left[\frac{g-1}{2}\right]$	$\left[\frac{g-1}{2}\right](+1,$ if g is even).

Choosing the smaller genus in each case, we get that X is γ-hyperelliptic for some $\gamma \leq \left[\frac{g-1}{2}\right]$.

Supporting lemma

Lemma

Let Γ be the free product of $g+1$ copies of $\mathbb{Z}_2$ and $g \geq 1$. Then it contains a normal free subgroup of index 4 on $2g-1$ generators. Moreover, if $F < \Gamma$ is a free subgroup of index 4 on $2g-1$ generators, then $F \lhd \Gamma$.

How to prove:

For existence: Construct a homomorphism σ of Γ onto the Dihedral group of order 4. Show by Reidemeister-Schreier method that $\operatorname{Ker}\sigma$ is free and on $2g-1$ generators.

For the second part: By letting Γ act on the cosets of F, have transitive representation $\theta : \Gamma \to S_4$. Understand that actually $\theta : \Gamma \to D_4$ and $\operatorname{Ker}\theta = F$.

corollaries

The immediate consequences of the theorem are the assertions below. The first one has been proved by I. A. Mednykh by sophisticated methods.

Corollary 1

Suppose X is a graph of genus three which is a two-fold covering of a graph Y of genus two. Then X is hyperelliptic.

Corollary 1

If X is a graph of genus five which is a two-fold covering of a hyperelliptic graph of genus three, then X is hyperelliptic or 1-hyperelliptic.