Algorithmic aspects related to the structure of lifted groups along covering projections

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Joint work with Rok Požar

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Regular covering projection of connected graphs

A surjective mapping $p\colon \tilde{X} \to X$ of connected graphs s.t.
fibers $p^{-1}(v)$ and $p^{-1}(a)$ = **orbits of a semi-regular subgroup** CT_p

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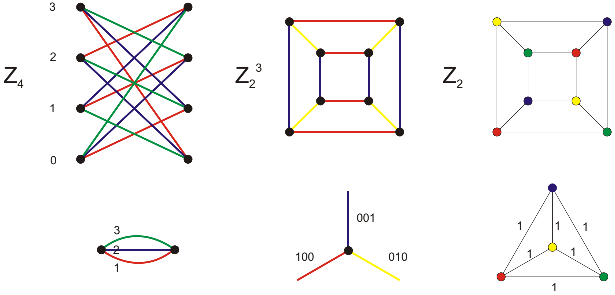

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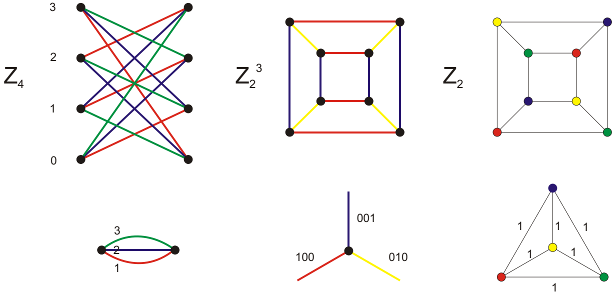

Construction/reconstruction
by a **regular voltage function** $\zeta\colon A(X) \to \Gamma \cong \mathrm{CT}_p$

Topological context

realization of graphs on surfaces: the genus problem

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realization of graphs on surfaces: the genus problem

Algebraic context

studying symmetry of graphs and maps on surfaces

Algebraic context: studying symmetry, I

Tutte, 1947: Graphs of valency 3 **are at most** 5**-arc transitive**.

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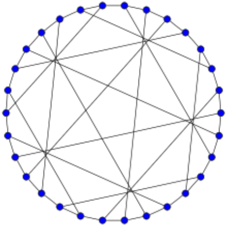

Tutte's 8-cage

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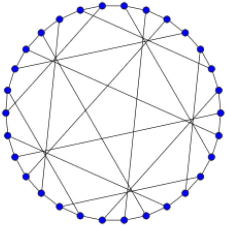

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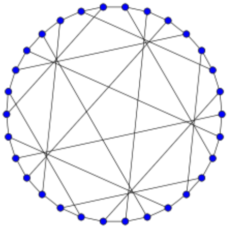

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Algebraic Graph Theory. Biggs, 1974 (Conway)
Djoković, 1974

Algebraic context: studying symmetry, II

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Building catalogs of graphs with prescribed degree of symmetry

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$\geq$ 1999: Boben, Marušič, Orbanić, Pisanski, Žitnik, and others

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$X \to X/N$, $N \triangleleft \mathrm{Aut}\, X$, intransitive

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knowledge of stabilizers
Djoković-Miller, 1980. Conder-Nedela, 2009, Goldschmidt, 1980.

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Reconstruction of the family

yes – **if** reduction involves **covers**.

Lifting automorphisms along regular covering projections

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{g} & X. \end{array}$$

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the lift of G is a again a group, denoted $\tilde{G} \leq \mathrm{Aut}\tilde{X}$

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Djoković, 1974: G s-**arc transitive** $\Rightarrow$ $\tilde{G}$ s-**arc transitive**

Lifting automorphisms – main questions

Lifting conditions in terms of voltages. Well understood
For a given $p\colon \tilde{X} \to X$, does $G \leq \mathrm{Aut}\, X$ have a lift?

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algorithm for testing G-admissibility, for finding the presentation of $\tilde{G}$, for testing whether the extension is split whenever CT_p is solvable, for testing sectional splittings in the abelian case, for the generation of all solvable G-admissible covers up to a given size, for the generation of all G-admissible covers up to a given size

Split extensions with sectional complements over Ω

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Suppose that G lifts as $\tilde{G} \cong \mathrm{CT}_p \rtimes G$

Different complements to CT_p can have very different actions on $\tilde{X}$!

transitive / intransitive

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A_4 lifts as $\mathbb{Z}_2 \times A_4$. All complements are sectional

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S_4 lifts as $\mathbb{Z}_2 \times S_4$. There are intransitive and transitive complements

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S_4 lifts as $\mathbb{Z}_2 \times S_4$. There are intransitive and transitive complements

$\Omega = V(X)$, p: Dodecahedron $\to$ Petersen

A_5 lifts to $\mathbb{Z}_2 \times A_5$. The unique copy of A_5 is transitive

Sectional split lifts: characterization via stabilizers

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Thm. A complement $\bar{\mathrm{G}}$ is sectional over $\Omega = \mathrm{G(b)}$ if and only if the intersection $\bar{\mathrm{G}} \cap \widetilde{\mathrm{G_b}}$ is the stabilizer of some $\tilde{\mathrm{u}} \in \mathrm{fib_b}$.

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Derivations

$$\mathrm{Der}(\mathrm{G}, \mathrm{CT}_\mathrm{p}) = \{\delta \colon \mathrm{G} \to \mathrm{CT}_\mathrm{p} \mid \delta_{\mathrm{gh}} = \delta_\mathrm{g} \theta_\mathrm{g}(\delta_\mathrm{h})\}$$

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Inner derivations

$$\mathrm{Inn}(G, CT_p) = \{\delta^c \colon G \to CT_p \mid \delta^c_g = c^{-1}\theta_g(c)\}$$

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All complements $\leftrightarrow \mathrm{Der}(G, CT_p)$. If CT_p is **abelian**, then $\mathrm{Der}(G, CT_p)$ is a group with $\mathrm{Inn}(G, CT_p)$ a normal subgroup, and

$$\text{complements/conj} \leftrightarrow H^1(G, CT_p) = \mathrm{Der}(G, CT_p)/\mathrm{Inn}(G, CT_p).$$

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Inner derivations

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$$\text{complements/conj} \leftrightarrow \mathrm{H}^1(\mathrm{G}, \mathrm{CT}_\mathrm{p}) = \mathrm{Der}(\mathrm{G}, \mathrm{CT}_\mathrm{p}) / \mathrm{Inn}(\mathrm{G}, \mathrm{CT}_\mathrm{p}).$$

Thm. Let G lift as a sectional split extension over $\Omega = \mathrm{G}(\mathrm{b})$. Then sectional complements $\leftrightarrow \{\delta \in \mathrm{Der}(\mathrm{G}, \mathrm{CT}_\mathrm{p}) \mid \delta|_{\mathrm{G}_\mathrm{b}} \in \mathrm{Inn}(\mathrm{G}_\mathrm{b}, \mathrm{CT}_\mathrm{p})\}$.
If CT_p is **abelian**, then

$$\text{sectional complements/conj} \leftrightarrow \leq \mathrm{H}^1(\mathrm{G}, \mathrm{CT}_\mathrm{p})$$

Counting invariant sections

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Thm. All sectional complements over $\Omega = \mathrm{G}(\mathrm{b})$ have the same number of invariant sections, namely

$$|\mathrm{Fix}(\tilde{\mathrm{G}}_{\tilde{\mathrm{u}}})| = |\mathrm{C}_{\mathrm{CT}_\mathrm{p}}(\tilde{\mathrm{G}}_{\tilde{\mathrm{u}}})|, \text{ where } \tilde{\mathrm{u}} \in \mathrm{fib}_\mathrm{b} \text{ is arbitrary.}$$

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Pro. Let $\bar{\mathrm{G}}$ be a sectional complement with an invariant section $\overline{\Omega}$ over Ω. Then the conjugate subgroup $\mathrm{c}\,\bar{\mathrm{G}}\,\mathrm{c}^{-1}$, $\mathrm{c} \in \mathrm{CT}_\mathrm{p}$, is also a sectional complement, with $\mathrm{c}(\overline{\Omega})$ as an invariant section over Ω.

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Thm. Let $\mathcal{C}$ be a conjugacy class of a sectional complements over $\mathrm{G}(\mathrm{u})$. Then the number of invariant sections through $\tilde{\mathrm{u}} \in \mathrm{fib}_\mathrm{b}$ that belong to some sectional complement from $\mathcal{C}$ does not depend on $\tilde{\mathrm{u}} \in \mathrm{fib}_\mathrm{b}$, and is equal to

$$|\mathrm{Fix}(\tilde{\mathrm{G}}_{\tilde{\mathrm{u}}})| \,/\, |\mathrm{C}_{\mathrm{CT}_\mathrm{p}}(\bar{\mathrm{G}})| = |\mathrm{C}_{\mathrm{CT}_\mathrm{p}}(\tilde{\mathrm{G}}_{\tilde{\mathrm{u}}})| \,/\, |\mathrm{C}_{\mathrm{CT}_\mathrm{p}}(\bar{\mathrm{G}})|.$$

Sectional splits: characterization via a cone over Ω

Define

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Thm (Požar, 2013). Suppose that G lifts along $p\colon Y \to \mathrm{Cone}_X(\Omega)$. If $Z = Y \setminus \mathrm{fib}_*$ is connected, then $\tilde{G}$ along $p_Z\colon Z \to X$ splits with an invariant section over Ω. Also, any $\tilde{X} \to X$ s.t. $\tilde{G}$ splits with an invariant section over Ω arises in this way.

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Note.

We can explicitly find all $\mathbb{Z}_p$-elementary abelian regular coverings along which G lifts in this manner. The problem is reduced to finding invariant subspaces of matrix group linearly representing the action of G on the first homology group $H_1(X, \mathbb{Z}_p)$. (M, Marušič, Potočnik, 2004).

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Note. For $\Omega = V(X)$ we get Biggs' compatibility condition, AGT 1974. Biggs' compatibility condition implies that $\bar{G}$ preserves all vertices in $\tilde{X}$ that are labeled by 1.

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$$\zeta_W = 1 \Rightarrow \zeta_{gW} = 1, \quad \text{for all} \quad W\colon \Omega \to \Omega.$$

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Example

Along $p\colon C_6 \to C_3$ the group $\mathbb{Z}_3$ lifts as $\mathbb{Z}_2 \times \mathbb{Z}_3$

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Note.
Finding the right voltage assignment is exponentially difficult ! However, for **abelian** covers **there is an efficient algorithm**.

Abelian covers: Finding sectional complements over $G(b)$

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Note.
Symbolic computation can be avoided., and can be carried out over $\mathbb{Z}$. The set of solutions, reduced modulo the defining relations for Γ, are in bijective correspondence with all sectional complements, which, when reduced modulo inner derivations correspond to a subgroup in $H^1(G, \Gamma)$.

Thank you!