On isospectrality of genus two graphs

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Isospectral surfaces and graphs

Peter Buser (1992) posed the following interesting problem: are two isospectral Riemann surfaces of genus two isometric? Up to our knowledge the problem is still open but, probably, can be solved positively. The aim of this paper is to give a positive solution of this problem for graphs of genus two. Because of the close link between Riemann surfaces and graphs we hope that our result will be helpful to make a progress in solution of the Buser problem.

Laplace operator on graphs

In this report graphs are supposed to be unoriented, but they may have loops and multiple edges. Denote by $V(G)$ and $E(G)$ the set of vertices and edges of a graph G respectively. For each $u, v \in V(G)$, we set a_{uv} to be equal to the number of edges between u and v.
The matrix $A = A(G) = [a_{uv}]_{u,v \in V(G)}$, is called the *adjacency matrix* of the graph G. Let $d(v)$ denote the valency of $v \in V(G)$, $d(v) = \sum_u a_{uv}$, and let $D = D(G)$ be the diagonal matrix indexed by $V(G)$ and with $d_{vv} = d(v)$.
The matrix $L = L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G.
Throughout the paper we shall denote by $\mu(G, x)$ the characteristic polynomial of $L(G)$. For brevity, we will call $\mu(G, x)$ the *Laplacian polynomial* of G.

Laplace operator on graphs

The roots of $\mu(G,x)$ will be called the Laplacian eigenvalues (or sometimes just eigenvalues) of G. They will be denoted by

$$\lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_n(G),\ (n = |V(G)|),$$

always enumerated in increasing order and repeated according to their multiplicity. Recall that for connected graph G we always have $\lambda_1(G) = 0$ and $\lambda_2(G) > 0$.
Two graphs G and H are called *Laplacian isospectral* (or just *isospectral*) if their Laplacian polynomials coincide $\mu(G,x) = \mu(H,x)$.

Genus of graphs

We define genus of graph G as

$$g = |E(G)| - |V(G)| + 1.$$

In graph theory, the term "genus" is traditionally used for a different concept, namely, the smallest genus of any surface in which the graph can be embedded, and the integer $g = g(G)$ is called the cyclomatic or the Betti number of G. We call g the genus of G in order to highlight the analogy with Riemann surfaces.
A *bridge* is an edge of a graph G whose deletion increases the number of connected components. A graph is said to be *bridgeless* if it contains no bridges.

Laplacian polynomial

A. K. Kel'mans (1967) gave a combinatorial interpretation to all the coefficients of $\mu(G, x)$ in terms of the numbers of certain spanning trees of the graph. We present the result in the following form.

Theorem

If $\mu(G, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^i c_i x^{n-i} + \ldots + (-1)^{n-1} c_{n-1} x$ *then*

$$c_i = \sum_{S \subset V(G),\, |S| = n-i} T(G_S),$$

where $T(H)$ *is the number of spanning trees of* H, *and* G_S *is obtained from graph* G *by identifying all points of* S *to a single point.*

Theta graphs

Let u and v are two (not necessary distinct) vertices. Denote by $\Theta(k, l, m)$ the graph consisting of three internally disjoint paths joining u to v with lengths $k, l, m \geq 0$ (see Fig. 1). We set
$\sigma_1 = \sigma_1(k, l, m) = k + l + m, \sigma_2 = \sigma_1(k, l, m) = k\,l + l\,m + k\,m$, and
$\sigma_3 = \sigma_1(k, l, m) = k\,l\,m$.
It is easy to see that two graphs $\Theta(k, l, m)$ and $\Theta(k', l', m')$ are isomorphic if and only if the unordered triples $\{k, l, m\}$ and $\{k', l', m'\}$ coincide.
We start with the following lemma.

Lemma 1

Let G be an arbitrary bridgeless graph of genus two. Then G is isomorphic to $\Theta(k, l, m)$ for some k, l, m with $\sigma_2 = k\,l + l\,m + k\,m > 0$.

Theta graphs

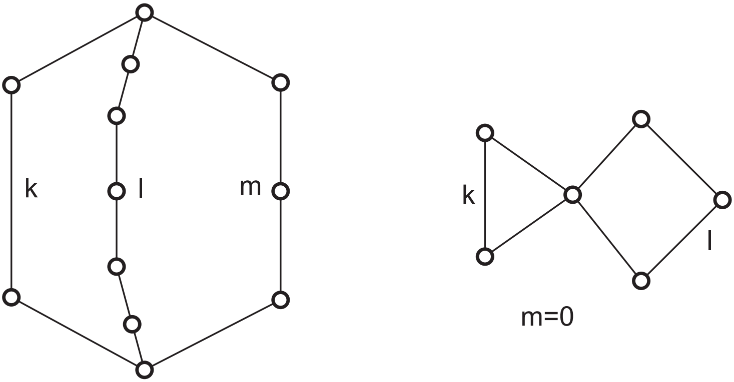

Fig.1. Theta graph $\Theta(k, l, m)$.

The main result

The main result of the presentaion is the following theorem.

Theorem

Two genus two bridgeless graphs are Laplacian isospectral if and only if they are isomorphic.

The proof of the theorem is based on the following three lemmas.

Lemma 2

Lemma 2

Let $G = \Theta(k, l, m)$ be a theta graph and

$$\mu(G, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^{n-1} c_{n-1} x$$

is its Laplacian polynomial. Then $n = k + l + m - 1, c_1 = 2(k + l + m)$ and $c_{n-1} = (k\,l + l\,m + k\,m)(k + l + m - 1)$.

Proof. The number of vertices, edges and spanning trees of graph G are given by

$$|V(G)| = k + l + m - 1,\ |E(G)| = k + l + m,\ T(G) = k\,l + l\,m + k\,m.$$

Then by the Kel'mans theorem we have
$n = |V(G)| = k + l + m - 1$, $c_1 = 2|E(G)| = 2(k + l + m)$ and
$c_{n-1} = |V(G)| \cdot T(G) = (k\,l + l\,m + k\,m)(k + l + m - 1)$.

Lemma 3

Let $G = \Theta(k, l, m)$ *be a theta graph and*
$\mu(G, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^{n-1} c_{n-1} x$ *is its Laplacian polynomial.*
Then

$$c_{n-2} = A(\sigma_1, \sigma_2) + B(\sigma_1, \sigma_2)\sigma_3$$

where $A(s,t) = (4t - 3st - 2s^2 t + s^3 t + 4t^2 - st^2)/12$, $B(s,t) = (3 - 4s + s^2 - 3t)/12$,
$\sigma_1 = k + l + m$, $\sigma_2 = k\,l + l\,m + k\,m$, *and* $\sigma_3 = k\,l\,m$.

By the Kelmans theorem we have

$$c_{n-2} = \sum_{S \subset V,\, |S|=2} T(X_S),$$

where X_S runs through all graphs obtained from $G = \Theta(k, l, m)$ by gluing two vertices. There are exactly four types of such graphs shown on Fig.2.

Lemma 3

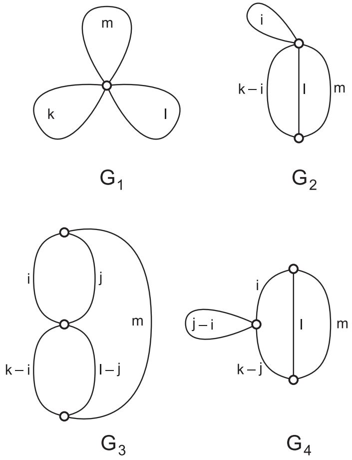

Fig. 2. The graphs obtained from $\Theta(k, l, m)$ by gluing two vertices

Lemma 3

Type G_1. Glue two 3-valent vertices of graph G. As a result we obtain the graph G_1 shown on Fig. 2. The number of spanning trees of this graph is $T_1 = T(C_k) \cdot T(C_l) \cdot T(C_m) = k\,l\,m$.

Type G_2. Glue one 3-valent and one 2-valent vertices of graph G. For given i, $1 \leq i \leq k-1$ we have $T(G_2) = i\sigma_2(k-i,\, l,\, m)$. We set $F(k,\, l,\, m) = \sum\limits_{i=1}^{k-1} i\sigma_2(k-i,\, l,\, m)$.

Then the total number of spanning trees for graphs of type G_2 is

$$T_2 = 2(F(k,\, l,\, m) + F(l,\, m,\, k) + F(m,\, k,\, l)).$$

In a similar way we calculate the numbers T_3 and T_4. Finally,

$$c_{n-2} = T_1 + T_2 + T_3 + T_4 = A(\sigma_1, \sigma_2) + B(\sigma_1, \sigma_2)\sigma_3.$$

Lemma 4

The proof of the following lemma is based on similar arguments.

Lemma 4

Let $G = \Theta(k, l, m)$ *be a theta graph and* $\mu(G, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^{n-1} c_{n-1} x$ *is its Laplacian polynomial. Then*

$$c_{n-3} = C(\sigma_1, \sigma_2) + D(\sigma_1, \sigma_2)\sigma_3 + E(\sigma_1, \sigma_2)\sigma_3^2,$$

where

$$\begin{aligned}
C(s,t) &= (-34t + 21st + 25s^2t - 10s^3t - 3s^4t + s^5t - 50t^2 + 10st^2 \\
&+ 12s^2t^2 - 2s^3t^2 - 16t^3 + st^3)/360, \\
D(s,t) &= (-45 + 50s + 5s^2 - 12s^3 + 2s^4 + 24st - 9s^2t + 15t^2)/360, \\
E(s,t) &= -3(-8 + 3s)/360.
\end{aligned}$$

One can consider the six cases shown on the Fig. 3.

Lemma 4

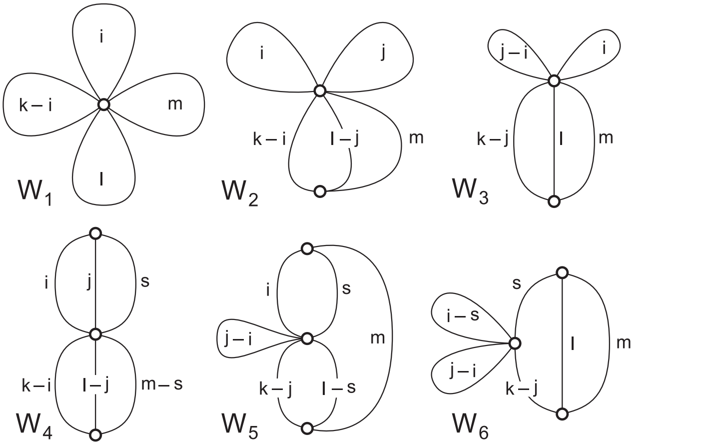

Fig. 3. The graphs obtained from $\Theta(k, l, m)$ by gluing three vertices

Proof of the Main Theorem

Let G and G' be two bridgeless graphs of genus two. Then by Lemma 1 for suitable $\{k, l, m\}$ and $\{k', l', m'\}$ we have

$$G = \Theta(k, l, m) \text{ and } G' = \Theta(k', l', m')$$

Denote by $\mu(G, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^{n-1} c_{n-1} x$ and $\mu(G', x) = x^{n'} - c_1 x^{n'-1} + \ldots + (-1)^{n'-1} c_{n'-1} x$ their Laplacian polynomials.

Suppose that the graphs G and G' are isospectral. Then $n' = n, c_1' = c_1, \ldots, c_{n-1}' = c_{n-1}$. By Lemma 2 we obtain

$$2\sigma_1 = 2\sigma_1' \text{ and } \sigma_2(\sigma_1 - 1) = \sigma_2'(\sigma_1' - 1).$$

Since both graphs are of genus 2 we have $\sigma_1 > 1$ and $\sigma_1' > 1$. Then the obtained system of equations gives $\sigma_1 = \sigma_1'$ and $\sigma_2 = \sigma_2'$. The theorem will be proved if we show that $\sigma_3 = \sigma_3'$. We will do this in two steps.

Proof of the Main Theorem

By Lemma 3

$$c_{n-2} = A(\sigma_1, \sigma_2) + B(\sigma_1, \sigma_2)\sigma_3,$$

where $A(s,t) = (4t - 3st - 2s^2t + s^3t + 4t^2 - st^2)/12$ and $B(s,t) = (3 - 4s + s^2 - 3t)/12.$

Step 1. $B(\sigma_1, \sigma_2) \neq 0$. Since $c'_{n-2} = c_{n-2}, \sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$ we obtain

$$B(\sigma_1, \sigma_2)\sigma'_3 = B(\sigma_1, \sigma_2)\sigma_3.$$

Hence $\sigma_3 = \sigma'_3$ and the theorem is proved.
Step 2. Case $B(\sigma_1, \sigma_2) = 0$ is covered by Lemma 4 and some elementary number theoretical arguments.

The isospectral property does not hold for graphs with bridges and for graphs of genus $g = 3$.

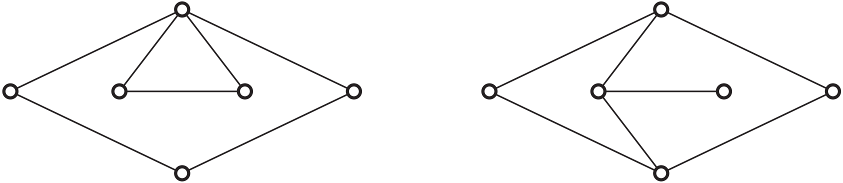

Fig. 4. Genus 2 graphs with the same spectrum. The second has a bridge.

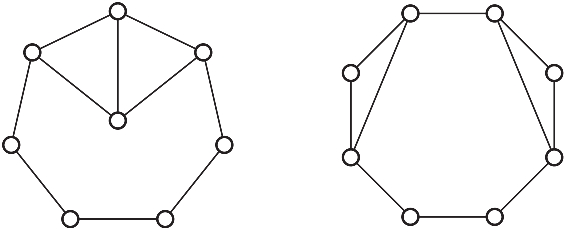

Fig. 5. Two nonisomorphic isospectral graphs of genus 3.