Isomorphism of isospectral genus three graphs

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Novosibirsk, 2014

The genus of graph

Let G be a finite connected multigraph. Denote by $V(G)$ and $E(G)$ the set of vertices and edges of graph G, respectively.

- We denote the genus of graph G by $g = |E(G)| - |V(G)| + 1$ – the dimension of the first homology group of G.

Laplacian matrix

- For each $u, v \in V(G)$, a_{uv} is equal to the number of edges between u and v.
- The matrix $A = A(G) = [a_{uv}], u, v \in V(G)$ is called the adjacency matrix of graph G.
- Let $d(v)$ denote the degree of $v \in V(G)$, $d(v) = \sum_u a_{uv}$.
- $D = D(G)$ is a diagonal matrix, $d_{vv} = d(v), v \in V(G)$.
- The matrix $L = L(G) = D(G) - A(G)$ is called the Laplacian matrix of G. We denote by $\mu(G, x)$ the characteristic polynomial of $L(G)$.

Two graphs G and H are called isospectral if their Laplacian polynomials coincide: $\mu(G, x) = \mu(H, x)$.

Laplacian matrix

A. K. Kel'mans (1967) gave a combinatorial interpretation to the coefficients of $\mu(X, x)$ in terms of the numbers of certain subforests of the graph.

Theorem (Kel'mans, 1967)

If $\mu(X, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^i c_i x^{n-i} + \ldots + (-1)^{n-1} c_{n-1} x$, then $c_i = \sum_{S \subset V, |S| = n-i} T(X_S)$, where $T(H)$ is the number of spanning trees of H and X_S is obtained from X by merging all the vertices of S.

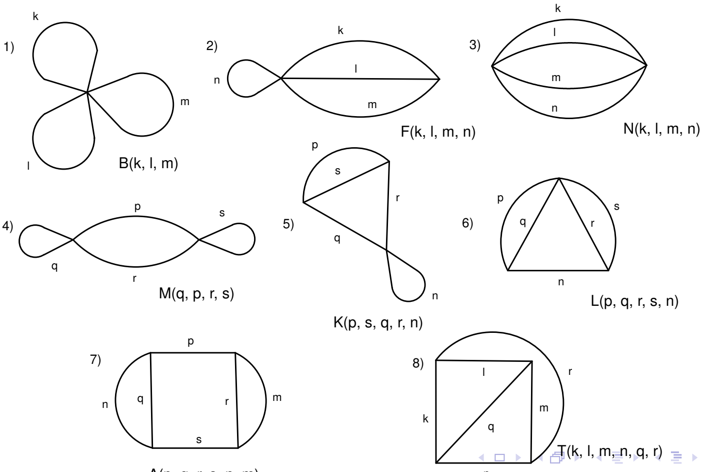
Corollary

$n = V(G), c_1 = 2E(G), c_{n-1} = V(G)T(G)$, $c_{n-2} = \sum_{S \subset V, |S| = 2} T(G_S)$, where graph G_S is obtained from graph G by merging the two vertices.

Classification of genus three graphs

Theorem

Let G be a finite connected bridgeless genus three multigraph. Then G is isomorphic to the graph of one of eight types.



Let $V(G) = \{v_1, v_2, ..., v_i, ..., v_N\}$.

Since there is no bridges then $d_i = deg(v_i) \geq 2 \ \forall i$.

If we remove a vertex u, $deg(u) = 2$, the number of vertices and edges decreases by one respectively $\Rightarrow$ removal of the vertex which degree is equal two won't change the genus of graph.

So we could suppose that $d_i \geq 3 \ \forall i$.

$\sum_i^N d_i = 2E$

$E - V + 1 = g(G) = 3 \Rightarrow E = V + 2$

$3V = 3N \leq \sum_i^N d_i = 2E = 2V + 4 \Rightarrow V \leq 4$

Isomorphism of isospectral genus three

Hypothesis

Two bridgeless genus three graphs belonging to the same type are isospectral if and only if they are isomorphic.

Graph B(k, l, m)

Two graphs $B(k,l,m)$ and $B(k',l',m')$ are isomorphic $\Leftrightarrow$ the unordered triples $\{k,l,m\}$ and $\{k',l',m'\}$ coincide.

Equivalently,

- $\sigma_1(k,l,m) = \sigma_1(k',l',m')$
- $\sigma_2(k,l,m) = \sigma_2(k',l',m')$
- $\sigma_3(k,l,m) = \sigma_3(k',l',m')$,

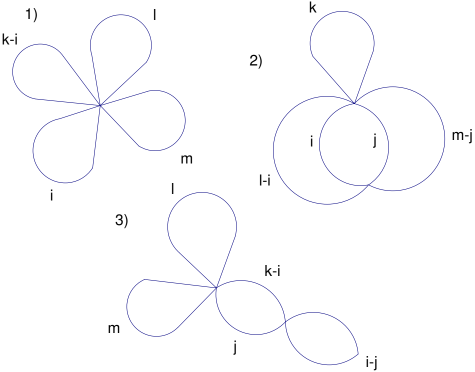
where $\sigma_1(x,y,z) = x+y+z$; $\sigma_2(x,y,z) = xy+yz+xz$; $\sigma_3(x,y,z) = xyz$

$B = B(k, l, m)$,
$\mu(B, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^i c_i x^{n-i} + \ldots + (-1)^{n-1} c_{n-1} x$ - Laplacian polynomial of B.

The number of vertices, edges and spanning trees of graph B could be calculated by

- $V(B) = k + l + m - 2$
- $E(B) = k + l + m$
- $T(B) = klm$

c_{n-2} for graph B



The number spanning trees for the graphs obtained from graph B by merging two vertices:

- $H1[k,l,m] = \sum_{i=1}^{k-1} iklm$
- $T1 = H1[k,l,m] + H1[l,m,k] + H1[m,k,l]$

- $H2[k,l,m] = \sum_{i=1}^{l-1} \sum_{j=1}^{m-1} k(mi(l-i) + lj(m-j))$
- $T2 = H2[k,l,m] + H2[l,m,k] + H2[m,k,l]$

- $H3[k,l,m] = \sum_{i=2}^{k-1} \sum_{j=1}^{i-1} ml(i-j)(k-i+j)$
- $T3 = H3[k,l,m] + H3[l,m,k] + H3[m,k,l]$

If graphs B and B' are isospectral then

- $n = V(B) = V(B') = n' \Rightarrow k + l + m = k' + l' + m' \Rightarrow \sigma_1 = \sigma_1^{'}$
- $c_1(B) = c_1(B') \Rightarrow V(B)T(B) = V(B')T(B') \Rightarrow (\sigma_1 - 2)\,\sigma_3 = \left(\sigma_1^{'} - 2\right)\sigma_3^{'}$
  $\Rightarrow \sigma_3 = \sigma_3^{'}$
- We can prove that $\sigma_2 = \sigma_2^{'}$ by calculation of c_{n-2}.

For graph B

$c_{n-2} = 1/12klm(12 - 5k - 4k^2 + k^3 - 5l + 2k^2l - 4l^2 + 2kl^2 + l^3 - 5m + 2k^2m + 2l^2m - 4m^2 + 2km^2 + 2lm^2 + m^3)$

We can rewrite this using symmetric polynomials:

$c_{n-2} = 1/12(12\sigma_3 - 5\sigma_1\sigma_3 - 4\sigma_1^2\sigma_3 + \sigma_1^3\sigma_3 + 8\sigma_2\sigma_3 - \sigma_1\sigma_2\sigma_3 - 3\sigma_3^2)$

Since we proved previously that

- $c_{n-2}(B) = c_{n-2}(B')$
- $\sigma_1 = \sigma_1^{'}$
- $\sigma_3 = \sigma_3^{'}$

we obtain that $\sigma_2 = \sigma_2^{'}$.

We proved that three symmetric polynomials coincide $\Longrightarrow$ the unordered triples (k, l, m) and (k', l', m') coincide $\Longrightarrow$ graphs B and B' are isomorphic.