Groebner - Shirshov bases for groups, semigroups, categories, and Lie algebras

Leonid A. Bokut
Sobolev Institute of Mathematics, Novosibirsk, Russia
bokut(at)math.nsc.ru

This is joint work with Yuqun Chen

Groebner bases and Groebner-Shirshov bases were invented independently in 1960th by A.I. Shirshov for ideals of free (commutative, anti-commutative) non-associative algebras, free Lie algebras, and implicitly free associative algebras, by H. Hironaka for ideals of the power series algebras (both formal and convergent), and by B. Buchberger for ideals of the polynomial algebras. Groebner bases and Groebner-Shirshov bases theories have been proved to be useful in different branches of mathematics, including commutative algebras; non-commutative and non-associative (super-, Lie, Kac-Moody, Drinfeld quantum group, conformal, ...) algebras; modules; semigroups (groups, categories, semirings,...) via semigroup (group, category, semiring, ...) algebras; operads; etc.. It is a powerful tool to study the classical subjects in algebra: relations; representations; free linear universal algebras; normal forms; word problems; conjugacy problems; algorithmic problmsconfluence rewriting systems; automaton groups and semigroups; embedding theorems; PBW type theorems; extensions; homologies; growth functions; Dehn functions; complexities; etc. We will emphasize on Groebner - Shirsov bases and normal forms for finite Coxeter and braid groups, plactic monoid, simplicial and cyclic categories, semisimple Lie algebras, Shirshov-Cartier-Cohn counter examples to PBW theorem for Lie algebras over a commutative algebra.

Composition-Diamond Lemma for associative algebras over a field. Let S be a monic subset of an algebra $k < X >$ (free associative), and $(X*, \leq)$ be the deg-lex order of on words $X*$. Then the following conditions are equivalent:

(i) S is a GS basis (any composition $(f,g)_w = fb - -ag$, where $w = \bar{f}b = a\bar{g}$, $\bar{f}$ be the maximal word in f, is “trivial” $mod(S,w)$;

(ii) $f \in Ideal(S) \longrightarrow \bar{f} = u\bar{s}v$ for some $s \in S$;

(iii) $Irr(S) = \{w, w \neq u\bar{s}v$ for any $s \in S\}$ is a linear basis of $k < X|S >= k < X > /Ideal(S)$.